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A framed f-structure on the tangent bundle of a Finsler manifold

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Abstract. Let (M,F) be a Finsler manifold, that is, M is a smooth manifold endowed with a Finsler metric F. In this paper, we introduce on the slit tangent bundle $\widetilde{TM}$ a Riemannian metric $\widetilde{G}$ which is naturally induced by F, and a family of framed f-structures which are parameterized by a real parameter $c \neq 0$. We prove that (i) the parameterized framed f-structure reduces to an almost contact structure on IM; (ii) the almost contact structure on IM is a Sasakian structure iff (M,F) is of constant flag curvature $K = c$; (iii) if $\mathcal{S} = y^i\delta_i$ is the geodesic spray of F and $R(\cdot,\cdot)$ the curvature operator of the Sasaki–Finsler metric which is induced by F, then $R(\cdot,\cdot)\mathcal{S} = 0$ iff (M,F) is a locally flat Riemannian manifold.

1. Introduction. Recently, the geometry of the tangent bundle of a smooth manifold attracts some people's interest [BF, O1, O2, OP]. As is well known, a Riemannian metric g on a smooth manifold M gives rise to several Riemannian metrics on the tangent bundle TM. The best known example is the Sasaki metric g_S, which was first introduced and studied in [S]. Although the Sasaki metric g_S is naturally induced by a Riemannian metric g on M, it is very rigid. For example, TM endowed with the Sasaki metric g_S is not locally symmetric unless the metric g is flat [K]. Moreover, the Sasaki metric g_S is not a good metric in the sense of [B] since its Ricci curvature is not constant, that is, the Sasaki metric g_S is generally not an Einstein metric.

To overcome this defect, V. Oproiu and his collaborators [O1, O2, OP] constructed on TM a family of Riemannian metrics with respect to which TM is a locally symmetric Riemannian manifold and has constant Ricci curvature (or is an Einstein manifold). It is natural to ask whether we can construct some nice metrics on $\widetilde{TM}$ under the condition that M is endowed with a Finsler metric. Recently, using the Sasaki–Finsler metric on the tan-

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gent bundle of a Finsler manifold, M. Anastasiei [A1] showed that the indicatrix bundle of a Finsler manifold carries a contact structure. A. Bejancu and H. R. Farran [BF] proved that a Finsler manifold (M, F) is of constant curvature $K = 1$ if and only if the horizontal Liouville vector field $\mathcal{S} = y^i \delta_i$ (also called the geodesic spray field associated to F) is a Killing vector field on the indicatrix bundle IM of (M, F).

Let (M, F) be a *Finsler manifold*, i.e., M is a smooth manifold and F is a Finsler metric on M. Denote $\widetilde{TM}$ the *slit tangent bundle* of M, i.e., the complement of the zero section in TM. In this paper, we introduce the following lift metric $\widetilde{G}$ on $\widetilde{TM}$ (cf. Section 3):

$$\widetilde{G} = G_{ij}dx^i dx^j + H_{ij}\delta y^i \delta y^j \tag{1.1}$$

where

$$G_{ij} = \frac{1}{\beta}g_{ij} + \frac{v}{\alpha\beta}y_i y_j, \qquad H_{ij} = \beta g_{ij} + w y_i y_j,$$

α and β are constants, and v and w are nonnegative functions of $\tau = F^2$.

We construct a parameterized framed f-structure on $\widetilde{TM}$. When restricted to the indicatrix bundle IM of (M, F), the framed f-framed structure reduces to an almost contact structure. We prove that the almost contact structure on IM is a Sasakian structure if and only if (M, F) is of constant flag curvature $K = c \neq 0$.

The main results of this paper are (cf. Theorems 4.9, 5.2, 5.4, 5.5 and 6.4 for details):

THEOREM 1.1. *Let $\widetilde{G}$, $\widetilde{f}$, $(\widetilde{\xi}_a)$, $(\widetilde{\eta}^a)$, $a = 1, 2$, be defined respectively by* (3.6), (4.7), (4.1) *and* (4.2). *Then the ensemble $(\widetilde{f}, (\widetilde{\xi}_a), (\widetilde{\eta}^a))$, $a = 1, 2$, provides a framed f-structure on $\widetilde{TM}$ if and only if*

$$\widetilde{G} = \left(\frac{1}{\beta}g_{ij} + \frac{\beta\tau - 1}{\beta\tau}y_i y_j\right)dx^i dx^j + \left(\beta g_{ij} + \frac{1 - \beta\tau}{\tau^2}y_i y_j\right)\delta y^i \delta y^j. \tag{1.2}$$

THEOREM 1.2. *Let $(\widetilde{f}, (\widetilde{\xi}_a), (\widetilde{\eta}^a)), a = 1, 2$, be the framed f-structure given by Theorem* 4.8. *Then the triple $(\widetilde{f}, \widetilde{\xi}_1, \widetilde{\eta}^1)$ defines an almost contact structure on IM, that is,*

$$\widetilde{\eta}^1(\widetilde{\xi}_1) = 1, \qquad \widetilde{f}(\widetilde{\xi}_1) = 0, \qquad \widetilde{\eta}^1 \circ \widetilde{f} = 0,$$
$$\widetilde{f}^2 = -I + \widetilde{\eta}^1 \otimes \widetilde{\xi}_1,$$
$$\widetilde{f}^3 + \widetilde{f} = 0, \qquad \operatorname{rank} \widetilde{f} = 2n - 2.$$

THEOREM 1.3. *Let (M, F) be a Finsler manifold endowed with the Chern–Rund connection ∇. Let also $c \neq 0$ be a parameter and*

(1.3)

$$\widetilde{G} = \sqrt{|c|}\left[g_{ij} + \left(\frac{1}{\sqrt{|c|}} - 1\right)y_i y_j\right]dx^i dx^j + \left[\frac{1}{\sqrt{|c|}}g_{ij} + \left(1 - \frac{1}{\sqrt{|c|}}\right)y_i y_j\right]\delta y^i \delta y^j$$

be the Riemannian metric on IM. Then $(IM, \widetilde{f}, \widetilde{\xi}_1, \widetilde{\eta}^1, \widetilde{G})$ is a contact Riemannian manifold.

THEOREM 1.4. *Let (M, F) be a Finsler manifold endowed with the Chern–Rund connection ∇. Let also $c \neq 0$ be a parameter and*

$$\widetilde{G} = \sqrt{|c|}\left[g_{ij} + \left(\frac{1}{\sqrt{|c|}} - 1\right) y_i y_j\right] dx^i dx^j + \left[\frac{1}{\sqrt{|c|}} g_{ij} + \left(1 - \frac{1}{\sqrt{|c|}}\right) y_i y_j\right] \delta y^i \delta y^j \tag{1.4}$$

be the Riemannian metric on IM. Then $(IM, \widetilde{f}, \widetilde{\xi}_1, \widetilde{\eta}^1, \widetilde{G})$ is a Sasakian manifold if and only if (M, F) is of constant flag curvature $K = c$.

THEOREM 1.5. *Let (M, F) be a Finsler manifold, $\mathcal{S} = y^i \delta_i$ be the geodesic spray field of F and $R(\cdot, \cdot)$ be the curvature operator of the Sasaki–Finsler metric*

$$\widetilde{G}_S = g_{ij} dx^i dx^j + g_{ij} \delta y^i \delta y^j.$$

Then

$$R(X, Y)\mathcal{S} = 0 \qquad \forall X, Y \in \mathcal{X}(TM)$$

if and only if (M, F) is locally a flat Riemannian manifold.

The organization of this paper is as follows. In Section 2, we fix some definitions and notation, and introduce the Chern–Rund connection ∇ of (M, F). In Section 3, we introduce a Riemannian metric $\widetilde{G}$ on the slit tangent bundle $\widetilde{TM}$, and define an almost complex structure $\widetilde{F}$ on $\widetilde{TM}$. In Section 4, we first define two vector fields $\widetilde{\xi}_1, \widetilde{\xi}_2$ and two 1-forms $\widetilde{\eta}^1, \widetilde{\eta}^2$ on $\widetilde{TM}$, and give some basic properties of these objects. Then we prove that there is a framed f-structure on $\widetilde{TM}$ if and only if $\tau(\beta + w\tau) = 1$. In Section 5, we prove that the framed f-structure on $\widetilde{TM}$ naturally induces an almost contact structure $(\widetilde{f}, \widetilde{\xi}_1, \widetilde{\eta}^1)$ on the indicatrix bundle IM such that $(IM, \widetilde{f}, \widetilde{\xi}_1, \widetilde{\eta}^1, \widetilde{G})$ is a contact Riemannian manifold. Furthermore, we prove that $(IM, \widetilde{f}, \widetilde{\xi}_1, \widetilde{\eta}^1, \widetilde{G})$ is a Sasakian manifold if and only if (M, F) is of constant flag curvature $K = c \neq 0$. In Section 6, we prove that $R(\cdot, \cdot)\mathcal{S} = 0$ if and only if (M, F) is a locally flat Riemannian manifold, and prove that IM with the contact Riemannian metric structure $(\widetilde{f}, \widetilde{\xi}_1, \widetilde{\eta}^1, \widetilde{G}_S)$ is locally isometric to $E^n \times S^{n-1}(4)$ for $n > 2$ and flat for $n = 2$ if the Finsler metric F comes from a locally flat Riemannian metric on M.

