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Eventually positive solutions for nonlinear impulsive differential equations with delays

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Abstract. Several recent oscillation criteria are obtained for nonlinear delay impulsive differential equations by relating them to linear delay impulsive differential equations or inequalities, and then comparison and oscillation criteria for the latter are applied. However, not all nonlinear delay impulsive differential equations can be directly related to linear delay impulsive differential equations or inequalities. Moreover, standard oscillation criteria for linear equations cannot be applied directly since continuous coefficient functions and initial functions are required. Therefore we establish oscillation criteria for linear or nonlinear impulsive equations with piecewise continuous coefficients and initial functions. Our technique is based on transforming our problem into a fixed point problem in Banach spaces, and then establishing comparison theorems. Our results extend, improve and correct some well known results in the literature.

1. Introduction. Impulsive differential equations are mathematical apparatus for simulation of different dynamical processes and phenomena observed in nature (see e.g. [10]). For this reason, many impulsive differential equations are studied and their qualitative properties are investigated. Among different qualitative theories relating to these equations, oscillation theory belongs to the more developed ones (see e.g. [1]–[12], [15]–[17]). One reason is that some oscillation criteria can be obtained for nonlinear delay impulsive differential equations by relating them to linear delay impulsive differential equations or inequalities (see e.g. [2], [3], [5], [6], [12], [15]), and then comparison and oscillation criteria for the latter can be applied to obtain numerous oscillation criteria.

Oscillation criteria for linear equations such as those in [8] cannot be applied directly since continuous coefficient functions and initial functions are required. Indeed, such requirements seem to be neglected in some recent results (see e.g. [5]). In the last section, we will discuss this in more detail.

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For the above reasons, it is important to establish oscillation criteria for linear or nonlinear impulsive equations with piecewise continuous coefficients and initial functions. We will see that after such criteria are established, existence of eventually positive solutions of several impulsive equations can be established with ease and a number of well known results in the literature can be generalized (such as Lemmas 1, 2 and 4 in [5], the main Theorem in [17], Theorem 1 in [4], Theorem 1 in [11], etc.).

To this end, we first recall some usual notation. $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real numbers and positive integers respectively. $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the intervals $(0,+\infty)$ and $(-\infty,0)$ respectively. We set $\mathbb{N}_n = \{1,\dots,n\}$. Assuming I_1 and I_2 are any two intervals in $\mathbb{R}$, we define $PC(I_1,I_2)$ to be the set of all functions $\varphi : I_1 \to I_2$ which are piecewise left continuous in I_1 with discontinuities of the first kind, and we define

$$\|\varphi\|_{I'} = \sup\{|\varphi(x)| : x \in I'\}$$

where $\varphi \in PC(I_1,I_2)$ and I' is a closed subinterval of I_1.

We let

$$\Upsilon = \{t_1, t_2, \dots\}$$

be a set of real numbers with $0 = t_0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. Also, $D^-y(t)$ will denote the left derivative of the function y at t.

We investigate the following nonlinear delay differential systems with impulsive effects:

$$D^-x(t) + \sum_{i=1}^{n} q_i(t) f_i(x(g_i(t))) = 0, \qquad t \in [0,\infty)\setminus\Upsilon, \tag{1.1}$$

$$x(t_k^+) - x(t_k) = I_k(x(t_k)), \qquad k \in \mathbb{N}, \tag{1.2}$$

and

$$D^-x(t) + \sum_{i=1}^{n} p_i(t) F_i(x(g_i(t))) \le 0, \qquad t \in [0,\infty)\setminus\Upsilon, \tag{1.3}$$

$$x(t_k^+) - x(t_k) = J_k(x(t_k)), \qquad k \in \mathbb{N}, \tag{1.4}$$

under the following conditions:

(A1) for each $i \in \mathbb{N}_n$, f_i and F_i are continuous functions on $\mathbb{R}$;
(A2) for each $i \in \mathbb{N}_n$, g_i is continuous on $[0,\infty)$ with $g_i(t) \le t$ for $t \in [0,\infty)$ and $\lim_{t\to\infty} g_i(t) = \infty$;
(A3) $0 = t_0 < t_1 < t_2 < \cdots$ are fixed numbers with $\lim_{k\to\infty} t_k = \infty$;
(A4) for each $k \in \mathbb{N}$, I_k and J_k are continuous functions on $\mathbb{R}$ such that $I_k(0) = J_k(0) = 0$, $\mu^2 + \mu J_k(\mu) > 0$ and $\mu^2 + \mu I_k(\mu) > 0$ for $\mu \neq 0$;
(A5) for each $i \in \mathbb{N}_n$, $q_i, p_i \in PC([0,\infty),\mathbb{R})$.

For convenience, we assume throughout this paper that $d \in \mathbb{R} \cup \{\infty\}$, and for any $\sigma \geq t_0 = 0$, we set

$$r_\sigma = \min_{1\leq i\leq n} \inf_{t\geq\sigma} g_i(t). \tag{1.5}$$

Definition 1.1. For any $d > \sigma \geq t_0$ and $\phi \in PC([r_\sigma, \sigma], \mathbb{R})$, a function $x \in PC([r_\sigma, d), \mathbb{R})$ is said to be a *solution* of system (1.1)–(1.2) on $[\sigma, d)$ satisfying the initial condition

$$x(t) = \phi(t), \qquad t \in [r_\sigma, \sigma],$$

if the following conditions are satisfied:

(i) x is absolutely continuous on each interval $(t_k, t_{k+1}] \cap [\sigma, d)$ where $k \in \mathbb{N} \cup \{0\}$;
(ii) $D^- x \in PC((\sigma, d), \mathbb{R})$ and x satisfies (1.1) in (σ, d);
(iii) for any $t_k \in (\sigma, d)$, (1.2) is satisfied.

We note that if x is a solution of system (1.1)–(1.2), then the absolute continuity of x implies that x is differentiable almost everywhere on $(t_k, t_{k+1}] \cap [\sigma, d)$ for each $k \in \mathbb{N} \cup \{0\}$. Furthermore, if there is an open interval $I \subset (t_k, t_{k+1}] \cap [\sigma, d)$ for some $k \in \mathbb{N} \cup \{0\}$ such that $D^- x$ is continuous on I, then x is differentiable on I. Hence, in such an interval, $D^- x$ in (1.1) can be replaced by x'.

We note that the definition of solutions of (1.3)–(1.4) is similar to Definition 1.1 and hence will not be repeated.

Definition 1.2. We say that a function $y = y(t)$ defined for all sufficiently large t is *eventually positive* (or *negative*) if there exists a number T such that $y(t) > 0$ (respectively $y(t) < 0$) for every $t \geq T$.

Given an arbitrary function φ defined on $\mathbb{R}$, we will need to impose some or all of the following conditions:

(B1) $\mu\varphi(\mu) > 0$ for $\mu \neq 0$;
(B2) φ is differentiable on $\mathbb{R} \setminus \{0\}$;
(B3) $\varphi(\mu_1)/\mu_1 \geq \varphi(\mu_2)/\mu_2$ if $\mu_1\mu_2 > 0$ and $|\mu_1| \leq |\mu_2|$;
(B4) there exist $0 < \theta_1, \theta_2 \leq \infty$ such that $|\varphi'(\mu)\mu| \leq M|\varphi(\mu)|$ on $(-\theta_1, \theta_2) \setminus \{0\}$ for some $M > 0$, and φ is continuously differentiable on $\mathbb{R} \setminus (-\theta_1, \theta_2)$.

Remark. There are many functions that satisfy (B1), (B2), (B3) or (B4). For instance, the functions $f(\mu) = \mu$, $f(\mu) = \operatorname{sgn}(\mu)\sqrt{|\mu|}$, and

$$f(\mu) = \begin{cases} \dfrac{2\sqrt{\mu}}{\mu + 2} & \text{if } \mu \geq 0, \\ -\log(1-\mu) & \text{if } \mu < 0, \end{cases}$$

satisfy (B1)–(B4). The function

$$f(\mu) = \begin{cases} \frac{1}{2\pi}\sqrt{\mu(\pi-\mu)} & \text{if } 0 \le \mu < \pi/2, \\ 1.5 - \sin(\mu) & \text{if } \mu \ge \pi/2, \\ \frac{\mu}{\mu^2+1} & \text{if } \mu < 0, \end{cases}$$

satisfies (B1), (B2) and (B4). Note that a function defined on $\mathbb{R}$ which is concave on $\mathbb{R}^+$ and convex on $\mathbb{R}^-$ satisfies (B3), and a continuously differentiable function defined on $\mathbb{R}$ satisfies (B4).

Let $\{b_k\}_{k\in\mathbb{N}}$ be a real sequence with $b_k > -1$ for $k \in \mathbb{N}$. For $t \ge s \ge 0$, we define a function

$$B(s,t) = \begin{cases} \prod_{s\le t_k<t} (1+b_k) & \text{if } [s,t)\cap \Upsilon \ne \emptyset, \\ 1 & \text{if } [s,t)\cap \Upsilon = \emptyset. \end{cases} \tag{1.6}$$

Let $\sigma \ge 0$. We set

$$h_i(t) = \min\{\sigma, g_i(t)\} \quad \text{and} \quad H_i(t) = \max\{\sigma, g_i(t)\}.$$

It is obvious that $h_i(t) \le \sigma \le H_i(t)$, and $B(\sigma,t)$ is a positive step function on $[\sigma,\infty)$. Let $\phi \in PC([r_\sigma,\sigma],\mathbb{R})$ with $\phi(\sigma) \ne 0$, f_i continuous on $\mathbb{R}$ and $q_i \in PC([0,\infty),\mathbb{R})$. For $\delta \in PC([\sigma,d),\mathbb{R})$, we define an operator

$$T_\phi(\delta)(t) = -\sum_{i=1}^{n} q_i(t)\frac{\Phi(\sigma,\phi,f_i,\delta)(t)}{B(\sigma,t)\phi(\sigma)}e^{-\int_\sigma^t \delta(s)\,ds}, \qquad \sigma \le t < d, \tag{1.7}$$

where

$$\Phi(\sigma,\phi,f_i,\delta)(t) = f_i\big(B(\sigma,H_i(t))\phi(h_i(t))e^{\int_\sigma^{H_i(t)} \delta(s)\,ds}\big). \tag{1.8}$$

Then $T_\phi(\delta) \in PC([\sigma,d),\mathbb{R})$ for any $\delta \in PC([\sigma,d),\mathbb{R})$.

2. Main theorems.

We begin by establishing equivalent and/or sufficient conditions for the existence of positive (or negative) solutions of (1.1)–(1.2).

THEOREM 2.1. *Let $d > \sigma \ge 0$, $\phi \in PC([r_\sigma,\sigma],\mathbb{R})$ with $\phi(\sigma) \ne 0$, and T_ϕ be defined by* (1.7). *Assume that* (A1)–(A5) *hold and f_i satisfy* (B1), (B2) *and* (B4) *for $i \in \mathbb{N}_n$. The following statements are equivalent:*

(a) *There exists $\alpha \in PC([\sigma,d),\mathbb{R})$ such that $T_\phi(\alpha)(t) = \alpha(t)$ on (σ,d).*
(b) *There exist $\beta,\gamma \in PC([\sigma,d),\mathbb{R})$ such that $\beta(t) \le \gamma(t)$ on $[\sigma,d)$, and for $\delta \in PC([\sigma,d),\mathbb{R})$ with $\beta(t) \le \delta(t) \le \gamma(t)$ on (σ,d), we have*

$$\beta(t) \le T_\phi(\delta)(t) \le \gamma(t) \qquad \text{on } (\sigma,d). \tag{2.1}$$

Proof. Assume (a) holds. We may take $\beta(t) \equiv \gamma(t) \equiv \alpha(t)$ on $[\sigma,d)$. It is obvious that (b) holds because $T_\phi(\alpha)(t) = \alpha(t)$ on (σ,d).

