ANNALES
POLONICI MATHEMATICI
104.1 (2012)

Entire functions that share a function with their first and second derivatives

by Feng Lü (Qingdao) and Junfeng Xu (Jiangmen)

Abstract. Applying the normal family theory and the theory of complex differential equations, we obtain a uniqueness theorem for entire functions that share a function with their first and second derivative, which generalizes several related results of G. Jank, E. Mues & L. Volkmann (1986), C. M. Chang & M. L. Fang (2002) and I. Lahiri & G. K. Ghosh (2009).

1. Introduction and main results. The subject of sharing values between entire functions and their derivatives was first studied by Rubel and Yang [17]. They proved in 1977 that if a non-constant entire function f and its first derivative f' share two distinct finite numbers a, b CM, then $f = f'$. Since then, sharing value problems have been studied by many authors and a number of profound results have been obtained (see, e.g., [2, 8]).

In order to state our main results, we need the following concepts and definitions.

Definition. The *order* of a meromorphic function f is defined by

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r}.$$

Let f, g be two entire functions, and let α be a function or a constant. If $f-\alpha$ and $g-\alpha$ have the same zeros, then we say f and g *share* α *IM* and write $f(z)=\alpha(z) \Leftrightarrow g(z)=\alpha(z)$. Moreover, if, for all z, $f(z)-\alpha(z)=0$ implies $g(z)-\alpha(z)=0$ then we write $f(z)=\alpha(z) \Rightarrow g(z)=\alpha(z)$. In what follows, we assume that the reader is familiar with the basic notation and results of the Nevanlinna value distribution theory (see [20]).

In 1986, G. Jank, E. Mues and L. Volkmann [9] proved

2010 *Mathematics Subject Classification*: Primary 30D35; Secondary 30D45.
Key words and phrases: entire functions, Nevanlinna theory, uniqueness, normal family, differential equation.

DOI: 10.4064/ap104-1-6

Theorem A. *Let f be an entire function. If f and f' share a finite non-zero value a IM, and if $f''(z) = a$ whenever $f(z) = a$, then $f = f'$.*

In 2002, J. M. Chang and M. L. Fang [4] replaced the constant a by the function z in Theorem A and derived

Theorem B. *Let f be a non-constant entire function. If*

$$f(z) = z \Leftrightarrow f'(z) = z, \qquad f'(z) = z \Rightarrow f''(z) = z,$$

then $f = f'$.

In 2003, J. M. Chang [3] improved Theorem B and proved

Theorem C. *Let f be a non-constant entire function and α be a meromorphic function satisfying $T(r,\alpha) = S(r,f)$ and $\alpha \not\equiv \alpha'$. If*

$$f(z) = \alpha \Leftrightarrow f'(z) = \alpha, \qquad f'(z) = \alpha \Rightarrow f''(z) = \alpha,$$

then $f = f'$.

Recently, I. Lahiri and G. K. Ghosh [10] extended Theorem B in another direction, replacing the function z by a polynomial of degree 1:

Theorem D. *Let f be a non-constant entire function and $a = \alpha z + \beta$, where α $(\neq 0)$ and β are constants. If*

$$f(z) = a \Rightarrow f'(z) = a, \qquad f'(z) = a \Rightarrow f''(z) = a,$$

then either $f(z) = A\exp\{z\}$ or

$$f(z) = \alpha z + \beta + (\alpha z + \beta - 2\alpha)\exp\left\{\frac{\alpha z + \beta - 2\alpha}{\alpha}\right\}.$$

In 2010, F. Lü and H. X. Yi [14] obtained a similar result:

Theorem E. *Let f be a non-constant transcendental meromorphic function with finitely many poles, and let R be a non-zero rational function. If*

$$f(z) = R(z) \Rightarrow f'(z) = R(z), \qquad f'(z) = R(z) \Rightarrow f''(z) = R(z),$$

then $f = f'$ or $f'(z) = A[R(z) - R'(z)]e^z + R'(z)$, where A is a non-zero constant.

It is natural to ask whether the conditions of Theorems D and E can be weakened or not. In this work, we derive the following result.

Theorem 1.1. *Let f be a non-constant entire function, and let $\alpha = Pe^Q$ $(\alpha \not\equiv \alpha')$ be an entire function satisfying $\rho(\alpha) < \rho(f)$, where P $(\not\equiv 0)$ and Q are polynomials. If $f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z)$ and $f'(z) = \alpha(z) \Rightarrow f''(z) = \alpha(z)$, then one of the following cases holds:*

(a) *$f = f'$;*

(b) *$f'(z) = A[\alpha(z) - \alpha'(z)]e^z + \alpha'(z)$ and α reduces to a polynomial, where A is a non-zero constant.*

Remark 1. The condition $\rho(\alpha) < \rho(f)$ plays an important part in the proof of Theorem 1.1. But we do not know whether it is necessary or not.

Remark 2. By a refined calculation, we can deduce that case (b) in Theorem 1.1 cannot occur if $\deg P \leq 2$. This will be proved in the last section. But, if $\deg P \geq 3$, case (b) cannot be deleted, as shown by the following example.