2. Preliminaries. In this section we fix some definitions and notation, and introduce the Chern–Rund connection ∇ of (M, F).

Suppose M is an n-dimensional C^∞ manifold with local coordinates $(x^1, \ldots, x^n)$. Denote by $T_x M$ the tangent space at $x \in M$ and $\pi : TM \to M$ the tangent bundle of M. Let $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ be the induced local coordinates on TM and $\mathcal{X}(\widetilde{TM})$ be the set of sections of the tangent bundle $T\widetilde{TM}$ of $\widetilde{TM}$.

Definition 2.1. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ such that

(i) F is C^∞ on $\widetilde{TM}$;
(ii) F is positively 1-homogeneous on the fibers of TM, i.e., $F(x, \lambda y) = \lambda F(x, y)$ for $\lambda > 0$;
(iii) for each $y \in \widetilde{T_xM}$, the quadratic form $g_y : T_xM \times T_xM \to \mathbb{R}$ defined by $g_y(u_1, u_2) := g_{ij}(y)u_1^i u_2^j$ is positive definite, i.e., the Finsler fundamental tensor
$$g_{ij}(y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$$
is positive definite on $\widetilde{TM}$.

We denote by (g^{ij}) the inverse matrix of (g_{ij}), i.e., $g_{ij}g^{ik} = \delta_j^k$.

Let $x \in M$ and denote by F_x the restriction of F to the fiber T_xM. To measure the non-Euclidean features of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by $\mathbf{C}_y(u_1, u_2, u_3) := C_{ijk}(y)u_1^i u_2^j u_3^k$ where
$$C_{ijk}(y) := \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$
By the 1-homogeneity of F, it is easy to check that

$$C_{ijk}(y)y^i = C_{ijk}(y)y^j = C_{ijk}(y)y^k = 0. \tag{2.1}$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in \widetilde{TM}}$ is called the *Cartan torsion* of (M, F). Using the Cartan torsion $\mathbf{C}$, one can define the *mean Cartan torsion* $\mathbf{I}_y : T_xM \to \mathbb{R}$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i(y) := g^{jk}C_{ijk}(y)$. It is well known that $\mathbf{I} = 0$ if and only if F comes from a Riemannian metric on M.

Let π^*TM be the pull-back tangent bundle over $\widetilde{TM}$. It is well known [CS] that π^*TM admits a unique linear connection ∇ called the *Chern–Rund connection*, which is torsion free and almost metric compatible. In the following we shall recall the connection coefficients and curvature components of ∇.

First, using the Finsler fundamental tensor g_{ij} and the Cartan tensor C_{ijk} associated to F, one defines the tensor $C^i{}_{jk} := g^{is}C_{sjk}$, which is actually the vertical connection coefficients of the Cartan connection of (M, F). If we define the formal Christoffel symbols $\gamma^k{}_{ij}$ of the second kind of F by
$$\gamma^k{}_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right),$$

then there is a canonical nonlinear connection on $\widetilde{TM}$, which is locally characterized by its connection coefficients N^i_j, i.e.,

$$N^i_j := \gamma^i{}_{jk} y^k - C^i{}_{jk} \gamma^k{}_{rs} y^r y^s.$$

We denote by $G^i := \frac{1}{2}\gamma^i{}_{jk} y^j y^k$ the spray coefficients of F and by $G^i{}_{jk} := \partial^2 G^i / \partial y^j \partial y^k$ the Berwald connection coefficients of F. The connection coefficients Γ^k_{ij} of the Chern–Rund connection ∇ are given by (see [CS])

$$\Gamma^k{}_{ij}(x,y) = \tfrac{1}{2} g^{kl} [\delta_i(g_{jl}) + \delta_j(g_{il}) - \delta_l(g_{ij})];$$

here and in the following we denote $\dot{\partial}_j := \partial/\partial y^j$ and $\delta_i := \partial/\partial x^i - N^j_i \partial/\partial y^j$. It is clear that

$$\Gamma^k{}_{ij} = \Gamma^k_{ji}, \qquad y^i \Gamma^k{}_{ij} = N^k_j. \tag{2.2}$$

Note that $\{\delta_i, \dot{\partial}_i\}$ is the natural local frame on $\widetilde{TM}$, and its dual coframe is $\{dx^i, \delta y^i\}$, where we denote $\delta y^i = dy^i + N^i_j dx^j$. In terms of the natural local frame $\{dx^i, \delta y^i\}$ on $\widetilde{TM}$, the connection 1-form of ∇ is $\omega_j{}^i = \Gamma^i{}_{jk} dx^k$, and the curvature 2-form of ∇ is

$$\Omega_j{}^i = d\omega_j{}^i - \omega_j{}^k \wedge \omega_k{}^i.$$

More precisely [CS],

$$\Omega_j{}^i = \tfrac{1}{2} R_j{}^i{}_{kl} dx^k \wedge dx^l + P_j{}^i{}_{kl} dx^k \wedge \delta y^l,$$

where

$$R_j{}^i{}_{kl} = \delta_k(\Gamma^i{}_{jl}) - \delta_l(\Gamma^i{}_{jk}) + \Gamma^i{}_{ks}\Gamma^s{}_{jl} - \Gamma^s{}_{jk}\Gamma^i{}_{ls}, \tag{2.3}$$

$$P_j{}^i{}_{kl} = -\dot{\partial}_l(\Gamma^i{}_{jk}). \tag{2.4}$$

It is clear that

$$R_j{}^i{}_{kl} + R_j{}^i{}_{lk} = 0, \tag{2.5}$$

$$R_j{}^i{}_{kl} + R_k{}^i{}_{lj} + R_l{}^i{}_{jk} = 0, \tag{2.6}$$

$$P_j{}^i{}_{kl} = P_k{}^i{}_{jl}. \tag{2.7}$$

Denote

$$R^i{}_{kl} := y^j R_j{}^i{}_{kl}, \qquad L^i{}_{kl} := -y^j P_j{}^i{}_{kl}. \tag{2.8}$$

Clearly,

$$R^i{}_{kl} = -R^i{}_{lk}, \qquad L^i{}_{kl} = L^i{}_{lk}, \qquad L^i{}_{kl} y^k = 0, \tag{2.9}$$

and it is easy to check that (cf. [CS])

$$R^i{}_{kl} = \delta_k(N^i_l) - \delta_l(N^i_k), \qquad L^i{}_{kl} = G^i{}_{kl} - \Gamma^i{}_{kl}. \tag{2.10}$$

Set

$$R_{jikl} := g_{is} R_j{}^s{}_{kl}, \qquad R_{ikl} := g_{is} R^s{}_{kl}, \qquad R^i{}_k := R^i{}_{kl} y^l, \qquad R_{ij} := g_{im} R^m{}_j.$$

Clearly,

$$y^j R_{jikl} = R_{ikl}, \quad y^l R_{ikl} = -y^l R_{ilk} = R_{ik} \tag{2.11}$$

and it is easy to check that (cf. [CS])

$$R^i{}_k y^k = 0, \quad R_{ij} = R_{ji}. \tag{2.12}$$

The flag curvature of the Chern–Rund connection ∇ associated to F is a geometrical invariant, which generalizes the sectional curvature in Riemannian geometry. Let $x \in M$ and $0 \neq y \in T_xM$. Then $V := V^i \frac{\partial}{\partial x^i}$ is called the *transverse edge*. The flag curvature is obtained by carrying out the following computation at the point $(x, y) \in \widetilde{TM}$, and viewing y and V as sections of π^*TM:

$$K(y, V) := \frac{R_{ik} V^i V^k}{g(y, y)g(V, V) - [g(y, V)]^2}.$$

If $K(y, V)$ is independent of the transverse edge V, i.e., there is a scalar function $\lambda(x, y)$ on $\widetilde{TM}$ such that $K(y, V) = \lambda(x, y)$, then (M, F) is called of *scalar flag curvature*. If furthermore $\lambda(x, y)$ is constant on $\widetilde{TM}$, then the Finsler manifold (M, F) is called of *constant flag curvature*.

A framed f-structure is a natural generalization of an almost contact structure. It was introduced by S. I. Goldberg and K. Yano [GY]. We recall its definition following [MR].

Definition 2.2. Let $\widetilde{M}$ be a $(2n + s)$-dimensional manifold endowed with an endomorphism f of rank $2n$ of the tangent bundle satisfying $f^3 + f = 0$. If there are vector fields (ξ_b) and 1-forms η^a $(a, b = 1, \dots, s)$ on $\widetilde{M}$ such that

$$\eta^a(\xi_b) = \delta^a_b, \quad f(\xi_a) = 0, \quad \eta^a \circ f = 0, \quad f^2 = -I + \sum_{a=1}^{s} \eta^a \otimes \xi_a, \tag{2.13}$$

where I is the identity automorphism of the tangent bundle, then $\widetilde{M}$ is said to be a *framed f-manifold*.