Next, assume that (b) holds. Take $\delta_0 \in PC([\sigma,d),\mathbb{R})$ and $\beta(t) \le \delta_0(t) \le \gamma(t)$ on (σ,d). Set

$$\delta_k(t) = \begin{cases} T_\phi(\delta_{k-1})(t) & \text{if } \sigma < t < d, \\ T_\phi(\delta_{k-1})(\sigma^+) & \text{if } t = \sigma, \end{cases} \qquad k \in \mathbb{N}.$$

We note that by induction, for any $k \in \mathbb{N}$,

$$\delta_k(\sigma) = -\sum_{i=1}^{n} q_i(\sigma^+) \frac{f_i(B(\sigma,\sigma^+)\phi(h_i(\sigma^+)))}{B(\sigma,\sigma^+)\phi(\sigma)} \quad \text{exists.}$$

So δ_k is well-defined for all $k \in \mathbb{N}$. In view of the definition of δ_k and (2.1), by induction, we see that $\delta_k \in PC([\sigma,d),\mathbb{R})$ and $\beta(t) \le \delta_k(t) \le \gamma(t)$ for $\sigma < t < d$ and all $k \in \mathbb{N}$. We will prove that the sequence $\{\delta_k\}$ converges uniformly on any compact subinterval $[\sigma,A]$ of $[\sigma,d)$. Let

$$M' = \max\{\|\beta\|_{[\sigma,A]}, \|\gamma\|_{[\sigma,A]}\}.$$

We note that for any $k \in \mathbb{N}$,

$$\beta(\sigma^+) \le \delta_k(\sigma) = T_\phi(\delta_{k-1})(\sigma^+) \le \gamma(\sigma^+).$$

Thus

$$\|\delta_k\|_{[\sigma,A]} \le M' \quad \text{for } k \in \mathbb{N}. \tag{2.2}$$

By (2.2), we see that there exists $M_1 > 0$ such that

$$B(\sigma,H_i(t))\phi(h_i(t))e^{\int_\sigma^{H_i(t)} \delta_k(s)\,ds} \le M_1, \qquad \sigma \le t \le A,$$

for all $k \in \mathbb{N}$ and $i \in \mathbb{N}_n$. We note that $B(\sigma,t)$ has a positive lower bound in $[\alpha,A]$. We apply the Mean-Value Theorem to the function e^μ and the bi-variate functions $f_i(\mu)/\nu$, $i \in \mathbb{N}_n$. Then there exists $\widetilde{M} > 0$ such that for each $k \in \mathbb{N}$,

$$\begin{aligned} |\delta_{k+1}(t) - \delta_k(t)| &= |T_\phi(\delta_k)(t) - T_\phi(\delta_{k-1})(t)| \\ &\le \widetilde{M} \int_\sigma^t |\delta_k(s) - \delta_{k-1}(s)|\,ds, \qquad \sigma \le t \le A. \end{aligned}$$

Therefore the sequence $\{\delta_k\}_{k\in\mathbb{N}}$ where

$$\delta_k(t) = \delta_0(t) + \sum_{j=0}^{k-1} [\delta_{j+1}(t) - \delta_j(t)], \qquad \sigma \le t \le A,$$

also converges uniformly. Let

$$\overline{\delta}(t) = \lim_{k\to\infty} \delta_k(t), \qquad \sigma \le t \le A. \tag{2.3}$$

Then $\beta(t) \le \overline{\delta}(t) \le \gamma(t)$ on $(\sigma,A]$. By the definition of T_ϕ, the points of discontinuity of δ_k for each $k \in \mathbb{N}$ only depend on the points of discontinuity of the two functions $B(\sigma,t)$ and $\phi(t)$. So the points of discontinuity of $\overline{\delta}$ are the same as those of δ_k for each $k \in \mathbb{N}$. It is clear that $\overline{\delta} \in PC([\sigma, A], \mathbb{R})$. By uniform convergence,

$$\overline{\delta}(t) = \lim_{k\to\infty} \delta_{k+1}(t) = \lim_{k\to\infty} T_\phi(\delta_k)(t) = T_\phi(\overline{\delta})(t), \qquad \sigma < t \le A.$$

Since A is an arbitrary point in $[\sigma, d)$, it follows that $\alpha(t)$ defined by (2.3) satisfies $\alpha(t) = T_\phi(\alpha)(t)$ on (σ, d). Thus (a) holds. ∎

In the next result, we alter the conditions that must be satisfied by f_i.

THEOREM 2.2. *Let $d > \sigma \ge 0$, $\phi \in PC([r_\sigma, \sigma], \mathbb{R})$ with $\phi(\sigma) \ne 0$, and T_ϕ be defined by* (1.7). *Assume that* (A1)–(A5) *hold, that ϕ has constant sign on $[r_\sigma, \sigma]$, $q_i(t) \ge 0$ for $t \ge 0$, and that f_i satisfies* (B1) *and* (B3) *for each $i \in \mathbb{N}_n$. Then statements* (a) *and* (b) *of Theorem* 2.1 *are equivalent.*

Proof. Assume (a) holds. Similar to the proof of Theorem 2.1, we may take $\beta(t) \equiv \gamma(t) \equiv \alpha(t)$ on $[\sigma, d)$. It is then obvious that (b) holds. Conversely, assume (b) holds. Let $\sigma < A < d$ and

$$\Omega_A = \{\delta \in PC([\sigma, A], \mathbb{R}) : \beta(t) \le \delta(t) \le \gamma(t) \text{ on } (\sigma, A]\}.$$

We shall use the Knaster–Tarski fixed point theorem to prove that T_ϕ has a fixed point in Ω_A for any $\sigma < A < d$. Clearly, Ω_A endowed with the supremum norm and the usual linear structure is a Banach space. If δ_1 and δ_2 belong to Ω_A, let us say that $\delta_1 \le \delta_2$ if and only if $\delta_1(t) \le \delta_2(t)$ on $(\sigma, A]$. Clearly, with this ordering, Ω_A is a partially ordered set. Furthermore, $\beta \in \Omega_A$ is the infimum of Ω_A, and every nonempty subset of Ω_A has a supremum that belongs to Ω_A. By (2.1), we see that $\beta(t) \le T_\phi(\delta)(t) \le \gamma(t)$ on $(\sigma, A]$, and $T_\phi(\delta) \in PC([\sigma, A], \mathbb{R})$ for any $\delta \in \Omega_A$. Thus $T_\phi(\Omega_A) \subseteq \Omega_A$. Given $\delta_1, \delta_2 \in \Omega_A$ with $\delta_1 \le \delta_2$, we have

$$e^{-\int_{H_i(t)}^t \delta_1(s)\,ds} \ge e^{-\int_{H_i(t)}^t \delta_2(s)\,ds} \quad \text{for } \sigma \le t \le A. \tag{2.4}$$

For each $i \in \mathbb{N}_n$, we claim that

$$\frac{\Phi(\sigma, \phi, f_i, \delta_1)(t)}{B(\sigma, t)\phi(\sigma)} e^{-\int_\sigma^t \delta_1(s)\,ds} \ge \frac{\Phi(\sigma, \phi, f_i, \delta_2)(t)}{B(\sigma, t)\phi(\sigma)} e^{-\int_\sigma^t \delta_1(s)\,ds}, \quad \sigma \le t \le A. \tag{2.5}$$

To see this, fix $i \in \mathbb{N}_n$. Assume that $t \in [\sigma, A]$ and $\phi(h_i(t)) = 0$. In view of $f_i(0) = 0$, we have

$$\Phi(\sigma, \phi, f_i, \delta_1)(t) = \Phi(\sigma, \phi, f_i, \delta_2)(t) = 0$$

for $\sigma \le t \le A$, so (2.5) holds. Assume now $t \in [\sigma, A]$ and $\phi(h_i(t)) \ne 0$. Since

$$\big(B(\sigma, H_i(t))\phi(h_i(t))\big)^2 e^{\int_\sigma^{H_i(t)} (\delta_1(s)+\delta_2(s))\,ds} > 0$$

and

$$B(\sigma, H_i(t))|\phi(h_i(t))| e^{\int_\sigma^{H_i(t)} \delta_2(s)\,ds} \ge B(\sigma, H_i(t))|\phi(h_i(t))| e^{\int_\sigma^{H_i(t)} \delta_1(s)\,ds}$$

for $\sigma \le t \le A$, by (B3) we see that

$$\frac{\Phi(\sigma,\phi,f_i,\delta_1)(t)}{\phi(h_i(t))}e^{-\int_\sigma^{H_i(t)}\delta_1(s)\,ds} \ge \frac{\Phi(\sigma,\phi,f_i,\delta_2)(t)}{\phi(h_i(t))}e^{-\int_\sigma^{H_i(t)}\delta_2(s)\,ds}$$

for $\sigma \le t \le A$. It follows from (2.4) that (2.5) holds again.

By (2.5), we see that $T_\phi(\delta_1)(t) \le T_\phi(\delta_2)(t)$ on $[\sigma, A]$ because $q_i(t) \ge 0$ for $t \ge 0$. Thus T_ϕ is increasing. By the Knaster–Tarski fixed point theorem, there exists $\delta \in \Omega_A$ such that $T_\phi(\delta) = \delta$. Let

$$A^* = \sup\{A \in (\sigma, d) : T_\phi(\delta) = \delta \text{ for some } \delta \in \Omega_A\}.$$

We claim that $A^* = d$. Indeed, if not, there exists $\delta \in \Omega_A$ such that $T_\phi(\delta) = \delta$ for any $A \in (A^*, d)$, a contradiction. Therefore, there exists $\alpha \in PC([\sigma, d), \mathbb{R})$ such that $\alpha(t) = T_\phi(\alpha)(t)$ on (σ, d), and so (a) holds. ∎

Theorem 2.3. *Let $d > \sigma \ge 0$, $\phi \in PC([r_\sigma, \sigma], \mathbb{R})$ with $\phi(\sigma) \neq 0$, and T_ϕ be defined by* (1.7). *Assume that* (A1)–(A5) *hold and $I_k(\mu) = b_k\mu$ for $\mu \in \mathbb{R}$ and $k \in \mathbb{N}$ where each b_k is a constant.*

(a) *Assume that $\alpha \in PC([\sigma, d), \mathbb{R})$ satisfies $T_\phi(\alpha)(t) = \alpha(t)$ on (σ, d). Then the function x defined by*

$$x(t) = \begin{cases} \phi(t) & \text{if } r_\sigma \le t \le \sigma, \\ B(\sigma,t)\phi(\sigma)e^{\int_\sigma^t \alpha(s)\,ds} & \text{if } \sigma < t < d, \end{cases} \tag{2.6}$$

is a solution of system (1.1)–(1.2) *satisfying the initial condition $x(t) = \phi(t)$ on $[r_\sigma, \sigma]$.*

(b) *Assume that x is a solution of* (1.1)–(1.2)) *on $[\sigma, d)$ satisfying the initial condition $x(t) = \phi(t)$ on $[r_\sigma, \sigma]$, and $x(t) \neq 0$ on $[\sigma, d)$. Then there exists $\alpha \in PC([\sigma, d), \mathbb{R})$ such that $T_\phi(\alpha)(t) = \alpha(t)$ on (σ, d).*