Let $\widetilde{M}$ be a $(2n - 1)$-dimensional contact Riemannian manifold with a contact metric structure (f, ξ, η, g) and $R(\cdot, \cdot)$ be the curvature operator of the Riemannian metric g. It is well-known that the condition $R(\cdot, \cdot)\xi = 0$ has strong implications for a contact metric manifold, namely that $\widetilde{M}$ is locally the product of Euclidean space E^n and a sphere of constant curvature $+4$. In [B1] and [B2], D. E. Blair proved the following theorem.

Theorem 2.3. *A contact metric manifold $\widetilde{M}^{2n-1}$ satisfying $R(\cdot, \cdot)\xi = 0$ is locally isometric to $E^n \times S^{n-1}(4)$ for $n > 2$ and flat for $n = 2$.*

3. A Riemannian metric on $\widetilde{TM}$. In this section we shall introduce a Riemannian metric $\widetilde{G}$ and an almost complex structure $\widetilde{F}$ on $\widetilde{TM}$.

Let $v : [0, \infty) \to \mathbb{R}$ be a smooth function. Then it makes sense to consider the function $v(\tau)$, where $\tau := F^2$ is defined on TM and smooth on $\widetilde{TM}$. We define a symmetric M-tensor G_{ij} on $\widetilde{TM}$ such that

$$G_{ij} := \frac{1}{\beta} g_{ij} + \frac{v}{\alpha\beta} y_i y_j, \tag{3.1}$$

where α, β are real constants and $y_i = g_{ij} y^j$. It is easy to check that the matrix (G_{ij}) is positive definite on $\widetilde{TM}$ if and only if $\alpha, \beta > 0, \alpha + 2\tau v > 0$. Let (H^{kl}) be the inverse matrix of (G_{ij}), i.e., $G_{ij} H^{ik} = \delta_j^k$. Then

$$H^{kl} = \beta g^{kl} + w y^k y^l, \tag{3.2}$$

$$w = -\frac{\beta v}{\alpha + \tau v}. \tag{3.3}$$

The components H^{kl} define a symmetric M-tensor on $\widetilde{TM}$. It is easy to see that if the matrix (G_{ij}) is positive definite, then so is (H^{kl}). Denote by $H_{ij}(x, y)$ the symmetric M-tensor field of type (0,2) obtained from the components H^{kl} by lowering the indices, i.e.,

$$H_{ij} = g_{ik} H^{kl} g_{lj} = \beta g_{ij} + w y_i y_j. \tag{3.4}$$

We also need the following M-tensor fields on TM obtained by usual algebraic tensor operations:

$$\begin{cases} G^{kl} = g^{ki} G_{ij} g^{jl} = \dfrac{1}{\beta} g^{kl} + \dfrac{v}{\alpha\beta} y^k y^l, \\ G_k^i = G^{ih} g_{hk} = G_{kh} g^{hi} = \dfrac{1}{\beta} \delta_k^i + \dfrac{v}{\alpha\beta} y^i y_k, \\ H_k^i = H^{ih} g_{hk} = H_{kh} g^{hi} = \beta \delta_k^i + w y^i y_k, \end{cases} \tag{3.5}$$

where (H_k^i) is the inverse matrix of (G_k^i), i.e., $H_k^i G_i^j = \delta_k^j$.

We introduce the Riemannian metric

$$\widetilde{G} = G_{ij} dx^i dx^j + H_{ij} \delta y^i \delta y^j \tag{3.6}$$

on the slit tangent bundle $\widetilde{TM}$. Equivalently,

$$\widetilde{G}(\delta_i, \delta_j) = G_{ij}, \quad \widetilde{G}(\dot{\partial}_i, \dot{\partial}_j) = H_{ij}, \quad \widetilde{G}(\delta_i, \dot{\partial}_j) = \widetilde{G}(\dot{\partial}_i, \delta_j) = 0.$$

Now we define an endomorphism $\widetilde{F} : \mathcal{X}(\widetilde{TM}) \to \mathcal{X}(\widetilde{TM})$ such that

$$\widetilde{F}(\delta_i) = -G_i^k \dot{\partial}_k, \quad \widetilde{F}(\dot{\partial}_i) = H_i^k \delta_k. \tag{3.7}$$

It is easy to check that $\widetilde{F}^2 = -I$, where I is the identity endomorphism on $\mathcal{X}(\widetilde{TM})$. Thus $\widetilde{F}$ is an almost complex structure on $\widetilde{TM}$ [PT].

4. A framed f-structure on $\widetilde{TM}$. In this section we shall prove that there is a framed f-structure on $\widetilde{TM}$, which is parameterized by a real parameter. We do this by defining a tensor field $\widetilde{f}$ of type $(1,1)$ on $\widetilde{TM}$, and obtain a necessary and sufficient condition for $\widetilde{f}$ to be a framed f-structure on $\widetilde{TM}$.

As is well known, there are two remarkable vector fields defined on $\widetilde{TM}$. One is the *vertical Liouville vector field* $\mathcal{C} = y^i\dot{\partial}_i$, which is globally defined on $\widetilde{TM}$. The other is the *horizontal Liouville vector field* $\mathcal{S} = y^i\delta_i$ (also called the *geodesic spray field* of F).

Now we define the vector fields $\widetilde{\xi}_1, \widetilde{\xi}_2$ and 1-forms $\widetilde{\eta}^1, \widetilde{\eta}^2$ on $\widetilde{TM}$ respectively by

$$\widetilde{\xi}_1 := (\beta + w\tau)\mathcal{S}, \qquad \widetilde{\xi}_2 := \mathcal{C}, \tag{4.1}$$

$$\widetilde{\eta}^1 := y_i dx^i, \qquad \widetilde{\eta}^2 := (\beta + w\tau)y_i\delta y^i. \tag{4.2}$$

PROPOSITION 4.1. *Let* $\widetilde{G}$ *be defined by* (3.6) *and* $\widetilde{F}$ *be defined by* (3.7). *Then*

$$\widetilde{G}(\widetilde{F}(X), \widetilde{F}(Y)) = \widetilde{G}(X, Y) \quad \textit{for } X, Y \in \mathcal{X}(\widetilde{TM}). \tag{4.3}$$

LEMMA 4.2. *Let* $\widetilde{F}$ *be defined by* (3.7) *and* $\widetilde{\xi}_1, \widetilde{\xi}_2$ *be defined by* (4.1). *Then*

$$\widetilde{F}(\widetilde{\xi}_1) = -\xi_2, \qquad \widetilde{F}(\widetilde{\xi}_2) = \widetilde{\xi}_1. \tag{4.4}$$

Proof. This follows immediately from (3.7)–(4.2). ∎

LEMMA 4.3. *Let* $\widetilde{F}$ *be defined by* (3.7) *and* $\widetilde{\eta}_1, \widetilde{\eta}_2$ *be defined by* (4.2). *Then*

$$\widetilde{\eta}^1 \circ \widetilde{F} = \widetilde{\eta}^2, \qquad \widetilde{\eta}^2 \circ \widetilde{F} = -\widetilde{\eta}^1. \tag{4.5}$$

Proof. It is sufficient to check (4.5) with respect to the adapted frame $\{\delta_i, \dot{\partial}_i\}$ on $\widetilde{TM}$. In fact,

$$\begin{aligned}
&\widetilde{\eta}^1 \circ \widetilde{F}(\delta_i) = 0 = \widetilde{\eta}^2(\delta_i),\\
&\widetilde{\eta}^1 \circ \widetilde{F}(\dot{\partial}_i) = \widetilde{\eta}^1(H_i^k\delta_k) = H_i^k y_k = (\beta + w\tau)y_i = \widetilde{\eta}^2(\dot{\partial}_i)\\
&\widetilde{\eta}^2 \circ \widetilde{F}(\delta_i) = -y_i = -\widetilde{\eta}^1(\delta_i), \qquad \widetilde{\eta}^2 \circ \widetilde{F}(\dot{\partial}_i) = 0 = \widetilde{\eta}^1(\dot{\partial}_i). \ \blacksquare
\end{aligned}$$

LEMMA 4.4. *Let* $\widetilde{G}$ *be defined by* (3.6) *and* $\widetilde{\eta}_1, \widetilde{\eta}_2$ *be defined by* (4.2). *Then*

$$\widetilde{\eta}^1(X) = \widetilde{G}(X, \widetilde{\xi}_1), \qquad \widetilde{\eta}^2(X) = \widetilde{G}(X, \widetilde{\xi}_2) \quad \textit{for } X \in \mathcal{X}(\widetilde{TM}). \tag{4.6}$$

Proof. With respect to the adapted frame $\{\delta_i, \dot{\partial}_i\}$ on $\widetilde{TM}$, we have $\widetilde{\eta}^1(\delta_i) = y_i$ and

$$\widetilde{G}(\delta_i, \widetilde{\xi}_1) = (\beta + w\tau)\left(\frac{1}{\beta}g_{ij} + \frac{v}{\alpha\beta}y_iy_j\right)y^j = \frac{\alpha\beta + \beta\tau v + w\tau(\alpha + \tau v)}{\alpha\beta}y_i = y_i,$$

where in the last equality we use (3.3). Thus $\widetilde{G}(\delta_i, \widetilde{\xi}_1) = \widetilde{\eta}^1(\delta_i)$. It is clear that $\widetilde{G}(\dot{\partial}_i, \widetilde{\xi}_1) = 0 = \widetilde{\eta}^1(\dot{\partial}_i)$. Therefore $\widetilde{\eta}^1(X) = \widetilde{G}(X, \widetilde{\xi}_1)$. Similarly we can prove $\widetilde{\eta}^2(X) = \widetilde{G}(X, \widetilde{\xi}_2)$. ∎

LEMMA 4.5. *Let $\widetilde{\xi}_1, \widetilde{\xi}_2$ be defined by* (4.1) *and $\widetilde{\eta}^1, \widetilde{\eta}^2$ be defined by* (4.2). *Then*

$$\widetilde{\eta}^a(\widetilde{\xi}_b) = (\beta + w\tau)\tau\delta^a_b, \qquad a, b = 1, 2.$$

Proof. In fact, it is easy to check that

$$\widetilde{\eta}^1(\widetilde{\xi}_1) = (\beta + w\tau)\tau, \quad \widetilde{\eta}^1(\widetilde{\xi}_2) = 0,$$
$$\widetilde{\eta}^2(\widetilde{\xi}_2) = (\beta + w\tau)\tau, \quad \widetilde{\eta}^2(\widetilde{\xi}_1) = 0. \ ∎$$

Using the almost complex structure $\widetilde{F}$, we define a new tensor field $\widetilde{f}$ of type $(1, 1)$ on $\widetilde{TM}$ by

$$\widetilde{f} = \widetilde{F} + \widetilde{\eta}^1 \otimes \widetilde{\xi}_2 - \widetilde{\eta}^2 \otimes \widetilde{\xi}_1. \tag{4.7}$$

PROPOSITION 4.6. *The tensor field $\widetilde{f}$ satisfies*

$$\widetilde{f}(\widetilde{\xi}_1) = [(\beta + w\tau)\tau - 1]\widetilde{\xi}_2, \quad \widetilde{f}(\widetilde{\xi}_2) = [1 - (\beta + w\tau)\tau]\widetilde{\xi}_1, \tag{4.8}$$
$$\widetilde{\eta}^1 \circ \widetilde{f} = [1 - (\beta + w\tau)\tau]\widetilde{\eta}^2, \quad \widetilde{\eta}^2 \circ \widetilde{f} = [(\beta + w\tau)\tau - 1]\widetilde{\eta}^1, \tag{4.9}$$
$$\widetilde{f}^2 = -I + [2 - (\beta + w\tau)\tau](\widetilde{\eta}^1 \otimes \widetilde{\xi}_1 + \widetilde{\eta}^2 \otimes \widetilde{\xi}_2). \tag{4.10}$$

Proof. Equalities (4.8) follow from Lemmas 4.2 and 4.5; (4.9) follows from (4.8) and Lemma 4.5; and (4.10) follows from (4.8). ∎

PROPOSITION 4.7. *The Riemannian metric $\widetilde{G}$ satisfies*

$$\widetilde{G}(\widetilde{f}(X), \widetilde{f}(Y)) = \widetilde{G}(X, Y) - [2 - (\beta + w\tau)\tau][\widetilde{\eta}^1(X)\widetilde{\eta}^1(Y) + \widetilde{\eta}^2(X)\widetilde{\eta}^2(Y)]$$

for $X, Y \in \mathcal{X}(\widetilde{TM})$.

Proof. By Lemmas 4.4 and 4.5, we have

$$\widetilde{G}(\widetilde{\xi}_1, \widetilde{\xi}_1) = (\beta + w\tau)\tau = \widetilde{G}(\widetilde{\xi}_2, \widetilde{\xi}_2), \quad \widetilde{G}(\widetilde{\xi}_1, \widetilde{\xi}_2) = 0. \tag{4.11}$$

From (4.11) and Lemmas 4.3 and 4.4 we get

$$\begin{aligned}
\widetilde{G}(\widetilde{f}(X), \widetilde{f}(Y)) &= \widetilde{G}(\widetilde{F}(X), \widetilde{F}(Y)) + \widetilde{G}(\widetilde{F}(X), \widetilde{\xi}_2)\widetilde{\eta}^1(Y) \\
&\quad - \widetilde{G}(\widetilde{F}(X), \widetilde{\xi}_1)\widetilde{\eta}^2(Y) + \widetilde{G}(\widetilde{\xi}_2, \widetilde{F}(Y))\widetilde{\eta}^1(X) \\
&\quad + \widetilde{\eta}^1(X)\widetilde{\eta}^1(Y)\widetilde{G}(\widetilde{\xi}_2, \widetilde{\xi}_2) - \widetilde{\eta}^2(X)\widetilde{G}(\widetilde{\xi}_1, \widetilde{F}(Y)) \\
&\quad + \widetilde{\eta}^2(X)\widetilde{\eta}^2(Y)\widetilde{G}(\widetilde{\xi}_1, \widetilde{\xi}_1) \\
&= \widetilde{G}(X, Y) + \widetilde{\eta}^2(\widetilde{F}(X))\widetilde{\eta}^1(Y) - \widetilde{\eta}^1(\widetilde{F}(X))\widetilde{\eta}^2(Y) \\
&\quad + \widetilde{\eta}^2(\widetilde{F}(Y))\widetilde{\eta}^1(X) + \widetilde{\eta}^1(X)\widetilde{\eta}^1(Y)(\beta + w\tau)\tau \\
&\quad - \widetilde{\eta}^2(X)\widetilde{\eta}^1(\widetilde{F}(Y)) + \widetilde{\eta}^2(X)\widetilde{\eta}^2(Y)(\beta + w\tau)\tau \\
&= \widetilde{G}(X, Y) - [2 - (\beta + w\tau)\tau][\widetilde{\eta}^1(X)\widetilde{\eta}^1(Y) + \widetilde{\eta}^2(X)\widetilde{\eta}^2(Y)]. \ ∎
\end{aligned}$$

Theorem 4.8. *Let $\widetilde{G}$, $\widetilde{f}$, $(\widetilde{\xi}_a)$, $(\widetilde{\eta}^a)$, $a = 1,2$, be defined respectively by* (3.6), (4.7), (4.1) *and* (4.2). *Then the ensemble* $(\widetilde{f}, (\widetilde{\xi}_a), (\widetilde{\eta}^a))$, $a = 1,2$, *provides a framed f-structure on $\widetilde{TM}$ if and only if*

$$\tau(\beta + w\tau) = 1.$$

Proof. Let $(\widetilde{f}, (\widetilde{\xi}_a), (\widetilde{\eta}^b))$ be a framed f-structure on $\widetilde{TM}$. Then by Definition 2.2, we have $\widetilde{f}(\widetilde{\xi}_1) = \widetilde{f}(\widetilde{\xi}_2) = 0$. Thus by (4.8) we get $1-(\beta+w\tau)\tau = 0$. Conversely, if $\tau(\beta + w\tau) = 1$, then using Lemma 4.5 and Proposition 4.6 we obtain

$$\widetilde{\eta}^a(\widetilde{\xi}_b) = \delta^a_b, \quad \widetilde{f}(\widetilde{\xi}_a) = 0, \quad \widetilde{\eta}^a \circ \widetilde{f} = 0, \quad a,b = 1,2, \tag{4.12}$$

$$\widetilde{f}^2 = -I + \widetilde{\eta}^1 \otimes \widetilde{\xi}_1 + \widetilde{\eta}^2 \otimes \widetilde{\xi}_2. \tag{4.13}$$

In order to complete the proof, we need to prove $\widetilde{f}^3 + \widetilde{f} = 0$ and to show that $\widetilde{f}$ is of rank $2n-2$. It follows from (4.12) and (4.13) that

$$\widetilde{f}^3(X) = -\widetilde{f}(X) \quad \forall X \in \mathcal{X}(\widetilde{TM}).$$

Now we need to show that $\operatorname{Ker}\widetilde{f} = \operatorname{span}\{\widetilde{\xi}_1, \widetilde{\xi}_2\}$. The inclusion $\operatorname{span}\{\widetilde{\xi}_1, \widetilde{\xi}_2\} \subseteq \operatorname{Ker}\widetilde{f}$ follows from the second equation in (4.12). Now let $X \in \operatorname{Ker}\widetilde{f}$. Then $\widetilde{f}(X) = 0$ implies that

$$\widetilde{F}(X) + \widetilde{\eta}^1(X)\widetilde{\xi}_2 - \widetilde{\eta}^2(X)\widetilde{\xi}_1 = 0.$$