Proof. First of all, we note that $D^-B(\sigma, t) = 0$ in $[\sigma, d)$. We first verify (a). Let x be defined by (2.6). By definition, $y(t) = \phi(t)$ if $r_\sigma \le t \le \sigma$. Assume $\eta, \varsigma \in (\sigma, d) \cap (t_k, t_{k+1}]$ for some $k \in \mathbb{N} \cup \{0\}$ and $\varsigma < \eta$. We have

$$B(\sigma, \eta) = B(\sigma, \varsigma) = B(\sigma, t_{k+1}).$$

By the Mean-Value Theorem,

$$\begin{aligned} |x(\eta) - x(\varsigma)| &= B(\sigma, t_{k+1})\phi(\sigma)\big(e^{\int_\sigma^\eta \alpha(s)\,ds} - e^{\int_\sigma^\varsigma \alpha(s)\,ds}\big) \\ &\le B(\sigma, t_{k+1})\phi(\sigma)A_k e^{A_k(t_{k+1}-t_k)}(\eta - \varsigma) \end{aligned}$$

where $A_k = \|\alpha\|_{[\sigma,d)\cap[t_k,t_{k+1}]}$. So x is absolutely continuous on each interval $(t_k, t_{k+1}] \cap [\sigma, d)$ where $k \in \mathbb{N} \cup \{0\}$. Since

$$D^-x(t) = \alpha(t)x(t), \qquad \sigma < t < d,$$

we see that $D^-x \in PC((\sigma, d), \mathbb{R})$. Now, we assert that

$$\Phi(\sigma, \phi, f_i, \alpha)(t) = f_i\big(B(\sigma, H_i(t))\phi(h_i(t))e^{\int_\sigma^{H_i(t)} \alpha(s)\,ds}\big) = f_i(x(g_i(t))) \tag{2.7}$$

for $\sigma < t < d$ and $i \in \mathbb{N}_n$. Indeed, first, for each $i \in \mathbb{N}_n$, if $g_i(t) \geq \sigma$, then

$$B(\sigma, H_i(t))\phi(h_i(t))e^{\int_\sigma^{H_i(t)} \alpha(s)\,ds} = B(\sigma, g_i(t))\phi(\sigma)e^{\int_\sigma^{g_i(t)} \alpha(s)\,ds} = x(g_i(t));$$

and if $g_i(t) < \sigma$, then

$$B(\sigma, H_i(t))\phi(h_i(t))e^{\int_\sigma^{H_i(t)} \alpha(s)\,ds} = B(\sigma, \sigma)\phi(g_i(t))e^{\int_\sigma^{\sigma} \alpha(s)\,ds} = x(g_i(t)).$$

By the definition of $\Phi(\sigma, \phi, f_i, \alpha)(t)$, (2.7) holds.

In view of $T_\phi(\alpha)(t) = \alpha(t)$ in (σ, d) and (2.7), we see that

$$\begin{aligned} D^-x(t) = x(t)\alpha(t) &= x(t)\left(-\sum_{i=1}^n q_i(t)\frac{\Phi(\sigma, \phi, f_i, \alpha)(t)}{B(\sigma, t)\phi(\sigma)}e^{-\int_\sigma^t \alpha(s)\,ds}\right) \\ &= -\sum_{i=1}^n q_i(t) f_i(x(g_i(t))), \qquad \sigma < t < d. \end{aligned}$$

For any $\sigma \leq t_k < d$,

$$x(t_k^-) = \phi(t_k^-) = \phi(t_k) = x(t_k) \quad \text{ if } \sigma = t_k,$$

$$x(t_k^-) = B(\sigma, t_k^-)\phi(\sigma)e^{\int_\sigma^{t_k} \alpha(s)\,ds} = B(\sigma, t_k)\phi(\sigma)e^{\int_\sigma^{t_k} \alpha(s)\,ds} = x(t_k) \quad \text{ if } \sigma \neq t_k,$$

and

$$\begin{aligned} x(t_k^+) &= B(\sigma, t_k^+)\phi(\sigma)e^{\int_\sigma^{t_k} \alpha(s)\,ds} = (1 + b_k)B(\sigma, t_k)\phi(\sigma)e^{\int_\sigma^{t_k} \alpha(s)\,ds} \\ &= (1 + b_k)x(t_k). \end{aligned}$$

Therefore, x is a solution of (1.1)–(1.2) with initial condition $x(t) = \phi(t)$ on $[r_\sigma, \sigma]$.

Next we verify (b). Clearly, $x(\sigma^+) \neq 0$ and

$$\frac{D^-x(\sigma^+)}{x(\sigma^+)} = -\sum_{i=1}^n q_i(\sigma^+)\frac{f_i(x(g_i(\sigma^+)))}{x(\sigma^+)} \quad \text{ exists.}$$

Let

$$\alpha(t) = \begin{cases} \dfrac{D^-x(t)}{x(t)} & \text{if } \sigma < t < d, \\[2ex] \dfrac{D^-x(\sigma^+)}{x(\sigma^+)} & \text{if } t = \sigma. \end{cases}$$

Then $\alpha \in PC([\sigma, d), \mathbb{R})$. We observe that for $\sigma < t < d$,

$$x(t) = B(\sigma, t)\phi(\sigma)e^{\int_\sigma^t \alpha(s)\,ds},$$

from which and (2.7) we see that

$$\begin{aligned}\alpha(t) = \frac{D^{-}x(t)}{x(t)} &= -\sum_{i=1}^{n} q_i(t)\frac{f_i(x(g_i(t)))}{x(t)} \\ &= -\sum_{i=1}^{n} q_i(t)\frac{\Phi(\sigma,\phi,f_i,\alpha)(t)}{B(\sigma,t)\phi(\sigma)}e^{-\int_\sigma^t \alpha(s)\,ds} \\ &= T_\phi(\alpha)(t), \qquad \sigma < t < d. \ \blacksquare\end{aligned}$$

We remark that from Theorem 2.3 we may derive Corollary 1 of [14] and Theorem of [17].

THEOREM 2.4. *Let* $d > \sigma \geq 0$. *Assume that* (A1)–(A5) *hold and assume that for each* $i \in \mathbb{N}_n$ *and* $k \in \mathbb{N}$:

(i) $p_i(t) \geq q_i(t) \geq 0$ *for* $t \geq 0$;
(ii) $|F_i(\mu)| \geq |f_i(\mu)|$, $\mu \in \mathbb{R}$;
(iii) *either* [F_i *satisfies* (B1) *and* (B3), *and* f_i *satisfies* (B1), (B2) *and* (B4)], *or* [F_i *satisfies* (B1), *and* f_i *satisfies* (B1) *and* (B3)];
(iv) $I_k(\mu_2)/\mu_2 \geq J_k(\mu_1)/\mu_1$ *if* $0 < \mu_1 \leq \mu_2$.

Let $\phi_1, \phi_2 \in PC([r_\sigma,\sigma],\mathbb{R})$ *be such that*

$$(2.8)\qquad \phi_1(\sigma)\phi_2(\sigma)\neq 0, \quad \phi_2(t)\geq\phi_1(t)\geq 0 \text{ and } \frac{\phi_1(t)}{\phi_1(\sigma)}\geq\frac{\phi_2(t)}{\phi_2(\sigma)} \text{ for } r_\sigma\leq t\leq\sigma.$$

If x *is a positive solution of* (1.3)–(1.4) *on* $[\sigma, d)$ *satisfying the initial condition* $x(t) = \phi_1(t)$ *on* $[r_\sigma,\sigma]$, *then* (1.1)–(1.2) *has a positive solution* y *on* $[\sigma, d)$ *such that* $y(t) \geq x(t)$ *on* $[r_\sigma, d)$.

Before proving Theorem 2.4, we first prove the following lemma.

LEMMA 2.5. *Assume that the hypotheses of Theorem* 2.4 *and* (2.8) *hold and that* I_k *and* J_k *are linear homogeneous functions for all* $k \in \mathbb{N}$. *If* x *is a positive solution of* (1.3)–(1.4) *on* $[\sigma, d)$ *satisfying the initial condition* $x(t) = \phi_1(t)$ *on* $[r_\sigma,\sigma]$, *then* (1.1)–(1.2) *has a positive solution* y *on* $[\sigma, d)$ *such that* $y(t) \geq x(t)$ *on* $[r_\sigma, d)$.

Proof. We let $I_k(\mu) = b_k\mu$ and $J_k(\mu) = b'_k\mu$ for $\mu \in \mathbb{R}$ and $k \in \mathbb{N}$. By (A4) and (iv), we have $b_k \geq b'_k > -1$ for $k \in \mathbb{N}$. Since $\{b_k\}_{k\in\mathbb{N}}$ is a sequence with $b_k > -1$ for $k \in \mathbb{N}$, we can consider $B(s,t)$ and $\Phi(\sigma,\phi,f_i,\delta)(t)$ defined by (1.6) and (1.8) respectively. Similarly, $\{b'_k\}_{k\in\mathbb{N}}$ is also a sequence with $b'_k > -1$ for $k \in \mathbb{N}$, so we may define

$$\overline{B}(s,t) = \begin{cases} \prod_{s\leq t_k<t}(1+b'_k) & \text{if } [s,t)\cap\Upsilon\neq\emptyset, \\ 1 & \text{if } [s,t)\cap\Upsilon=\emptyset,\end{cases}$$

for $\sigma \le s \le t$. Then

$$0 < \bar{B}(s,t) \le B(s,t), \quad \sigma \le s \le t < d. \tag{2.9}$$

In view of $x(t) > 0$ on $[\sigma, d)$,

$$-\infty < -\sum_{i=1}^{n} q_i(\sigma^+)\frac{F_i(x(g_i(\sigma^+)))}{x(\sigma^+)} \le 0.$$

So

$$\beta(t) \equiv \begin{cases} \dfrac{D^- x(t)}{x(t)} & \text{if } \sigma < t < d, \\ -\displaystyle\sum_{i=1}^{n} q_i(\sigma^+)\frac{F_i(x(g_i(\sigma^+)))}{x(\sigma^+)} & \text{if } t = \sigma, \end{cases} \tag{2.10}$$

is well-defined. Then $\beta \in PC([\sigma, d), \mathbb{R})$ and

$$\beta(t) = \frac{D^- x(t)}{x(t)} \le -\sum_{i=1}^{n} p_i(t)\frac{\bar{\Phi}(\sigma, \phi_1, F_i, \beta)(t)}{\bar{B}(\sigma,t)\phi(\sigma)} e^{-\int_\sigma^t \beta(s)\,ds} \le 0 \tag{2.11}$$

where $\sigma < t < d$ and

$$\bar{\Phi}(\sigma, \phi, F_i, \beta)(t) = F_i\big(\bar{B}(\sigma, H_i(t))\phi(h_i(t))e^{\int_\sigma^{H_i(t)} \beta(s)\,ds}\big).$$

Let $\delta \in PC([\sigma, d), \mathbb{R})$ and $\beta(t) \le \delta(t) \le 0$ on (σ, d). We first claim that for each $i \in \mathbb{N}_n$,

$$\frac{\bar{\Phi}(\sigma, \phi_1, F_i, \beta)(t)}{\bar{B}(\sigma,t)\phi_1(\sigma)} e^{-\int_\sigma^{H_i(t)} \beta(s)\,ds} \ge \frac{\Phi(\sigma, \phi_2, f_i, \delta)(t)}{B(\sigma,t)\phi_2(\sigma)} e^{-\int_\sigma^{H_i(t)} \delta(s)\,ds}, \quad \sigma < t < d. \tag{2.12}$$

Indeed, fix $i \in \mathbb{N}_n$. Let $t \in (\sigma, d)$. Assume $\phi_1(h_i(t))\phi_2(h_i(t)) = 0$. By (2.8), we have $\phi_1(h_i(t)) = \phi_2(h_i(t)) = 0$. In view of $f_i(0) = F_i(0) = 0$, (2.12) holds. Assume now $\phi_1(h_i(t))\phi_2(h_i(t)) > 0$. By (2.9), we have

$$\bar{B}(\sigma, H_i(t))\phi_1(h_i(t))e^{\int_\sigma^{H_i(t)} \beta(s)\,ds} \le B(\sigma, H_i(t))\phi_2(h_i(t))e^{\int_\sigma^{H_i(t)} \delta(s)\,ds}.$$

We note that

$$\bar{B}(\sigma, H_i(t))\bar{B}(H_i(t), t) = \bar{B}(\sigma, t) \text{ and } B(\sigma, H_i(t))B(H_i(t), t) = B(\sigma, t).$$

By (2.8) and (2.9), we have

$$\frac{\phi_1(h_i(t))\bar{B}(\sigma, H_i(t))}{\phi_1(\sigma)\bar{B}(\sigma,t)} = \frac{\phi_1(h_i(t))}{\phi_1(\sigma)\bar{B}(H_i(t),t)} \ge \frac{\phi_2(h_i(t))}{\phi_2(\sigma)B(H_i(t),t)} = \frac{\phi_2(h_i(t))B(\sigma, H_i(t))}{\phi_2(\sigma)B(\sigma,t)}. \tag{2.13}$$

By condition (iii), we see that if F_i satisfies (B3), then

(2.14)
$$\frac{F_i\big(\bar{B}(\sigma, H_i(t))\phi_1(h_i(t))e^{\int_\sigma^{H_i(t)} \beta(s)\,ds}\big)}{\bar{B}(\sigma, H_i(t))\phi_1(h_i(t))e^{\int_\sigma^{H_i(t)} \beta(s)\,ds}} \ge \frac{F_i\big(B(\sigma, H_i(t))\phi_2(h_i(t))e^{\int_\sigma^{H_i(t)} \delta(s)\,ds}\big)}{B(\sigma, H_i(t))\phi_2(h_i(t))e^{\int_\sigma^{H_i(t)} \delta(s)\,ds}}$$

and if f_i satisfies (B3), then

(2.15)
$$\frac{f_i\big(\bar{B}(\sigma, H_i(t))\phi_1(h_i(t))e^{\int_\sigma^{H_i(t)} \beta(s)\,ds}\big)}{\bar{B}(\sigma, H_i(t))\phi_1(h_i(t))e^{\int_\sigma^{H_i(t)} \beta(s)\,ds}} \ge \frac{f_i\big(B(\sigma, H_i(t))\phi_2(h_i(t))e^{\int_\sigma^{H_i(t)} \delta(s)\,ds}\big)}{B(\sigma, H_i(t))\phi_2(h_i(t))e^{\int_\sigma^{H_i(t)} \delta(s)\,ds}}.$$

In view of condition (ii), (2.14) and (2.15) lead to

(2.16)
$$\frac{\bar{\Phi}(\sigma, \phi, F_i, \beta)(t)}{\bar{B}(\sigma, H_i(t))\phi_1(h_i(t))} e^{-\int_\sigma^{H_i(t)} \beta(s)\,ds} \ge \frac{\Phi(\sigma, \phi, f_i, \delta)(t)}{B(\sigma, H_i(t))\phi_2(h_i(t))} e^{-\int_\sigma^{H_i(t)} \delta(s)\,ds},$$

from which it follows by (2.13) that (2.12) holds again.

Therefore, by (i), (2.11) and (2.12),

$$0 \ge T_{\phi_2}(\delta)(t) \ge \beta(t), \quad \sigma < t < d.$$

By Theorems 2.1, 2.2 and $\phi_2(\sigma) > 0$, system (1.1)–(1.2) has a positive solution y defined by

$$y(t) = \begin{cases} \phi_2(t), & \text{if } r_\sigma \le t \le \sigma, \\ B(\sigma, t)\phi_2(\sigma)e^{\int_\sigma^t \alpha(s)\,ds} & \text{if } \sigma < t < d, \end{cases}$$

where $\alpha \in PC([\sigma, d), \mathbb{R})$, $T_{\phi_2}(\alpha)(t) = \alpha(t)$ and $0 \ge \alpha(t) \ge \beta(t)$ for $\sigma < t < d$. Obviously, $x(t) \le y(t)$ on $[r_\sigma, d)$. ■

Proof of Theorem 2.4. If $[\sigma, d) \cap \{t_k\}_{k\in\mathbb{N}} = \emptyset$, then by definition, x is also a positive solution of (1.3) and

$$x(t_k^+) - x(t_k) = b'_k x(t_k), \quad k \in \mathbb{N},$$

on $[\sigma, d)$ where b'_k are constants with $b'_k > -1$ for $k \in \mathbb{N}$. By Lemma 2.5, equations (1.1) and

$$y(t_k^+) - y(t_k) = b_k y(t_k), \quad k \in \mathbb{N},$$

have a positive solution y on $[\sigma, d)$ such that $x(t) \le y(t)$ on $[r_\sigma, d)$ where b_k is an arbitrary constant with $b_k \ge b'_k$ for each $k \in \mathbb{N}$. By definition, y is also a positive solution of (1.1)–(1.2) on $[\sigma, d)$. The proof of this case is complete.

If $[\sigma, d) \cap \{t_k\}_{k\in\mathbb{N}} \ne \emptyset$, we still need to use Lemma 2.5 in each interval $[\sigma, d) \cap [t_k, t_{k+1})$, $k \in \mathbb{N}$, to construct a solution of (1.1)–(1.2). For convenience, we assume that $\sigma = t_1$ and $d = \infty$. Let $b'_k = J_k(x(t_k))/x(t_k)$ for

$k \in \mathbb{N}$. Then x is a positive solution of the system

$$D^- x(t) + \sum_{i=1}^{n} p_i(t) F_i(x(g_i(t))) \le 0, \qquad t \in [0,\infty) \setminus \Upsilon,$$
$$x(t_k^+) - x(t_k) = b_k' x(t_k), \qquad k \in \mathbb{N},$$

on $[\sigma, \infty)$ with initial condition $x(t) = \phi_1(t)$ on $[r_\sigma, \sigma]$. By (A4) and $x(t_k) > 0$ for $k \in \mathbb{N}$, we further see that $b_k' > -1$ for $k \in \mathbb{N}$. If $\beta(t)$ is defined by (2.10), we see that $\beta \in PC([\sigma,\infty), \mathbb{R})$, (2.11) holds and

$$x(t) = \begin{cases} \phi_1(t) & \text{if } r_\sigma \le t \le \sigma, \\ \bar{B}(\sigma, t)\phi_1(\sigma) e^{\int_\sigma^t \beta(s)\, ds} & \text{if } t > \sigma. \end{cases}$$

Now, we construct a positive solution of (1.1)–(1.2) in four steps.

Step 1. Let $b_1 = I_1(\phi_2(t_1))/\phi_2(t_1)$. Since $\phi_2(t_1) \ge \phi_1(t_1) \ge 0$ and (iv) holds, we see that $b_1 \ge b_1' > -1$. By Lemma 2.5, there exists a function y_1, a positive solution of (1.1) and

$$y(t_1^+) - y(t_1) = b_1 y(t_1),$$

on $[t_1, t_2)$ with initial condition $y_1(t) = \phi_2(t)$ on $[r_{t_1}, t_1]$, such that $y_1(t) \ge x(t)$ on $[r_{t_1}, t_2)$. Then

$$y_1(t_1^+) - y_1(t_1) = b_1 y_1(t_1) = \frac{I_1(\phi_2(t_1))}{\phi_2(t_1)} y_1(t_1) = I_1(y_1(t_1)).$$

In fact, y_1 is of the form

$$y_1(t) = \begin{cases} \phi_2(t) & \text{if } r_{t_1} \le t \le t_1, \\ (I_1(\phi_2(t_1)) + \phi_2(t_1)) e^{\int_{t_1}^t \alpha_1(s)\, ds} & \text{if } t_1 < t < t_2, \end{cases}$$

where $\alpha_1 \in PC([t_1, t_2), \mathbb{R})$ and $\beta(t) \le \alpha_1(t) \le 0$ on $[t_1, t_2)$. Since $\alpha_1 \in PC([t_1, t_2), \mathbb{R})$ and $\alpha_1(t_2^-)$ exists, we see that $y_1(t_2^-)$ exists. Then we may define a function $\overline{y}_1$ on $[r_{t_1}, t_2]$ such that $\overline{y}_1(t) = y_1(t)$ on $[r_{t_1}, t_2)$ and $\overline{y}_1(t_2) = y_1(t_2^-)$. Clearly, $\overline{y}_1$ is also a positive solution of (1.1)–(1.2) on $[t_1, t_2)$ with initial condition $\overline{y}_1(t) = \phi_2(t)$ on $[r_{t_1}, t_1]$. Furthermore,

$$\overline{y}_1(t) \ge x(t) > 0 \quad \text{on } [r_{t_1}, t_2]. \tag{2.17}$$

Step 2. We first claim that for $t \in [r_{t_2}, t_2]$,

$$\frac{\overline{y}_1(t)}{\overline{y}_1(t_2)} \le \frac{x(t)}{x(t_2)}. \tag{2.18}$$

Indeed, since $r_{t_1} \le r_{t_2}$, we note that

$$[r_{t_2}, t_2] = [r_{t_2}, t_2] \cap ([r_{t_1}, t_1] \cup (t_1, t_2]).$$

For $t \in [r_{t_2}, t_2] \cap (t_1, t_2]$, by the definition of $\overline{y}_1$, we have

$$\frac{\overline{y}_1(t)}{\overline{y}_1(t_2)} = e^{-\int_t^{t_2} \alpha_1(s)\, ds} \le e^{-\int_t^{t_2} \beta(s)\, ds} = \frac{x(t)}{x(t_2)}.$$

Thus (2.18) holds. If $[r_{t_2}, t_2] \cap [r_{t_1}, t_1] \neq \emptyset$, let $t \in [r_{t_2}, t_2] \cap [r_{t_1}, t_1]$; then

$$\begin{aligned}\frac{I_1(\phi_2(t_1)) + \phi_2(t_1)}{\phi_2(t)} &= \left(\frac{I_1(\phi_2(t_1))}{\phi_2(t_1)} + 1\right)\frac{\phi_2(t_1)}{\phi_2(t)} \\ &\geq \left(\frac{J_1(\phi_1(t_1))}{\phi_1(t_1)} + 1\right)\frac{\phi_1(t_1)}{\phi_1(t)} = \frac{J_1(\phi_1(t_1)) + \phi_1(t_1)}{\phi_1(t)}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\overline{y}_1(t)}{\overline{y}_1(t_2)} &= \frac{\phi_2(t)}{(I_1(\phi_2(t_1)) + \phi_2(t_1))e^{\int_{t_1}^{t_2} \alpha_1(s)\,ds}} \\ &\leq \frac{\phi_1(t)}{(J_1(\phi_1(t_1)) + \phi_1(t_1))e^{\int_{t_1}^{t_2} \beta(s)\,ds}} = \frac{x(t)}{x(t_2)}.\end{aligned}$$

Thus (2.18) holds again.

Second, we let $b_2 = I_2(\overline{y}_1(t_2))/\overline{y}_1(t_2)$. By (2.17) and (iv), we have $b_2 \geq b_2' > -1$. By Lemma 2.5, there exists a function y_2, a positive solution of (1.1) and

$$y(t_2^+) - y(t_2) = b_2 y(t_2),$$

satisfying the initial condition $y_2(t) = \overline{y}_1(t)$ on $[r_{t_2}, t_2]$, such that $y_2(t) \geq x(t)$ on $[r_{t_2}, t_3)$. So

$$y_2(t_2^+) - y_2(t_2) = b_2 y_2(t_2) = \frac{I_2(\overline{y}_1(t_2))}{\overline{y}_1(t_2)}\overline{y}_1(t_2) = I_2(y_2(t_2)).$$

In fact, y_2 is of the form

$$y_2(t) = \begin{cases} \overline{y}_1(t) & \text{if } r_{t_2} \leq t \leq t_2, \\ (I_2(\overline{y}_1(t_2)) + \overline{y}_1(t_2))e^{\int_{t_2}^{t} \alpha_2(s)\,ds} & \text{if } t_2 < t < t_3, \end{cases}$$

where $\alpha_2 \in PC([t_2, t_3), \mathbb{R})$ and $\beta(t) \leq \alpha_2(t) \leq 0$ on $[t_2, t_3)$. Similarly, we may define a function $\overline{y}_2$ on $[r_{t_2}, t_3]$ such that $\overline{y}_2(t) = y_2(t)$ on $[r_{t_2}, t_3)$ and $\overline{y}_2(t_3) = y_2(t_3^-)$. Clearly, $\overline{y}_2$ is a positive solution of (1.1)–(1.2) on $[t_2, t_3)$ with initial condition $\overline{y}_2(t) = \overline{y}_1(t)$ on $[r_{t_2}, t_2]$. Furthermore,

$$\overline{y}_2(t) \geq x(t) > 0 \quad \text{on } [r_{t_2}, t_3]. \tag{2.19}$$

STEP 3. We first claim that

$$\frac{\overline{y}_2(t)}{\overline{y}_2(t_3)} \leq \frac{x(t)}{x(t_3)}, \qquad t \in [r_{t_3}, t_3]. \tag{2.20}$$