Thus

$$\widetilde{F}^2(X) = \widetilde{\eta}^1(X)\widetilde{F}(\widetilde{\xi}_2) - \widetilde{\eta}^2(X)\widetilde{F}(\widetilde{\xi}_1).$$

Since $\widetilde{F}^2 = -I$, it follows from Lemma 4.2 that

$$X = -\widetilde{\eta}^1(X)\widetilde{\xi}_1 - \widetilde{\eta}^2(X)\widetilde{\xi}_2,$$

that is, $X \in \operatorname{span}\{\xi_1, \xi_2\}$. ∎

Note that the condition $\tau(\beta + w\tau) = 1$ in Theorem 4.8 implies that

$$v = \frac{\alpha(\beta\tau - 1)}{\tau}, \quad w = \frac{1-\beta\tau}{\tau^2}. \tag{4.14}$$

Thus the functions v and w are related by

$$v = -\alpha\tau w. \tag{4.15}$$

Now if we substitute (4.14) into (3.1) and (3.4), we can restate Theorem 4.8 as follows:

Theorem 4.9. *Let $\widetilde{G}$, $\widetilde{f}$, $(\widetilde{\xi}_a)$, $(\widetilde{\eta}^a)$, $a = 1,2$, be defined respectively by* (3.6), (4.7), (4.1) *and* (4.2). *Then the ensemble* $(\widetilde{f}, (\widetilde{\xi}_a), (\widetilde{\eta}^a))$, $a = 1,2$, *provides a framed f-structure on $\widetilde{TM}$ if and only if*

$$\widetilde{G} = \left(\frac{1}{\beta}g_{ij} + \frac{\beta\tau - 1}{\beta\tau}y_iy_j\right)dx^idx^j + \left(\beta g_{ij} + \frac{1-\beta\tau}{\tau^2}y_iy_j\right)\delta y^i\delta y^j. \tag{4.16}$$

COROLLARY 4.10. *Assume that* $(\widetilde{f}, (\widetilde{\xi}_a), (\widetilde{\eta}^a))$, $a = 1, 2$, *provides a framed f-structure on* $\widetilde{TM}$. *Then*

$$\widetilde{G}(\widetilde{f}(X), \widetilde{f}(Y)) = \widetilde{G}(X, Y) - \widetilde{\eta}^1(X)\widetilde{\eta}^1(Y) - \widetilde{\eta}^2(X)\widetilde{\eta}^2(Y)$$

for $X, Y \in \mathcal{X}(\widetilde{TM})$.

Proof. This follows from Proposition 4.7 and Theorem 4.8. ∎

Now let the framed f- structure on $\widetilde{TM}$ be given by Theorem 4.8. Using (4.7), we can get the local expression of $\widetilde{f}$ as follows:

$$\widetilde{f}(\delta_i) = -\frac{1}{\beta}\left(\delta_i^k - \frac{1}{\tau}y_i y^k\right)\dot{\partial}_k, \tag{4.17}$$

$$\widetilde{f}(\dot{\partial}_i) = \beta\left(\delta_i^k - \frac{1}{\tau}y_i y^k\right)\delta_k. \tag{4.18}$$

If we set $\phi(X, Y) := \widetilde{G}(\widetilde{f}(X), Y)$ for $X, Y \in \mathcal{X}(\widetilde{TM})$, and use (4.17) and (4.18), we have

$$\phi(\delta_i, \dot{\partial}_j) = \widetilde{G}(\widetilde{f}(\delta_i), \dot{\partial}_j) = -g_{ij} + \frac{1}{\tau}y_i y_j, \tag{4.19}$$

$$\phi(\dot{\partial}_i, \delta_j) = \widetilde{G}(\widetilde{f}(\dot{\partial}_i), \delta_j) = g_{ij} - \frac{1}{\tau}y_i y_j, \tag{4.20}$$

$$\phi(\delta_i, \delta_j) = \phi(\dot{\partial}_i, \dot{\partial}_j) = 0. \tag{4.21}$$

Using (4.19)–(4.21) we get $\phi(X, Y) = -\phi(Y, X)$. Thus ϕ is a 2-form on $\widetilde{TM}$. On the other hand, by using (4.2) we obtain

$$d\widetilde{\eta}^1(\delta_i, \dot{\partial}_j) = \delta_i\widetilde{\eta}^1(\dot{\partial}_j) - \dot{\partial}_j\widetilde{\eta}^1(\delta_i) = -\dot{\partial}_j y_i = -g_{ij}. \tag{4.22}$$

Similarly we obtain

$$d\widetilde{\eta}^1(\dot{\partial}_i, \delta_j) = g_{ij}, \quad d\widetilde{\eta}^1(\delta_i, \delta_j) = d\widetilde{\eta}^1(\dot{\partial}_i, \dot{\partial}_j) = 0. \tag{4.23}$$

Relations (4.19)–(4.23) give us the following equality on $\widetilde{TM}$:

$$\phi = d\widetilde{\eta}^1 + \Omega, \quad \text{where} \quad \Omega = \frac{1}{\tau}y_i y_j dx^i \wedge \delta y^j. \tag{4.24}$$

5. Almost contact structure on the indicatrix bundle. In this section we assume that the framed f-structure on $\widetilde{TM}$ is given by Theorem 4.8. In this case,

$$\widetilde{\xi}_1 = \frac{1}{\tau}\mathcal{S}, \quad \widetilde{\eta}^2 = \frac{1}{\tau}y_i\delta y^i.$$

We shall prove that when we restrict the framed f-structure to the indicatrix bundle IM, we get a parameterized contact structure on IM. Moreover, we prove that the parameterized contact structure on IM is a Sasakian structure if and only if (M, F) is of constant flag curvature $K = c \neq 0$.

Let IM be the indicatrix bundle of (M, F), i.e.,

$$IM = \{(x, y) \in \widetilde{TM} \mid F(x, y) = 1\},$$

which is a submanifold of dimension $2n - 1$ of $\widetilde{TM}$.

Note that $\widetilde{\xi}_2$ is a unit vector field on IM since $\widetilde{G}(\widetilde{\xi}_2, \widetilde{\xi}_2) = 1$. It is easy to show that $\widetilde{\xi}_2$ is a normal vector field on IM with respect to the metric $\widetilde{G}$. Indeed, if the local equations of IM in $\widetilde{TM}$ are given by

$$x^i = x^i(u^\gamma), \qquad y^i = y^i(u^\gamma), \qquad \gamma \in \{1, \dots, 2m - 1\}, \tag{5.1}$$

then we have

$$\frac{\partial F}{\partial x^i} \frac{\partial x^i}{\partial u^\gamma} + \frac{\partial F}{\partial y^i} \frac{\partial y^i}{\partial u^\gamma} = 0. \tag{5.2}$$

Since F is a horizontal covariant constant, i.e., $\frac{\partial F}{\partial x^i} = N_i^k \frac{\partial F}{\partial y^k}$ we obtain

$$\left(N_i^k \frac{\partial x^i}{\partial u^\gamma} + \frac{\partial y^k}{\partial u^\gamma} \right) \frac{\partial F}{\partial y^k} = 0. \tag{5.3}$$

The natural frame field $\{\partial / \partial u^\gamma\}$ on IM is represented by

$$\frac{\partial}{\partial u^\gamma} = \frac{\partial x^i}{\partial u^\gamma} \frac{\partial}{\partial x^i} + \frac{\partial y^i}{\partial u^\gamma} \frac{\partial}{\partial y^i} = \frac{\partial x^i}{\partial u^\gamma} \delta_i + \left(N_i^k \frac{\partial x^i}{\partial u^\gamma} + \frac{\partial y^k}{\partial u^\gamma} \right) \frac{\partial}{\partial y^k}. \tag{5.4}$$

Thus by (5.3) and the condition $\tau(\beta + w\tau) = 1$, we have

$$\widetilde{G}\left(\frac{\partial}{\partial u^\gamma}, \widetilde{\xi}_2 \right) = \frac{1}{F} \left(N_i^k \frac{\partial x^i}{\partial u^\gamma} + \frac{\partial y^k}{\partial u^\gamma} \right) \frac{\partial F}{\partial y^k} = 0, \tag{5.5}$$

where we use the equality $y_k / F = \partial F / \partial y^k$. Therefore $\widetilde{\xi}_2$ is orthogonal to vectors that are tangent to IM. It is clear that the vector field $\widetilde{\xi}_1 = (1/\tau) y^i \delta_i$ is tangent to IM since $\widetilde{G}(\widetilde{\xi}_1, \widetilde{\xi}_2) = 0$.