Indeed, since $r_{t_2} \leq r_{t_3}$, we see that

$$[r_{t_3}, t_3] = [r_{t_3}, t_3] \cap ([r_{t_2}, t_2] \cup (t_2, t_3]).$$

For $t \in [r_{t_3}, t_3] \cap (t_2, t_3]$, by the definition of $\overline{y}_1$, we have

$$\frac{\overline{y}_2(t)}{\overline{y}_2(t_3)} = e^{-\int_t^{t_3} \alpha_2(s)\,ds} \leq e^{-\int_t^{t_3} \beta(s)\,ds} = \frac{x(t)}{x(t_3)}. \tag{2.21}$$

Thus (2.20) holds. If $[r_{t_3}, t_3] \cap [r_{t_2}, t_2] \neq \emptyset$, let $t \in [r_{t_3}, t_3] \cap [r_{t_2}, t_2]$. Since

$$\overline{y}_2(t_2^+) = \overline{y}_1(t_2) + I_2(\overline{y}_1(t_2)) \quad \text{and} \quad x(t_2^+) = x(t_2) + J_2(x(t_2)),$$

we see that

$$\frac{\overline{y}_2(t_2^+)}{\overline{y}_1(t_2)} = 1 + \frac{I_2(\overline{y}_1(t_2))}{\overline{y}_1(t_2)} \geq 1 + \frac{J_2(x(t_2))}{x(t_2)} = \frac{x(t_2^+)}{x(t_2)}. \tag{2.22}$$

By (2.18), (2.21) and (2.22), we see that

$$\frac{\overline{y}_1(t)}{\overline{y}_2(t_3)} = \frac{\overline{y}_1(t)}{\overline{y}_1(t_2)} \frac{\overline{y}_1(t_2)}{\overline{y}_2(t_2^+)} \frac{\overline{y}_2(t_2^+)}{\overline{y}_2(t_3)} \leq \frac{x(t)}{x(t_2)} \frac{x(t_2)}{x(t_2^+)} \frac{x(t_2^+)}{x(t_3)} = \frac{x(t)}{x(t_3)},$$

so (2.20) holds again.

Second, we let $b_3 = I_3(\overline{y}_2(t_3))/\overline{y}_2(t_3)$. By (2.19) and (iv), we have $b_3 \geq b_3' > -1$. By Lemma 2.5, there exists a function y_3, a positive solution of (1.1) and

$$y(t_3^+) - y(t_3) = b_3 y(t_3),$$

on $[t_3, t_4)$ with initial condition $y_3(t) = \overline{y}_2(t)$ on $[r_{t_3}, t_3]$. Similarly, there exists a function $\overline{y}_3$ which is a positive solution of (1.1)–(1.2) on $[t_3, t_4)$ with initial condition $\overline{y}_3(t) = \overline{y}_2(t)$ on $[r_{t_3}, t_3]$. Furthermore, $\overline{y}_3(t) \geq x(t) > 0$ on $[r_{t_3}, t_4]$.

Step 4. By induction, we have functions $\overline{y}_k$ defined on $[r_{t_k}, t_{k+1}]$ such that $\overline{y}_k$ is a positive solution of (1.1)–(1.2) on $[t_k, t_{k+1})$ with initial condition $\overline{y}_k(t) = \overline{y}_{k-1}(t)$ on $[r_{t_k}, t_k]$ for $k \geq 2$. Let

$$y(t) = \phi_2(t)\chi_{[r_{t_1}, t_1]}(t) + \sum_{k=1}^{\infty} \overline{y}_k(t)\chi_{(t_k, t_{k+1}]}(t), \qquad t \geq r_{t_1}.$$

Finally, we verify that y is a positive solution of (1.1)–(1.2) on $[t_1, \infty)$ with initial condition $y(t) = \phi_2(t)$ on $[r_{t_1}, t_1]$. Clearly, y is a positive function on $[t_1, \infty)$, $y(t) = \phi_2(t)$ on $[r_{t_1}, t_1]$, y is absolutely continuous on each interval $(t_k, t_{k+1}]$, $k \in \mathbb{N}$, $D^- y \in PC((t_1, \infty), \mathbb{R})$; and

$$D^- y(t) + \sum_{i=1}^{n} q_i(t) f_i(y(g_i(t))) = 0, \qquad t > t_1.$$

For $k \geq 0$,

$$y(t_k^-) = \overline{y}_{k-1}(t_k^-) = \overline{y}_{k-1}(t_k) = y(t_k)$$

and

$$y(t_k^+) = \overline{y}_k(t_k^+) = \overline{y}_{k-1}(t_k) + I_k(\overline{y}_{k-1}(t_k)) = y(t_k) + I_k(y(t_k)).$$

The proof is complete.

COROLLARY 2.6. *Assume that the hypotheses of Theorem* 2.4 *hold with* (iv) *replaced by*

(iv′) $I_k(\mu_2)/\mu_2 \geq J_k(\mu_1)/\mu_1$ *if* $\mu_2 \leq \mu_1 < 0$.

Let $\phi_3, \phi_4 \in PC([r_\sigma, \sigma], \mathbb{R})$ *be such that*

(2.23)
$$\phi_3(\sigma)\phi_4(\sigma) \neq 0, \quad \phi_4(t) \leq \phi_3(t) \leq 0 \text{ and } \frac{\phi_3(t)}{\phi_3(\sigma)} \geq \frac{\phi_4(t)}{\phi_4(\sigma)} \text{ for } r_\sigma \leq t \leq \sigma.$$

If x *is a negative solution of*

$$D^- x(t) + \sum_{i=1}^n p_i(t) F_i(x(g_i(t))) \geq 0, \quad t \in [0, \infty) \setminus \Upsilon,$$
$$x(t_k^+) - x(t_k) = J_k(x(t_k)), \quad k \in \mathbb{N},$$

on $[\sigma, d)$ *satisfying the initial condition* $x(t) = \phi_3(t)$ *on* $[r_\sigma, \sigma]$, *then system* (1.1)–(1.2) *has a negative solution* y *on* $[\sigma, d)$ *satisfying the initial condition* $y(t) = \phi_4(t)$ *on* $[r_\sigma, \sigma]$ *such that* $y(t) \leq x(t)$ *on* $[r_\sigma, d)$.

Proof. Let $\widetilde{\phi}_3(t) = -\phi_3(t)$ and $\widetilde{\phi}_4(t) = -\phi_4(t)$ on $[r_\sigma, \sigma]$. It is easy to see that $\widetilde{\phi}_3(\sigma)\widetilde{\phi}_4(\sigma) \neq 0$, and

$$\widetilde{\phi}_4(t) \geq \widetilde{\phi}_3(t) \geq 0 \quad \text{and} \quad \frac{\widetilde{\phi}_3(t)}{\widetilde{\phi}_3(\sigma)} \geq \frac{\widetilde{\phi}_4(t)}{\widetilde{\phi}_4(\sigma)} \quad \text{for } r_\sigma \leq t \leq \sigma.$$

Let $\widetilde{F}_i(\mu) = -F_i(-\mu)$, $\widetilde{f}_i(\mu) = -f_i(-\mu)$, $\widetilde{I}_k(\mu) = -I_k(-\mu)$ and $\widetilde{J}_k(\mu) = -J_k(-\mu)$ where $\mu \in \mathbb{R}$, $i \in \mathbb{N}_n$ and $k \in \mathbb{N}$. Let $z(t) = -x(t)$ on $[r_\sigma, d)$. Then z is a positive solution of

$$D^- x(t) + \sum_{i=1}^n p_i(t) \widetilde{F}_i(x(g_i(t))) \leq 0, \quad t \in [0, \infty) \setminus \Upsilon,$$
$$x(t_k^+) - x(t_k) = \widetilde{J}_k(x(t_k)), \quad k \in \mathbb{N},$$

satisfying the initial condition $z(t) = \widetilde{\phi}_3(t)$ on $[r_\sigma, \sigma]$. It is easy to see that all $\widetilde{F}_i$, $\widetilde{f}_i$, $\widetilde{I}_k$ and $\widetilde{J}_k$ satisfy the hypotheses (i)–(iv) of Theorem 2.4. In view of (iv′), we may further see that $\widetilde{I}_k(y)/y \geq \widetilde{J}_k(x)/x$ if $0 < x \leq y$. By Theorem 2.4, the system

$$D^- x(t) + \sum_{i=1}^n q_i(t) \widetilde{f}_i(x(g_i(t))) = 0, \quad t \in [0, \infty) \setminus \Upsilon,$$
$$x(t_k^+) - x(t_k) = \widetilde{I}_k(x(t_k)), \quad k \in \mathbb{N},$$

has a positive solution $\widetilde{y}$ on $[\sigma, d)$ such that $\widetilde{y}(t) = \widetilde{\phi}_4(t)$ on $[r_\sigma, \sigma]$ and $\widetilde{y}(t) \geq z(t)$ on $[r_\sigma, d)$. Let $y(t) = -\widetilde{y}(t)$ on $[r_\sigma, d)$. Then y is a negative solution of (1.1)–(1.2) on $[\sigma, d)$ such that $y(t) \leq x(t)$ on $[r_\sigma, d)$. ∎

Corollary 2.7. *Assume that* (A1)–(A5) *hold, that for each* $k \in \mathbb{N}$, $\mu I_k(\mu) \geq 0$ *for* $\mu \neq 0$, *and for each* $i \in \mathbb{N}_n$, q_i *is nonnegative, and* f_i *satisfies* (B1) *and* (B3). *If the equation*

$$y'(t) + \sum_{i=1}^{n} q_i(t) f_i(y(g_i(t))) = 0 \tag{2.24}$$

has an eventually positive (*or negative*) *solution, then system* (1.1)–(1.2) *has an eventually positive* (*respectively negative*) *solution as well.*

Proof. Assume that x is an eventually positive or negative solution of (2.24). Then there exists $T > 0$ such that $x(t) > 0$ for $t \geq T$ or $x(t) < 0$ for $t \geq T$. Let $\phi(t) = x(t)$ for $t \geq T$, $F_i(\mu) = f_i(\mu)$ and $J_k(\mu) = 0$ for $\mu \in \mathbb{R}$. In view of $\mu I_k(\mu) \geq 0$ for $\mu \neq 0$, we see that I_k satisfy (iv) and (iv′). By Theorem 2.4 and Corollary 2.6, system (1.1)–(1.2) has an eventually positive (respectively negative) solution. ∎

Corollary 2.8. *Assume that* (A1)–(A5) *hold, and for each* $i \in \mathbb{N}_n$, q_i *is nonnegative, and* f_i *satisfies* (B1) *and* (B3). *Assume further that for each* $k \in \mathbb{N}$,

$$I_k(\mu_2)/\mu_2 \geq I_k(\mu_1)/\mu_1 \quad \text{if } 0 < |\mu_1| \leq |\mu_2|.$$

Then the following statements are equivalent:

(a) *System* (1.1)–(1.2) *has an eventually positive* (*or negative*) *solution.*
(b) *For all* $(\varepsilon_1, \ldots, \varepsilon_n) \in (0,1]^n$, *the system*

$$D^- x(t) + \sum_{i=1}^{n} \varepsilon_i q_i(t) f_i(x(g_i(t))) = 0, \quad t \in [0,\infty) \setminus \Upsilon, \tag{2.25}$$

$$x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k \in \mathbb{N}, \tag{2.26}$$

has an eventually positive (*respectively negative*) *solution.*

Proof. It is obvious that if by taking all $\varepsilon_i = 1$ system (2.25)–(2.26) has an eventually positive (or negative) solution, then (1.1)–(1.2) also has an eventually positive (respectively negative) solution.