LEMMA 5.1. *Let the framed f-structure be given by Theorem 4.8. Then restricting to IM we have*

$$\widetilde{\xi}_1 = y^i \delta_i = \mathcal{S}, \qquad \widetilde{\eta}^2 = 0, \qquad \widetilde{f}(X) = \widetilde{F}(X) + \widetilde{\eta}^1(X) \widetilde{\xi}_2 \quad \text{for } X \in \mathcal{X}(IM).$$

Proof. It is clear since $\tau = F^2 = 1$ on IM and $\widetilde{\eta}^2(X) = \widetilde{G}(X, \widetilde{\xi}_2) = 0$. This completes the proof. ∎

Note that Corollary 4.10 and Lemma 5.1 implies the following theorem:

THEOREM 5.2. *Let the framed f-structure be given by Theorem 4.8. Then the triple $(\widetilde{f}, \widetilde{\xi}_1, \widetilde{\eta}^1)$ defines an almost contact structure on IM, that is,*

$$\widetilde{\eta}^1(\widetilde{\xi}_1) = 1, \qquad \widetilde{f}(\widetilde{\xi}_1) = 0, \qquad \widetilde{\eta}^1 \circ \widetilde{f} = 0,$$
$$\widetilde{f}^2 = -I + \widetilde{\eta}^1 \otimes \widetilde{\xi}_1,$$
$$\widetilde{f}^3 + \widetilde{f} = 0, \qquad \operatorname{rank} \widetilde{f} = 2n - 2.$$

Note that $\tau = 1$ on IM, thus by (1.2) we have

$$\widetilde{G} = \frac{1}{\beta}[g_{ij} + (\beta - 1)y_i y_j]dx^i dx^j + [\beta g_{ij} + (1 - \beta)y_i y_j]\delta y^i \delta y^j. \tag{5.6}$$

By Corollary 4.10, we have

THEOREM 5.3. *Let $\widetilde{G}$ be the Riemannian metric given by* (1.2). *Then*

$$\widetilde{G}(\widetilde{f}(X), \widetilde{f}(Y)) = \widetilde{G}(X, Y) - \widetilde{\eta}^1(X)\widetilde{\eta}^1(Y) \quad \textit{for } X, Y \in \mathcal{X}(IM).$$

One can check that $\{\delta_i, \widetilde{f}(\delta_j)\}, j = 1, \ldots, n-1$, is a local frame on a neighborhood U of the point $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n) \in IM$ with $y^n \neq 0$. Since points like $(x, 0)$ are not in IM, one may always consider such a local frame. Let $\phi(X, Y) = \widetilde{G}(\widetilde{f}(X), Y)$ be the 2-form associated to the almost contact structure $(\widetilde{f}, \widetilde{\xi}_1, \widetilde{\eta}^1, \widetilde{G})$ on IM. By using (4.24) we have $\phi = d\widetilde{\eta}^1 + \Omega$, where $\Omega = \frac{1}{\tau}y_i y_j dx^i \wedge \delta y^j$. Now we show that Ω is zero on IM. Since $\{\delta_i, \widetilde{f}(\delta_j)\}_{j=1}^{n-1}$ is a local frame on a neighborhood U of $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n) \in IM$ with $y^n \neq 0$, it is sufficient to prove $\Omega(\delta_i, \widetilde{f}(\delta_j)) = \Omega(\delta_i, \delta_j) = \Omega(\widetilde{f}(\delta_i), \widetilde{f}(\delta_j)) = 0$. By the definition of Ω, it is easy to see that $\Omega(\delta_i, \delta_j) = \Omega(\widetilde{f}(\delta_i), \widetilde{f}(\delta_j)) = 0$. But from (4.17) we obtain

$$\begin{aligned}\Omega(\delta_i, \widetilde{f}(\delta_j)) &= -\frac{1}{\beta}(\delta_j^k - y_j y^k)\Omega(\delta_i, \dot{\partial}_k) \\ &= -\frac{1}{\beta}(\delta_j^k - y_j y^k)y_i y_k = -\frac{1}{\beta}(y_i y_j - y_j y_i) = 0.\end{aligned}$$

Therefore $\Omega(X, Y) = 0$ for all $X, Y \in \mathcal{X}(IM)$ and consequently by using (4.24) we deduce that $\phi(X, Y) = d\widetilde{\eta}^1(X, Y)$ for all $X, Y \in \mathcal{X}(IM)$. Substituting $\beta = 1/\sqrt{|c|}$ with $c \neq 0$ a constant into (1.3), we obtain the following theorem.

THEOREM 5.4. *Let (M, F) be a Finsler manifold endowed with the Chern–Rund connection ∇. Let also $c \neq 0$ be a parameter and*

$$\widetilde{G} = \sqrt{|c|}\left[g_{ij} + \left(\frac{1}{\sqrt{|c|}} - 1\right)y_i y_j\right]dx^i dx^j + \left[\frac{1}{\sqrt{|c|}}g_{ij} + \left(1 - \frac{1}{\sqrt{|c|}}\right)y_i y_j\right]\delta y^i \delta y^j \tag{5.7}$$

be the Riemannian metric on IM. Then $(IM, \widetilde{f}, \widetilde{\xi}_1, \widetilde{\eta}^1, \widetilde{G})$ is a contact Riemannian manifold.

Let $(\widetilde{f}, \widetilde{\xi}_1, \widetilde{\eta}^1)$ be the contact structure on IM and $N_{\widetilde{f}}$ be the Nijenhuis tensor field of $\widetilde{f}$. The contact structure $(\widetilde{f}, \widetilde{\xi}_1, \widetilde{\eta}^1)$ is called *normal* if $N := N_{\widetilde{f}} + 2d\widetilde{\eta}^1 \otimes \widetilde{\xi}_1 = 0$, and *Sasakian* if it is normal and $\phi = d\widetilde{\eta}^1$.

Note that if $d\widetilde{\eta}^1(\delta_i, \delta_j) = 0$, then $N(\delta_i, \delta_j) = N_{\widetilde{f}}(\delta_i, \delta_j)$. By the definition of Nijenhuis tensor field we have

$$N_{\widetilde{f}}(\delta_i, \delta_j) = [\widetilde{f}(\delta_i), \widetilde{f}(\delta_j)] - \widetilde{f}[\widetilde{f}(\delta_i), \delta_j] - \widetilde{f}[\delta_i, \widetilde{f}(\delta_j)] + \widetilde{f}^2[\delta_i, \delta_j].$$

By a direct calculation one gets

$$[\widetilde{f}(\delta_i), \widetilde{f}(\delta_j)] = \frac{1}{\beta^2}(y_i\delta_j^k - y_j\delta_i^k)\dot{\partial}_k,$$
$$\widetilde{f}^2[\delta_i, \delta_j] = (R^k{}_{ij} - R_{tij}y^t y^k)\dot{\partial}_k,$$
$$\widetilde{f}[\widetilde{f}(\delta_i), \delta_j] + \widetilde{f}[\delta_i, \widetilde{f}(\delta_j)] = 0.$$

Thus we obtain

$$N(\delta_i, \delta_j) = N_{\bar{f}}(\delta_i, \delta_j) = \left[\frac{1}{\beta^2}(y_i\delta_j^k - y_j\delta_i^k) + R^k{}_{ij} - R_{tij}y^t y^k\right]\dot{\partial}_k.$$

Therefore $N(\delta_i, \delta_j) = 0$ is equivalent to

$$\frac{1}{\beta^2}(y_i\delta_j^k - y_j\delta_i^k) + R^k{}_{ij} - R_{tij}y^t y^k = 0. \tag{5.8}$$

Contracting (5.8) with g_{kl} we get

$$R_{lij} = R_{tij}y^t y_l - \frac{1}{\beta^2}(y_i g_{jl} - y_j g_{il}). \tag{5.9}$$

Since $R_{tij}y^t y^j = R_{ti}y^t = 0$, thus the flag curvature $K(y, V)$ of (M, F) is

$$K(y, V) = \frac{R_{li}V^l V^i}{(g_{li} - y_l y_i)V^l V^i} = \frac{-\frac{1}{\beta^2}(y_i y_l - g_{il})V^l V^i}{(g_{li} - y_l y_i)V^l V^i} = \frac{1}{\beta^2}.$$

Note that the vanishing of $N(\delta_i, \delta_j)$ also implies the vanishing of $N(\widetilde{f}(\delta_i), \widetilde{f}(\delta_j))$ and $N(\delta_i, \widetilde{f}(\delta_j))$. Thus if we take $\beta = 1/\sqrt{|c|}$ with $c \neq 0$ a constant, we obtain the following theorem.

Theorem 5.5. *Let (M, F) be a Finsler manifold endowed with the Chern–Rund connection ∇. Let also $c \neq 0$ be a parameter and*

$$\widetilde{G} = \sqrt{|c|}\left[g_{ij} + \left(\frac{1}{\sqrt{|c|}} - 1\right)y_i y_j\right]dx^i dx^j + \left[\frac{1}{\sqrt{|c|}}g_{ij} + \left(1 - \frac{1}{\sqrt{|c|}}\right)y_i y_j\right]\delta y^i \delta y^j \tag{5.10}$$

be the Riemannian metric on IM. Then $(IM, \widetilde{f}, \widetilde{\xi}_1, \widetilde{\eta}^1, \widetilde{G})$ is a Sasakian manifold if and only if (M, F) is of constant flag curvature $K = c$.