Conversely, assume that (1.1)–(1.2) has an eventually positive solution x. Then there is $\sigma \geq 0$ such that $x(t) > 0$ for $t \geq r_\sigma$. Given $(\varepsilon_1, \ldots, \varepsilon_n) \in (0,1]^n$, x is also a positive solution of

$$D^- x(t) + \sum_{i=1}^{n} \varepsilon_i q_i(t) f_i(x(g_i(t))) \leq 0, \quad t \in [0,\infty) \setminus \Upsilon,$$
$$x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k \in \mathbb{N},$$

on $[\sigma,\infty)$. By Theorem 2.4, system (2.25)–(2.26) has a positive solution on $[\sigma,\infty)$.

Second, assume that (1.1)–(1.2) has an eventually negative solution x. So there is $\sigma \geq t_0$ such that $x(t) < 0$ for $t \geq r_\sigma$. Given $(\varepsilon_1, \ldots, \varepsilon_n) \in (0,1]^n$, we see that x is a negative solution of

$$D^- x(t) + \sum_{i=1}^{n} \varepsilon_i q_i(t) f_i(x(g_i(t))) \geq 0, \qquad t \in [0,\infty) \setminus \Upsilon,$$

$$x(t_k^+) - x(t_k) = I_k(x(t_k)), \qquad k \in \mathbb{N},$$

on $[\sigma,\infty)$. By Corollary 2.6, system (2.25)–(2.26) has a negative solution on $[\sigma,\infty)$. ∎

Remark. If we assume that $\varepsilon_i = \varepsilon$ and $f_i(\mu) = \mu$ for all $i \in \mathbb{N}_n$ and $I_k(\mu) = 0$ for all $k \in \mathbb{N}$, then Corollary 2.8 is Lemma 4 in [5].

However, it does not seem clear why Lemma 4 in [5] can be applied to yield Theorem 2 in [5] (see the last three paragraphs in the last section). As will be explained in the last section, our next theorem will solve this problem under some special conditions.

Theorem 2.9. *Assume that* (A1)–(A5) *hold, each I_k is linear, each q_i is nonnegative, and each f_i satisfies* (B1). *Assume further that there is at least one $j \in \mathbb{N}_n$ such that $g_j(t) < t$ for $t \geq 0$ and the preimage $g_j^{-1}(0)$ does not contain an open interval. Let $\sigma \geq 0$ and $\phi \in PC([r_\sigma,\sigma],\mathbb{R}^+)$ be given. Assume that there exists a sequence $\{\varepsilon_j\}_{j=1}^{\infty} = \{(\varepsilon_{j1},\ldots,\varepsilon_{jn})\}_{j=1}^{\infty}$ in $(0,1)^n$ such that*

$$\lim_{j\to\infty} \varepsilon_j = (1,\ldots,1)$$

and for all $j \in \mathbb{N}$, the system

$$x'(t) + \sum_{i=1}^{n} \varepsilon_{ji} q_i(t) f_i(x(g_i(t))) = 0, \qquad t \in [0,\infty) \setminus \Upsilon, \tag{2.27}$$

$$x(t_k^+) - x(t_k) = I_k(x(t_k)), \qquad k \in \mathbb{N}, \tag{2.28}$$

has a positive solution y_{ε_j} on $[\sigma,\infty)$ with initial condition $y_{\varepsilon_j}(t) = \phi(t)$ on $[r_\sigma,\sigma]$. Then system (1.1)–(1.2) *has a positive solution $\overline{y}$ on $[\sigma,\infty)$ with initial condition $\overline{y}(t) = \phi(t)$ on $[r_\sigma,\sigma]$.*

Proof. We let $I_k(x) = b_k x$ on $\mathbb{R}$ where b_k are constants for $k \in \mathbb{N}$. By (A4), $b_k > -1$ for $k \in \mathbb{N}$. We recall $B(s,t)$ defined by (1.6). Without loss of generality, we may assume $\sigma = 0$. For each $j \in \mathbb{N}$, define

$$x_{\varepsilon_j}(t) = \begin{cases} \phi(t) & \text{if } r_0 \leq t \leq 0, \\ B(0,t)^{-1} y_{\varepsilon_j}(t) & \text{if } t > 0. \end{cases}$$

We note that for each $j \in \mathbb{N}$, $x_{\varepsilon_j}(t) > 0$ on $[r_0,\infty)$ and

$$x_{\varepsilon_j}(t_k^+) = \frac{y_{\varepsilon_j}(t_k^+)}{B(0,t_k^+)} = \frac{(1+b_k) y_{\varepsilon_j}(t_k)}{(1+b_k) B(0,t_k)} = x_{\varepsilon_j}(t_k), \qquad k \in \mathbb{N}.$$

Thus x_{ε_j} is a positive continuous function on $[0,\infty)$ for each $j\in\mathbb{N}$. By (2.27), it is easy to check that for any $j\in\mathbb{N}$, x_{ε_j} is a positive solution of

$$D^-x(t)+\sum_{i=1}^{n}\varepsilon_{ji}\frac{q_i(t)}{B(0,t)}f_i(B(0,g_i(t))x(g_i(t)))=0 \tag{2.29}$$

on $[0,\infty)$ with initial condition $x_{\varepsilon_j}(t)=\phi(t)$ on $[r_0,0]$. If we can find a positive solution $\overline{x}$ of

$$D^-x(t)+\sum_{i=1}^{n}\frac{q_i(t)}{B(0,t)}f_i(B(0,g_i(t))x(g_i(t)))=0 \tag{2.30}$$

on $[0,\infty)$ with initial condition $\overline{x}(t)=\phi(t)$ on $[r_{t_0},t_0]$, then

$$\overline{y}(t)=\begin{cases}\phi(t) & \text{if } r_0\le t\le 0,\\ B(0,t)\overline{x}(t) & \text{if } t>0,\end{cases}$$

as required. To this end, we first prove that for any $A>0$, equation (2.30) has a positive solution $\overline{x}_A$ on $[0,A]$ satisfying the initial condition $\overline{x}(t)=\phi(t)$ on $[r_0,0]$. In view of (2.29) and $x_{\varepsilon_j}(t)>0$ on $[r_0,\infty)$, it is easy to see that x_{ε_j} is decreasing on $[0,\infty)$ for each $j\in\mathbb{N}$. Thus for all $j\in\mathbb{N}$,

$$|x_{\varepsilon_j}(t)|\le m_1,\qquad r_0\le t\le A,$$

where $m_1=\|\phi\|_{[r_0,0]}$. So $\{x_{\varepsilon_j}(t):j\in\mathbb{N}\}$ is uniformly bounded in $[0,A]$. Let

$$M_1=\max_{i\in\mathbb{N}_n}\|q_i\|_{[0,A]}.$$

There are $m_2>0$ and $M_2>0$ such that $m_2\le B(0,t)\le M_2$ on $[0,A]$. Integrating (2.29) from 0 to t, we see that

$$x_{\varepsilon_j}(t)=\phi(0)-\sum_{i=1}^{n}\varepsilon_{ji}\int_0^t\frac{q_i(s)}{B(0,s)}f_i(B(0,g_i(s))x_{\varepsilon_j}(g_i(s)))\,ds \tag{2.31}$$

where $0\le t\le A$ and $j\in\mathbb{N}$. By continuity of f_i, $i\in\mathbb{N}_n$, there is $M_3>0$ such that for each $i\in\mathbb{N}_n$, $|f_i(\mu)|\le M_3$ if $|\mu|\le m_1M_2$. By (2.31), we have

$$\begin{aligned}|x_{\varepsilon_j}(\eta)-x_{\varepsilon_j}(\varsigma)|&=\sum_{i=1}^{n}\varepsilon_{ji}\int_\varsigma^\eta\frac{q_i(s)}{B(0,s)}f_i(B(0,g_i(s))x_{\varepsilon_j}(g_i(s)))\,ds\\&\le\sum_{i=1}^{n}\int_\varsigma^\eta\frac{q_i(s)}{B(0,s)}f_i(B(0,g_i(s))x_{\varepsilon_j}(g_i(s)))\,ds\\&\le n\frac{M_1}{m_2}M_3(\eta-\varsigma)\end{aligned}$$

where $0\le\varsigma\le\eta\le A$ and $j\in\mathbb{N}$. So $\{x_{\varepsilon_j}(t):j\in\mathbb{N}\}$ is equicontinuous in $[0,A]$. By Ascoli–Arzelà's Theorem, there exists a subsequence $\{\widetilde{\varepsilon}_j\}_{j=1}^{\infty}$ of

$\{\varepsilon_j\}_{j=1}^{\infty}$ and a function $\widetilde{x}_A$ defined on $[0, A]$ such that

$$\lim_{j\to\infty} x_{\widetilde{\varepsilon}_j}(t) = \widetilde{x}_A(t) \quad \text{uniformly on } [0, A]$$

where

$$\lim_{j\to\infty} \widetilde{\varepsilon}_j = (1, \ldots, 1).$$

Define

$$\overline{x}_A(t) = \begin{cases} \phi(t) & \text{if } r_0 \le t \le 0, \\ \widetilde{x}_A(t) & \text{if } 0 < t \le A. \end{cases}$$

By uniform convergence, $\overline{x}_A$ is decreasing on $[0, A]$ and $\overline{x}_A$ is a nonnegative solution of (2.30) on $[0, A]$ with initial condition $\overline{x}_A(t) = \phi(t)$ on $[r_0, 0]$. Let

$$A^* = \sup\{A > 0 : (2.30) \text{ has a decreasing nonnegative solution on } [0, A]\}.$$

Clearly, $A^* = \infty$. Thus there exists a function $\overline{x}$ that is decreasing on $[0, \infty)$ and is a nonnegative solution of (2.30) on $[0, \infty)$ with initial condition $\overline{x}(t) = \phi(t)$ on $[r_0, 0]$. Let

$$T = \inf\{t \ge 0 : \overline{x}(t) = 0\}.$$

If T is infinite, our proof is complete. Assume therefore that T is finite. We note that $T > 0$ due to $\phi(0) > 0$. Since $\overline{x}(t) \ge 0$ and $\overline{x}$ is decreasing on $[0, \infty)$, we see that $\overline{x}(t) = 0$ for $t \ge T$. Since $g_j(T) < T$, by continuity of $g_j(t)$, there exists $h_T > 0$ such that $g_j(t) < T$ for $T \le t \le T + h_T$. For $T < t < T + h_T$, we note that $\overline{x}(g_j(t)) > 0$ and

$$0 = D^-\overline{x}(t) = -\sum_{i=1}^{n} \frac{q_i(t)}{B(0,t)} f_i(\overline{x}(g_i(t))) \le \frac{q_j(t)}{B(0,t)} f_j(\overline{x}(g_j(t))) \le 0.$$

Then $q_j(t) = 0$ on $(T, T + h_T)$. ∎

REMARK. Under the same initial condition, one of the hypotheses of Theorem 2.9, that there is at least one $j \in \mathbb{N}_n$ such that $g_j(t) < t$ for $t \ge 0$ and the preimage $g_j^{-1}(0)$ does not contain an open interval, cannot be omitted. Otherwise, a counterexample can be given (see Example 3.2 below).

REMARK. From the proofs of our theorems and corollaries, we may see that if we only consider the existence of positive (or negative) solutions, it is sufficient that the functions F_i and f_i satisfy (B3) and (B4) on $\mathbb{R}^+$ (respectively $\mathbb{R}^-$).