If $c = \pm 1$, then using (1.3) we get the following Sasaki–Finsler metric $\widetilde{G}_S$, which was also studied in [BF]:

$$\widetilde{G}_S = g_{ij}dx^i dx^j + g_{ij}\delta y^i \delta y^j. \tag{5.11}$$

Thus by Theorem 5.5 we have

COROLLARY 5.6. *Let (M,F) be a Finsler manifold endowed with the Chern–Rund connection ∇. Then $(IM,\widetilde{f},\widetilde{\xi}_1,\widetilde{\eta}^1,\widetilde{G}_S)$ is a Sasakian manifold if and only if (M,F) is of constant flag curvature $K=\pm 1$.*

6. The Riemannian curvature of $\widetilde{G}_S$. In this section, we shall give an application of the Sasaki–Finsler metric (5.11), which is the special case of $\widetilde{G}$, i.e., $\beta=1$ and $\tau=1$. We first derive the curvature $R(X,Y)\mathcal{S}$ for $X,Y\in\mathcal{X}(\widetilde{TM})$, where $R(\cdot,\cdot)$ is the curvature operator of the Sasaki–Finsler metric (5.11) and $\mathcal{S}=y^i\delta_i$ is the geodesic spray field of F. Using the local formula for $R(X,Y)\mathcal{S}$, we show at the end of this section that IM endowed with the contact Riemannian metric structure $(\widetilde{f},\widetilde{\xi}_1,\widetilde{\eta}^1,\widetilde{G}_S)$ is locally isometric to $E^n\times S^{n-1}(4)$ for $n>2$ and flat for $n=2$ if the Finsler metric F comes from a locally flat Riemannian metric on M.

If we denote by $g_{jk;l}:=\delta_l(g_{jk})-g_{sk}G^s_{jl}-g_{js}G^s_{kl}$ the horizontal covariant derivative of g_{jk} with respect to the Berwald connection of (M,F), then we have

LEMMA 6.1 ([A2], [BF]). *In terms of the adapted frames $\{\delta_i,\dot{\partial}_i\}$ on $\widetilde{TM}$, the Levi-Civita connection D associated to the Sasaki–Finsler metric $\widetilde{G}_S$ is given by*

$$\begin{aligned} D_{\delta_k}\delta_j &= A^i_{j;k}\delta_i + A^i_{jk}\dot{\partial}_i,\\ D_{\delta_k}\dot{\partial}_j &= B^i_{j;k}\delta_i + B^i_{jk}\dot{\partial}_i,\\ D_{\dot{\partial}_k}\dot{\partial}_j &= E^i_{j;k}\delta_i + E^i_{jk}\dot{\partial}_i, \end{aligned}$$

where

$$A^i_{j;k}=\Gamma^i{}_{jk},\qquad A^i_{jk}=-\big(C^i_{jk}+\tfrac{1}{2}R^i_{jk}\big), \tag{6.1}$$

$$B^i_{j;k}=C^i_{jk}+\tfrac{1}{2}g^{il}R_{jlk},\qquad B^i_{jk}=\Gamma^i{}_{jk}, \tag{6.2}$$

$$E^i_{j;k}=-\tfrac{1}{2}g^{il}g_{jk;l},\qquad E^i_{jk}=C^i_{jk}. \tag{6.3}$$

THEOREM 6.2. *In terms of the adapted frame $\{\delta_i,\dot{\partial}_i\}$, the curvature operator $R(\cdot,\cdot)\mathcal{S}$ of the Levi-Civita connection D associated to the Sasaki–Finsler metric $\widetilde{G}_S$ is given by*

$$\begin{aligned} R(\delta_i,\delta_j)\mathcal{S} = \big(&\tfrac{1}{2}(R^s{}_jC^k_{si}-R^s{}_iC^k_{sj})+\tfrac{1}{4}g^{kt}(R_{sti}R^s{}_j-R_{stj}R^s{}_i)\\ &-R^k{}_{ij}-\tfrac{1}{2}g^{kt}R^l{}_{ij}R_{lt}\big)\delta_k+\tfrac{1}{2}(R^k{}_{j|i}-R^k{}_{i|j})\dot{\partial}_k, \end{aligned} \tag{6.4}$$

$$\begin{aligned} R(\dot{\partial}_i,\dot{\partial}_j)\mathcal{S} = \tfrac{1}{2}\big\{&g^{kl}y^s[\dot{\partial}_i(R_{jls})-\dot{\partial}_j(R_{ils})]-(C^{lk}_iR_{jl}-C^{lk}_jR_{il})\\ &+\tfrac{1}{2}g^{tl}g^{kr}(R_{jl}R_{irt}-R_{il}R_{jrt})\big\}\delta_k\\ &-\tfrac{1}{2}(R_{jl}L^{lk}_i-R_{il}L^{lk}_j)\dot{\partial}_k, \end{aligned} \tag{6.5}$$

$$R(\delta_i,\dot{\partial}_j)\mathcal{S}=\big(-L^k{}_{ji}+\tfrac{1}{2}g^{kl}R_{jl|i}-\tfrac{1}{4}R^s{}_i g^{kl}g_{rj}L^r{}_{sl}-\tfrac{1}{2}g^{kt}R_{lt}L^l_{ij} \\ -\tfrac{1}{4}g^{kt}R_{li}L^l{}_{jt}\big)\delta_k-\big[C^k_{ji}+\tfrac{1}{2}y^l\dot{\partial}_j(R^k{}_{il})+\tfrac{1}{2}R_{jl}C^{lk}_i \\ +\tfrac{1}{4}g^{tl}R_{jl}R^k{}_{ti}+\tfrac{1}{2}R^s{}_iC^k_{sj}\big]\dot{\partial}_k. \tag{6.6}$$

Proof. Note that ∇ is torsion free and $\widetilde{\xi}_1=\mathcal{S}=y^k\delta_k$. Thus by using Lemma 6.1 we get

$$D_{\delta_j}\mathcal{S}=\tfrac{1}{2}R^k{}_j\dot{\partial}_k \tag{6.7}$$

and

$$D_{[\delta_i,\delta_j]}\mathcal{S}=\big(R^k{}_{ij}+\tfrac{1}{2}R^l{}_{ij}g^{kt}R_{lt}\big)\delta_k. \tag{6.8}$$

Since $R(\delta_i,\delta_j)\mathcal{S}=D_{\delta_i}D_{\delta_j}\mathcal{S}-D_{\delta_j}D_{\delta_i}\mathcal{S}-D_{[\delta_i,\delta_j]}\mathcal{S}$, we get

$$R(\delta_i,\delta_j)\mathcal{S}=\tfrac{1}{2}\delta_i(R^k{}_j)\dot{\partial}_k+\tfrac{1}{2}R^s{}_j\big[\big(C^k_{si}+\tfrac{1}{2}g^{kt}R_{sti}\big)\delta_k+\varGamma^k{}_{si}\dot{\partial}_k\big] \\ -\tfrac{1}{2}\delta_j(R^k{}_i)\dot{\partial}_k-\tfrac{1}{2}R^s{}_i\big[\big(C^k_{sj}+\tfrac{1}{2}g^{kt}R_{stj}\big)\delta_k+\varGamma^k{}_{sj}\dot{\partial}_k\big] \\ -\big(R^k{}_{ij}+\tfrac{1}{2}R^l{}_{ij}g^{kt}R_{lt}\big)\delta_k.$$

It is clear that

$$\tfrac{1}{2}[\delta_i(R^k{}_j)+R^s{}_j\varGamma^k{}_{si}-\delta_j(R^k{}_i)-R^s{}_i\varGamma^k{}_{sj}]=\tfrac{1}{2}(R^k{}_{j|i}-R^k{}_{i|j})$$

and

$$\tfrac{1}{2}R^s{}_j\big(C^k_{si}+\tfrac{1}{2}g^{kt}R_{sti}\big)-\tfrac{1}{2}R^s{}_i\big(C^k_{sj}+\tfrac{1}{2}g^{kt}R_{stj}\big)-\big(R^k{}_{ij}+\tfrac{1}{2}R^l{}_{ij}g^{kt}R_{lt}\big) \\ =\tfrac{1}{2}(R^s{}_jC^k_{si}-R^s{}_iC^k_{sj})+\tfrac{1}{4}g^{kt}(R_{sti}R^s{}_j-R_{stj}R^s{}_i)-R^k{}_{ij}-\tfrac{1}{2}g^{kt}R^l{}_{ij}R_{lt}.$$

Thus we obtain (6.4).