3. Examples

EXAMPLE 3.1 (Lasota–Ważewska equation). Consider the equation

$$D^-N(t) = -aN(t) + pe^{-rN(g(t))}, \quad t \ge 0, \tag{3.1}$$

where $a, p, r > 0$ and g is continuous on $[0, \infty)$ such that $g(t) < t$ and $\lim_{t\to\infty} g(t) = \infty$. The unique equilibrium N^* of equation (3.1) is positive and satisfies the equation

$$N^* = \frac{p}{a} e^{-rN^*}.$$

Consider the following impulsive condition:

$$N(t_k^+) - N^* = (1 + b_k)(N(t_k) - N^*), \quad k \in \mathbb{N}, \tag{3.2}$$

where $b_k > -1$ and $\{t_k\}_{k\in\mathbb{N}}$ satisfies (A3). After the change of variables

$$r(N(t) - N^*) = x(t),$$

the impulsive system (3.1) and (3.2) takes the form

$$D^- x(t) + ax(t) + arN^*(1 - e^{-x(g(t))}) = 0, \quad t \in [0, \infty) \setminus \Upsilon, \tag{3.3}$$

$$x(t_k^+) - x(t_k) = b_k x(t_k), \quad k \in \mathbb{N}. \tag{3.4}$$

Consider the impulsive system

$$D^- x(t) + ax(t) + arN^* x(g(t)) = 0, \quad t \in [0, \infty) \setminus \Upsilon, \tag{3.5}$$

$$x(t_k^+) - x(t_k) = b_k x(t_k), \quad k \in \mathbb{N}. \tag{3.6}$$

The following statements are true:

(i) If the impulsive system (3.5)–(3.6) has an eventually positive (or negative) solution, then (3.3)–(3.4) has an eventually positive (respectively negative) solution.

(ii) Assume

$$\sum_{k:\, b_k>0} b_k < \infty. \tag{3.7}$$

If the impulsive system (3.3)–(3.4) has an eventually positive (or negative) solution, then the impulsive system (3.5)–(3.6) has an eventually positive (respectively negative) solution.

(iii) Assume $b_k \ge 0$ for $k \in \mathbb{N}$ and

$$arN^* \int_{g(t)}^{t} e^{a(s-g(s))}\, ds \le \frac{1}{e}, \quad t \ge \sigma, \tag{3.8}$$

for some $\sigma \ge 0$. Then the impulsive system (3.3)–(3.4) has eventually positive and eventually negative solutions.

Proof of (i). Let $q_1(t) = p_1(t) = a$, $q_2(t) = p_2(t) = arN^*$, $f_1(\mu) = F_1(\mu) = F_2(\mu) = \mu$ and $f_2(\mu) = 1 - e^{-\mu}$ for $\mu \in \mathbb{R}$ and $t \ge 0$. Clearly, f_1 and F_1 satisfy (A1) and (B1)–(B4) on $\mathbb{R}$, f_2 satisfies (A1), (B1), (B2) and (B4) on $\mathbb{R}$, F_2 satisfies (A1), (B1) and (B3) on $\mathbb{R}$. Furthermore, $p_i(t) \ge q_i(t) \ge 0$ on $[0, \infty)$ and $f_i(\mu) \le F_i(\mu)$ on $\mathbb{R}^+$ for $i = 1, 2$. By Theorem 2.4 and Corollary 2.6, it is easy to see that (i) is true.

Proof of (ii). If (3.3)–(3.4) has an eventually positive solution x_1, then there is $T>0$ such that $x_1(t)>0$ for $t\ge T$. We note that $\liminf_{\mu\to\infty} f_i(\mu)>0$ for $i=1,2$, and

$$\int_t^\infty arN^*\,ds = \int_t^\infty a\,ds = +\infty \qquad \text{for any } t>0.$$

Let

$$c_k = \begin{cases} 1+b_k & \text{if } b_k \ge 0,\\ 1 & \text{if } -1<b_k<0.\end{cases}$$

Then

$$|(1+b_k)\mu| \le c_k|\mu|$$

for all $k\in\mathbb{N}$ and $\mu\in\mathbb{R}$. By (3.7), we see that

$$\sum_{k\in\mathbb{N}}(c_k-1) = \sum_{b_k>0} b_k < \infty.$$

By Lemma 4 in [3], we see that $\lim_{t\to\infty} x_1(t)=0$. We note that $f_2'(0)=1$. Then for any $\varepsilon\in(0,1)$, there exists $\sigma>T$ such that $r_\sigma>T$ is defined by (1.5), and $f_2(x_1(t))/x_1(t)\ge(1-\varepsilon)$ for $t\ge T$. So the impulsive system

$$\begin{aligned} &D^-x(t)+ax(t)+arN^*(1-\varepsilon)x(g(t))\le 0, \qquad t\in[0,\infty)\setminus\Upsilon,\\ &x(t_k^+)-x(t_k)=b_kx(t_k), \qquad k\in\mathbb{N}, \end{aligned}$$

has an eventually positive solution x_1 for all $\varepsilon\in(0,1)$. By Theorem 2.4, the impulsive system

$$\begin{aligned} &D^-x(t)+ax(t)+arN^*(1-\varepsilon)x(g(t))= 0, \qquad t\in[0,\infty)\setminus\Upsilon,\\ &x(t_k^+)-x(t_k)=b_kx(t_k), \qquad k\in\mathbb{N}, \end{aligned}$$

has a positive solution y_ε on $[\sigma,\infty)$ with initial condition $y_\varepsilon(t)=x_1(t)$ on $[r_\sigma,\sigma]$ for $\varepsilon\in(0,1)$. Since $g(t)<t$, and $ar^2N^*>0$, by Theorem 2.9 system (3.5)–(3.6) has an eventually positive solution. If (3.3)–(3.4) has an eventually negative solution x_2, let $\widetilde{f_2}(\mu)=-f_2(-\mu)$. Then $\widetilde{f_2}'(0)=1$ and the impulsive system

$$\begin{aligned} &D^-x(t)+ax(t)+arN^*\widetilde{f_2}(x(g(t)))= 0, \qquad t\in[0,\infty)\setminus\Upsilon,\\ &x(t_k^+)-x(t_k)=b_kx(t_k), \qquad k\in\mathbb{N}, \end{aligned}$$

has an eventually positive solution $-x_2$. Similarly, (3.5)–(3.6) has an eventually positive solution. Since (3.5) and (3.6) are linear, the impulsive system (3.5)–(3.6) has an eventually negative solution as well. Therefore, (ii) is true.

Proof of (iii). By (3.8) and Theorem 3.3.1 in [8], the equation

$$D^-x(t)+ax(t)+arN^*x(g(t))=0$$

has eventually positive and eventually negative solutions. By Corollary 2.7, system (3.5)–(3.6) has eventually positive and eventually negative solutions. By (i), statement (iii) is true.

We recall that a function φ defined for all sufficiently large t is *nonoscillatory* if φ is eventually positive or eventually negative. In the rest of this section, we compare Theorems 9 and 10 in [3] with our results.

It is easy to see that Theorem 9 in [3] follows from (ii) of Example 3.1 and Corollary 2.8, and Theorem 10 in [3] follows from (i). In view of (i) and (ii) of Example 3.1, we can further improve the sufficient conditions of Theorem 9 and 10 in [3] so that their conclusions are still true. More specifically, the condition "$-1 < b_k \le 0$" in Theorem 9 of [3] can be replaced by (3.7), and the condition "$b_k \ge 0$" in Theorem 10 of [3] can be replaced by "$b_k > -1$". We note that if (3.7) holds, by statements (i) and (ii) in Example 3.1, the impulsive system (3.3)–(3.4) has a nonoscillatory solution if, and only if, (3.5)–(3.6) has a nonoscillatory solution. This means that our results can lead to more complete results.

The following example shows that the condition in Theorem 2.9 that there is at least one $j \in \mathbb{N}_n$ such that $g_j(t) < t$ for $t \ge 0$ and the preimage $g_j^{-1}(0)$ does not contain an open interval, cannot be removed.

Example 3.2. We consider a "nonimpulsive" equation

$$x'(t) + \varepsilon p_1(t)x(g_1(t)) + \varepsilon p_2(t)x(g_2(t)) = 0 \tag{3.9}$$

where $\varepsilon \in (0, 1]$, $g_1(t) = t$, $g_2(t) = t - 1$, $p_1(t) = 1$ for $t \ge 0$ and

$$p_2(t) = \begin{cases} 2(1-t)e^{-t} & \text{if } 0 \le t \le 1, \\ 0 & \text{if } t > 1. \end{cases}$$

Clearly, there does not exist $j \in \mathbb{N}_2$ such that $g_j(t) < t$ for $t \ge 0$ and the preimage $g_j^{-1}(0)$ does not contain an open interval. Let

$$a(\varepsilon) = (3\varepsilon^2 - 2\varepsilon + 1)e^{-\varepsilon} - 2\varepsilon e^{-1}, \quad \varepsilon \in (0, 1).$$

First, we claim that $a(\varepsilon) > 0$ for $0 < \varepsilon < 1$. Indeed, we have

$$a'(\varepsilon) = (-3\varepsilon^2 + 8\varepsilon - 3)e^{-\varepsilon} - 2e^{-1} \quad \text{and} \quad a''(\varepsilon) = (3\varepsilon^2 - 14\varepsilon + 11)e^{-\varepsilon}$$

for $0 < \varepsilon < 1$. Since $a''(\varepsilon) > 0$ for $0 < \varepsilon < 1$ and $a'(1^-) = 0$, we see that $a'(\varepsilon) < 0$ for $0 < \varepsilon < 1$. It follows from $a(1^-) = 0$ that $a(\varepsilon) > 0$ for $0 < \varepsilon < 1$. Second, by straightforward computation, the continuous functions

$$x_\varepsilon(t) = \begin{cases} 1 & \text{if } t \in [-1, 0], \\ \dfrac{3\varepsilon^2 - 2\varepsilon + 1}{(\varepsilon-1)^2}e^{-\varepsilon t} - \left(\dfrac{2\varepsilon}{\varepsilon - 1}(t-1) + \dfrac{2\varepsilon}{(\varepsilon-1)^2}\right)e^{-t} & \text{if } t \in (0, 1], \\ \dfrac{a(\varepsilon)}{(\varepsilon-1)^2}e^{-\varepsilon t} & \text{if } t \in (1, \infty), \end{cases}$$

where $\varepsilon \in (0,1)$, and

$$x_{\varepsilon=1}(t) = \begin{cases} 1 & \text{if } t \in [-1,0], \\ e^{-t}(1-t)^2 & \text{if } t \in (0,1], \\ 0 & \text{if } t \in (1,\infty), \end{cases}$$

are solutions of (3.9) with initial condition $x_\varepsilon(t) = x_{\varepsilon=1}(t) = 1$ on $[-1,0]$. By our claim, we see that x_ε are positive for $0 < \varepsilon < 1$, but the solution x_1 of (3.9) (where $\varepsilon = 1$) is not eventually positive.

4. Discussions. One of the motivations of our study is to obtain results that can help us fill in the gap caused by some mistakes of [5], mentioned in the introduction. To see that such gaps exist, we need to explain briefly what details are involved and what our previous results can achieve.

The first occurs in the proofs of Lemmas 1 and 2 in [5]. More specifically, the authors used Theorem 3.2.1 of [8], but as will be seen below, the hypotheses of that theorem are quite restrictive and hence the proofs in [5] are not justified.

For ease of discussion, we first recall Theorem 3.2.1 of [8].

THEOREM 4.1 (see [8, Theorem 3.2.1]). *Suppose that p_i, q_i, r_i, τ_i are positive continuous functions on $[t_0, T)$ where $t_0 < T \le \infty$ and $i \in \mathbb{N}_n$, and that*

$$p_i(t) \ge q_i(t) \ge r_i(t), \qquad t_0 \le t < T \text{ and } i \in \mathbb{N}_n.$$

Assume that x, y and z are continuous solutions of

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t-\tau_i(t)) \le 0,$$

$$x'(t) + \sum_{i=1}^{n} q_i(t)x(t-\tau_i(t)) = 0,$$

and

$$x'(t) + \sum_{i=1}^{n} r_i(t)x(t-\tau_i(t)) \ge 0,$$

respectively, such that $x(t) > 0$ for $t_0 \le t < T$, $z(t_0) \ge y(t_0) \ge x(t_0)$,

$$\frac{x(t)}{x(t_0)} \ge \frac{y(t)}{y(t_0)} \ge \frac{z(t)}{z(t_0)} \ge 0, \qquad r_{t_0} \le t < t_0. \tag{4.1}$$

Then $z(t) \ge y(t) \ge x(t)$ for $t_0 \le t < T$.