Next we prove (6.5). It is easy to check that

$$D_{\dot{\partial}_j}\mathcal{S}=\delta_j+\tfrac{1}{2}g^{il}R_{jl}\delta_i. \tag{6.9}$$

Thus

$$D_{\dot{\partial}_i}D_{\dot{\partial}_j}\mathcal{S}=D_{\delta_j}\dot{\partial}_i+[\dot{\partial}_i,\delta_j]+\tfrac{1}{2}\dot{\partial}_i(g^{tl}R_{jl})\delta_t+\tfrac{1}{2}g^{tl}R_{jl}\{D_{\delta_t}\dot{\partial}_i+[\dot{\partial}_i,\delta_t]\} \\ =\big[C^k_{ij}+\tfrac{1}{2}g^{kl}R_{ilj}+\tfrac{1}{2}\dot{\partial}_i(g^{kl}R_{jl})+\tfrac{1}{2}g^{tl}R_{jl}\big(C^k_{it}+\tfrac{1}{2}g^{kr}R_{irt}\big)\big]\delta_k \\ -\big(L^k_{ij}+\tfrac{1}{2}g^{tl}R_{jl}L^k_{it}\big)\dot{\partial}_k \\ =\big[C^k_{ij}+\tfrac{1}{2}(g^{kl}R_{ilj}-C^{lk}_iR_{jl}+g^{kl}\dot{\partial}_i(R_{jl}))+\tfrac{1}{4}g^{tl}g^{kr}R_{jl}R_{irt}\big]\delta_k \\ -\big(L^k_{ij}+\tfrac{1}{2}R_{jl}L^{lk}_i\big)\dot{\partial}_k,$$

where we denote $C_i^{lk} := g^{lt} C_{ti}^k$ and $L_i^{lk} := g^{tl} L_{it}^k$. Therefore,

$$\begin{aligned} R(\dot\partial_i, \dot\partial_j)\mathcal{S} &= \tfrac{1}{2}\big\{g^{kl}(R_{ilj} - R_{jli}) - (C_i^{lk} R_{jl} - C_j^{lk} R_{il}) \\ &\quad + g^{kl}[\dot\partial_i(R_{jl}) - \dot\partial_j(R_{il})] + \tfrac{1}{2} g^{tl} g^{kr}(R_{jl}R_{irt} - R_{il}R_{jrt})\big\}\delta_k \\ &\quad - \tfrac{1}{2}(R_{jl}L_i^{lk} - R_{il}L_j^{lk})\dot\partial_k \\ &= \tfrac{1}{2}\big\{g^{kl} y^s[\dot\partial_i(R_{jls}) - \dot\partial_j(R_{ils})] - (C_i^{lk}R_{jl} - C_j^{lk}R_{il}) \\ &\quad + \tfrac{1}{2} g^{tl} g^{kr}(R_{jl}R_{irt} - R_{il}R_{jrt})\big\}\delta_k - \tfrac{1}{2}(R_{jl}L_i^{lk} - R_{il}L_j^{lk})\dot\partial_k. \end{aligned}$$

Now we prove (6.6). It follows from (6.7) and (6.9) that

$$\begin{aligned} R(\delta_i, \dot\partial_j)\mathcal{S} &= D_{\delta_i}\big(\delta_j + \tfrac{1}{2} g^{tl} R_{jl}\delta_t\big) - \tfrac{1}{2} D_{\dot\partial_j}(R^k{}_i\dot\partial_k) - G^l_{ji} D_{\dot\partial_l}\widetilde{\xi}_1 \\ &= \Gamma^k{}_{ji}\delta_k - \big(C^k_{ji} + \tfrac{1}{2}R^k{}_{ji}\big)\dot\partial_k + \tfrac{1}{2}\delta_i(g^{kl}R_{jl})\delta_k \\ &\quad + \tfrac{1}{2} g^{tl} R_{jl}\big[\Gamma^k{}_{ti}\delta_k - \big(C^k_{ti} + \tfrac{1}{2}R^k{}_{ti}\big)\dot\partial_k\big] - \tfrac{1}{2}\dot\partial_j(R^k{}_i)\dot\partial_k \\ &\quad - \tfrac{1}{2}R^s{}_i\big(-\tfrac{1}{2} g^{kl} g_{sj;l}\delta_k + C^k_{sj}\dot\partial_k\big) - G^k_{ji}\delta_k - \tfrac{1}{2} G^l_{ij} g^{kt} R_{lt}\delta_k. \end{aligned}$$

Since $\delta_i(g_{st}) = g_{rt}\Gamma^r{}_{si} + g_{sr}\Gamma^r{}_{ti}$, we have

$$\begin{aligned} \delta_i(g^{kl}R_{jl}) &= -g^{ks} g^{lt}\delta_i(g_{st})R_{jl} + g^{kl}\delta_i(R_{jl}) \\ &= -(g^{ks}\Gamma^l{}_{si} + g^{lt}\Gamma^k{}_{ti})R_{jl} + g^{kl}\delta_i(R_{jl}). \end{aligned}$$

Thus

$$\begin{aligned} R(\delta_i, \dot\partial_j)\mathcal{S} &= \big[\Gamma^k{}_{ji} - \tfrac{1}{2}(g^{ks}\Gamma^l{}_{si} + g^{lt}\Gamma^k{}_{ti})R_{jl} + \tfrac{1}{2} g^{kl}\delta_i(R_{jl}) + \tfrac{1}{2} g^{tl} R_{jl}\Gamma^k{}_{ti} \\ &\quad + \tfrac{1}{4}R^s{}_i g^{kl} g_{sj;l} - G^k_{ji} - \tfrac{1}{2} G^l_{ij} g^{kt} R_{lt}\big]\delta_k \\ &\quad - \big[\big(C^k_{ji} + \tfrac{1}{2}R^k{}_{ji}\big) + \tfrac{1}{2}R_{jl}C_i^{lk} \\ &\quad + \tfrac{1}{4} g^{tl} R_{jl} R^k{}_{ti} + \tfrac{1}{2}\dot\partial_j(R^k{}_i) + \tfrac{1}{2}R^s{}_i C^k_{sj}\big]\dot\partial_k \\ &= \big(-L^k{}_{ji} + \tfrac{1}{2} g^{kl} R_{jl|i} - \tfrac{1}{4}R^s{}_i g^{kl} g_{rj} L^r{}_{sl} \\ &\quad - \tfrac{1}{2} g^{kt} R_{lt} L^l_{ij} - \tfrac{1}{4} g^{kt} R_{li} L^l{}_{jt}\big)\delta_k \\ &\quad - \big[C^k_{ji} + \tfrac{1}{2} y^l\dot\partial_j(R^k{}_{il}) + \tfrac{1}{2}R_{jl}C_i^{lk} + \tfrac{1}{4} g^{tl} R_{jl} R^k{}_{ti} + \tfrac{1}{2}R^s{}_i C^k_{sj}\big]\dot\partial_k. \ \blacksquare \end{aligned}$$

THEOREM 6.3. *Let (M, F) be a Finsler manifold and $R(\cdot,\cdot)\mathcal{S}$ be the curvature operator of the Sasaki–Finsler metric*

$$\widetilde{G}_S = g_{ij}dx^i dx^j + g_{ij}\delta y^i\delta y^j.$$

Then

$$R(X, Y)\mathcal{S} = 0 \quad \forall X, Y \in \mathcal{X}(TM)$$

if and only if (M, F) is a locally flat Riemannian manifold.

Proof. Let (M, F) be a locally flat Riemannian manifold. Then

$$C^i_{jk} = 0, \quad R^i{}_j = 0, \quad R^i{}_{jk} = 0, \quad R_{ijk} = 0, \quad R_{ij} = 0, \quad L^k_{ji} = 0.$$

Thus using (6.4)–(6.6), we obtain

$$R(\delta_i,\delta_j)\mathcal{S}=R(\dot\partial_i,\dot\partial_j)\mathcal{S}=R(\delta_i,\dot\partial_j)\mathcal{S}=0,$$

which implies that $R(X,Y)\mathcal{S}=0$ for all $X,Y\in\mathcal{X}(TM)$. Conversely, if $R(X,Y)\mathcal{S}=0$ then by using (6.6) we obtain

$$C^k_{ji}+\tfrac{1}{2}y^l\dot\partial_j(R^k{}_{il})+\tfrac{1}{2}R_{jl}C^{lk}_i+\tfrac{1}{4}g^{tl}R_{jl}R^k{}_{ti}+\tfrac{1}{2}R^s{}_iC^k_{sj}=0. \tag{6.10}$$

Contracting (6.10) with y^j we get $y^jy^l\dot\partial_j(R^k{}_{il})=0$. Since $R^k{}_{il}$ is homogeneous of degree 1 with respect to y, it follows that $y^lR^k{}_{il}=0$, or equivalently $R^k{}_i=0$. So $R_{jl}=0$. Furthermore, by (2.50) of [CS],

$$R^i{}_{kl}=\tfrac{1}{3}[\dot\partial_l(R^i{}_k)-\dot\partial_k(R^i{}_l)]=0.$$

Consequently, by (6.10) we get $C^k_{ji}=0$, which implies that F comes from a Riemannian metric on M and (M,F) is locally flat. ∎

From Theorems 6.3 and 2.3 we have following theorem

THEOREM 6.4. *The $(2n-1)$-dimensional manifold IM with the contact Riemannian metric structure $(\widetilde f,\widetilde\xi_1,\widetilde\eta^1,\widetilde G_S)$ is locally isometric to $E^n\times S^{n-1}(4)$ for $n>2$ and flat for $n=2$ if the Finsler metric F comes from a locally flat Riemannian metric on M.*

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