We note that the initial functions in the hypotheses of Theorem 4.1 are assumed to be continuous. But the initial functions used in the proofs of Lemmas 1 and 2 in [5] may not. Therefore, it is not justified to cite Theorem 4.1. Besides, when the authors cite Theorem 4.1 to construct the solutions on each interval $(t_k, t_{k+1}]$, they seem to ignore the necessity to test whether the initial functions satisfy the hypotheses of Theorem 4.1 (in particular, condition (4.1) above). Furthermore, if we follow their proofs of Lemmas 1 and 2 in [5], we will find that Theorem 4.1 cannot be used in general. We will give an example to explain our observation. Consider the system

$$x'(t) + \frac{1}{2e}x(t-1) = 0, \qquad t \in [0,\infty) \setminus \Upsilon, \tag{4.2}$$

$$x(t_k^+) - x(t_k) = I_k(x(t_k)), \qquad k \in \mathbb{N}, \tag{4.3}$$

where $t_k = 0.5k$ and $I_k(\mu) = \operatorname{sgn}(\mu)\sqrt[4]{|\mu|}$ for $\mu \in \mathbb{R}$ and $k \in \mathbb{N}$. Then I_k is nondecreasing for each $k \in \mathbb{N}$. We know that the equation $x'(t)+e^{-1}x(t-1) = 0$ has a positive solution e^{-t}. By Corollary 2.7, there exists a positive solution x of

$$\begin{aligned} &x'(t) + \frac{1}{e}x(t-1) = 0, \qquad t \in [0,\infty) \setminus \Upsilon, \\ &x(t_k^+) - x(t_k) = I_k(x(t_k)), \qquad k \in \mathbb{N}, \end{aligned}$$

on $[1,\infty)$ with initial condition $x(t) = e^{-t}$ on $[0, 1]$. Then x is also a positive solution of

$$\begin{aligned} &x'(t) + \frac{1}{2e}x(t-1) \le 0, \qquad t \in [0,\infty) \setminus \Upsilon, \\ &x(t_k^+) - x(t_k) = I_k(x(t_k)), \qquad k \in \mathbb{N} \end{aligned}$$

on $[1,\infty)$ with initial condition $x(t) = e^{-t}$ on $[0, 1]$. Thus the hypotheses of Lemma 1 in [5] are satisfied. By the proof of Lemma 1 in [5], there would exist a solution y of (4.2)–(4.3) on $[1, 2]$ with initial condition $y(t) = e^{-t}$ on $[0, 1]$. By straightforward computations,

$$\begin{aligned} x(t) &= \frac{1}{e^t} + \frac{1}{\sqrt[4]{e}} - \frac{1}{e} && \text{on } (1, 1.5], \\ x(t) &= \frac{1}{e^t} + \sqrt[4]{x(1.5)} - \frac{1}{e^{1.5}} && \text{on } (1.5, 2], \end{aligned}$$

and

$$\begin{aligned} y(t) &= \frac{1}{2e^t} + \frac{1}{\sqrt[4]{e}} - \frac{1}{2e} && \text{on } (1, 1.5], \\ y(t) &= \frac{1}{2e^t} + \sqrt[4]{y(1.5)} - \frac{1}{2e^{1.5}} && \text{on } (1.5, 2]. \end{aligned}$$

Now if we were to use Theorem 4.1 to construct a solution on $(2, 2.5]$, we need to verify

$$\frac{y(t)}{y(2)} \le \frac{x(t)}{x(2)} \qquad \text{for } 1 \le t < 2.$$

But

$$\frac{y(1.5)}{y(2)} \approx 0.809 \geq 0.788 \approx \frac{x(1.5)}{x(2)},$$

so the proof of Lemma 1 in [5] is incomplete. Similarly, the proof of Lemma 2 in [5] is also incomplete.

Although Lemma 1 in [5] may still be true under the original hypotheses, our Theorem 2.4 can be used to obtain the same conclusion if the hypothesis "I_k is nondecreasing" of that lemma is replaced by "I_k satisfies (A4) and

$$\frac{I_k(\mu_2)}{\mu_2} \geq \frac{I_k(\mu_1)}{\mu_1} \quad \text{if } 0 < \mu_1 \leq \mu_2 \tag{4.4}$$

for all $k \in \mathbb{N}$". We may also use our Theorem 2.4 to verify Lemma 2 in [5] under the same hypotheses. Moreover, the hypothesis "I_k is nondecreasing" of Lemma 2 in [5] can be replaced by the general condition "$\mu I_k(\mu) > 0$ for $x \neq 0$".

Lemma 4.2 (cf. [5, Lemma 1]). *Assume that*

(1) $0 \leq t_0 < t_1 < t_2 < \cdots$ *are fixed points with* $\lim_{k\to\infty} t_k = \infty$;
(2) *for all* $i \in \mathbb{N}_n$, p_i *and* τ_i *are continuous functions from* $[t_0, \infty)$ *to* $\mathbb{R}^+$ *and* $\lim_{t\to\infty}\{t - \tau_i(t)\} = \infty$;
(3) *for all* $k \in \mathbb{N}$, I_k *satisfies* (A4) *and* (4.4) (*replacing the condition that* I_k *is not decreasing in Lemma* 1 *of* [5]).

If the system

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) \leq 0, \quad t \in [0,\infty) \setminus \Upsilon, \tag{4.5}$$

$$x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k \in \mathbb{N}, \tag{4.6}$$

has an eventually positive solution x, then the systems

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = 0, \quad t \in [0,\infty) \setminus \Upsilon, \tag{4.7}$$

$$x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k \in \mathbb{N}, \tag{4.8}$$

and

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) \geq 0, \quad t \in [0,\infty) \setminus \Upsilon, \tag{4.9}$$

$$x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k \in \mathbb{N}, \tag{4.10}$$

have eventually positive solutions y and z respectively, which satisfy $z(t) \geq y(t) \geq x(t)$ for all large t.

Proof. Let $p_i(t) = q_i(t)$, $g_i(t) = t - \tau_i(t)$, $F_i(\mu) = f_i(\mu) = \mu$ and $J_k(\mu) = I_k(\mu)$ where $t \geq t_0$, $\mu \in \mathbb{R}$, $i \in \mathbb{N}_n$ and $k \in \mathbb{N}$. Since x is eventually positive, there exists $\sigma > 0$ such that $x(t) > 0$ for $t \geq r_\sigma$. Let $\phi_1(t) = \phi_2(t) = x(t)$ on $[r_\sigma, \sigma]$. By Theorem 2.4, there exists a positive solution y of (4.7)–(4.8) with initial condition $y(t) = x(t)$ on $[r_\sigma, \sigma]$. Let $z(t) = y(t)$ for $[r_\sigma, \sigma]$. Then z is a positive solution of (4.9)–(4.10) with initial condition $z(t) = x(t)$ on $[r_\sigma, \sigma]$. ∎

LEMMA 4.3 (cf. [5, Lemma 2]). *Assume that the hypotheses* (1) *and* (2) *of Lemma* 4.2 *hold and that for* $k \in \mathbb{N}$, $\mu I_k(\mu) \geq 0$ *on* $\mathbb{R}\backslash\{0\}$ *and* $|I_k(\mu)| \leq b_k|\mu|$ *on* $\mathbb{R}$ *where* $b_k \geq 0$. *If system* (4.5)–(4.6) *has an eventually positive solution* x, *then the system*

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = 0, \qquad t \in [0, \infty) \setminus \Upsilon,$$
$$x(t_k^+) - x(t_k) = b_k x(t_k), \qquad k \in \mathbb{N},$$

has an eventually positive solution y *which satisfies* $y(t) \geq x(t)$ *for all large* t.

Proof. Let $p_i(t) = q_i(t)$, $g_i(t) = t - \tau_i(t)$, $F_i(\mu) = f_i(\mu) = \mu$, $\widetilde{I}_k(\mu) = b_k\mu$, and $\widetilde{J}_k(\mu) = I_k(\mu)$ where $t \geq t_0$, $\mu \in \mathbb{R}$, $i \in \mathbb{N}_n$ and $k \in \mathbb{N}$. Since $\mu I_k(\mu) \geq 0$ on $\mathbb{R} \setminus \{0\}$ and $|I_k(\mu)| \leq b_k|\mu|$ on $\mathbb{R}$ for each $k \in \mathbb{N}$, we see that

$$\widetilde{I}_k(0) = \widetilde{J}_k(0) = 0,$$
$$\mu^2 + \mu\widetilde{I}_k(\mu) = (1 + b_k)\mu^2 > 0,$$

and

$$\mu^2 + \mu\widetilde{J}_k(\mu) = \mu^2 + \mu I_k(\mu) > 0$$

for $\mu \neq 0$.

We further note that

$$\frac{\widetilde{J}_k(\mu_1)}{\mu_1} = \frac{I_k(\mu_1)}{\mu_1} \leq b_k = \frac{\widetilde{I}_k(\mu_2)}{\mu_2}, \qquad 0 < \mu_1 \leq \mu_2.$$

Since $x(t) > 0$ eventually, there exists $\sigma > 0$ such that $x(t) > 0$ for $t \geq r_\sigma$. By Theorem 2.4, there exists a positive solution y of

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i(t)) = 0, \qquad t \in [0, \infty) \setminus \Upsilon,$$
$$x(t_k^+) - x(t_k) = \widetilde{I}_k(x(t_k)), \qquad k \in \mathbb{N},$$

with initial condition $y(t) = x(t)$ on $[r_\sigma, \sigma]$, and $y(t) \geq x(t)$ for $t \geq r_\sigma$. ∎

We remark that once Lemmas 1 and 2 in [5] are replaced by our Lemmas 4.2 and 4.3, the subsequent results in [5] with corresponding modifications that employ Lemmas 1 and 2 will be correct. However, in the proof of

Theorem 2 in [5] (see lines 5 to 7 on page 1273 of [5]), it is stated that

$$x'(t) + \sum_{i=1}^{n} (1-\varepsilon) p_i(t) \prod_{\sigma \le t_k < t-\tau_i(t)} (1+b_k)^{-1} x(t-\tau_i(t)) = 0 \tag{4.11}$$

has an eventually positive solution and hence by Lemma 4 in [5],

$$x'(t) + \sum_{i=1}^{n} p_i(t) \prod_{\sigma \le t_k < t-\tau_i(t)} (1+b_k)^{-1} x(t-\tau_i(t)) = 0 \tag{4.12}$$

has an eventually positive solution. This statement is not justified. Before explaining this, let us state Lemma 4 in [5].

LEMMA 4.4 (see [5, Lemma 4]). *Assume p_i and τ_i are continuous functions from $[t_0, \infty)$ to $\mathbb{R}^+$ and $\lim_{t\to\infty}\{t-\tau_i(t)\} = \infty$ for all $i \in \mathbb{N}_n$. Then the following statements are equivalent.*

(i) *The equation*

$$x'(t) + \sum_{i=1}^{n} p_i(t) x(t-\tau_i(t)) = 0$$

has an eventually positive solution.

(ii) *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in [0, \varepsilon_0]$,*

$$x'(t) + \sum_{i=1}^{n} (1-\varepsilon) p_i(t) x(t-\tau_i(t)) = 0$$

has an eventually positive solution.

We note that in the above lemma, the number ε may equal zero and the coefficient functions are assumed to be continuous. But in (4.11), the number ε in [5] is required to be nonzero and the functions

$$p_i(t) \prod_{\sigma \le t_k < t-\tau_i(t)} (1+b_k)^{-1}, \qquad i \in \mathbb{N}_n,$$

are not continuous. So Lemma 4.4 cannot be applied. However, p_i and τ_i satisfy the hypotheses of our Theorem 2.9, and so it can be applied to yield the correct result. More specifically, if the notation $\mathbb{R}^+$ in [5] denotes the interval $(0, \infty)$ (not $[0, \infty)$), then the hypotheses of Theorem 2.9 hold and thus this problem is solved.

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