Example 1. Let $\alpha(z) = z^3 + 6z^2 + 12z + 12$ and $f(z) = z^3Ae^z + z^3 + 6z^2 + 12z + 12$, where $A = e^3$ is a constant. Differentiating f twice yields

$$f'(z) = (z^3+3z^2)Ae^z+3z^2+12z+12, \quad f''(z) = (z^3+6z^2+6z)Ae^z+6z+12.$$

It is not difficult to deduce that

$$f(z) - \alpha(z) = 0 \Rightarrow f'(z) - \alpha(z) = 0, \quad f'(z) - \alpha(z) = 0 \Rightarrow f''(z) - \alpha(z) = 0.$$

Thus, case (b) occurs.

The following corollary is an immediate consequence of Theorem 1.1 and Remark 2.

Corollary 1.2. *Let f be a transcendental entire function, and let P $(\neq 0)$ be a polynomial with* $\deg P \leq 2$. *If*

$$f(z) = P(z) \Rightarrow f'(z) = P(z), \qquad f'(z) = P(z) \Rightarrow f''(z) = P(z),$$

then $f = f'$.

Remark 3. The following example shows that the assumption in Corollary 1.2 that f is a transcendental entire function is necessary.

Example 2. Let $f(z) = 2z^2-4z+4$ and $P(z) = z^2$. Then $f(z)-P(z) = (z-2)^2$, $f'(z)-P(z) = -(z-2)^2$ and $f''(z)-P(z) = (2-z)(2+z)$. It is easy to see that $f(z) = P(z) \Rightarrow f'(z) = P(z)$ and $f'(z) = P(z) \Rightarrow f''(z) = P(z)$, but $f \neq f'$.

In the proof of Theorem 1.1, we need that f is of finite order. Therefore, we will first prove it. In fact, using the theory of normal families we will obtain the following result of independent interest.

Theorem 1.3. *Let f be a non-constant entire function, and let $\alpha = Pe^Q$ $(\alpha \neq \alpha')$ where P $(\neq 0)$ and Q are polynomials. If $f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z)$ and $f'(z) = \alpha(z) \Rightarrow f''(z) = \alpha(z)$, then f is of finite order.*

Remark 4. With a similar analysis, if the first derivative f' is replaced by the kth derivative $f^{(k)}$, then Theorem 1.3 still holds.

Remark 5. The proof of Theorem 1.1 is based on [4] and [19]. The proof of Theorem 1.3 is based on [7] and [12].

2. Some lemmas. In the proofs of our main results, we need some key lemmas, recalled below for the convenience of the reader.

Using the ideas of [12, Lemma 1] and the famous Pang–Zalcman Lemma [16], F. Lü, J. F. Xu and A. Chen [13] obtained the following result, which plays an important part in the proof of Theorem 1.3.

LEMMA 2.1 ([13]). *Let $\{f_n\}$ be a family of functions meromorphic (resp. analytic) on the unit disc $\triangle$. If $a_n \to a$, $|a| < 1$, $f_n^\sharp(a_n) \to \infty$, and if there exists $A \geq 1$ such that $|f'(z)| \leq A$ whenever $f(z) = 0$, then there exist*

(a) *a subsequence of f_n (still denoted $\{f_n\}$),*
(b) *points $z_n \to z_0$, $|z_0| < 1$,*
(c) *positive numbers $\rho_n \to 0$,*

such that $\rho_n^{-1} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where g is a non-constant meromorphic (resp. entire) function on $\mathbb{C}$ such that $\rho(g) \leq 2$ (resp. $\rho(g) \leq 1$), $g^\sharp(\xi) \leq g^\sharp(0) = A + 1$ and

$$\rho_n \leq \frac{M}{f_n^\sharp(a_n)},$$

where M is a constant which is independent of n.

Here, as usual, $g^\sharp(\xi) = |g'(\xi)|/(1 + |g(\xi)|^2)$ is the spherical derivative.

LEMMA 2.2 ([12]). *Let f be a meromorphic function of infinite order on $\mathbb{C}$. Then there exist points $z_n \to \infty$ such that for every $N > 0$, $f^\sharp(z_n) > |z_n|^N$ if n is sufficiently large.*

LEMMA 2.3 ([5]). *Let g be a non-constant entire function with order $\rho(g) \leq 1$, let $k \geq 2$ be an integer, and let a be a non-zero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$ and $g'(z) = a \Rightarrow g^{(k)}(z) = 0$, then $g(z) = a(z - z_0)$, where z_0 is a constant.*

LEMMA 2.4 ([20]). *Let f be an entire function of finite order and k be a positive integer. Then*

$$m(r, f^{(k)}/f) = O(\log r) \quad \text{as } r \to \infty.$$

We also need a result from the theory of differential equations. First, we give a definition and a notation.

Consider a rational function R which behaves asymptotically as cr^β as $r \to \infty$, where $c \neq 0$, β are constants. Define the *degree* of R *at infinity* as $\deg_\infty R = \max\{0, \beta\}$.

We consider the linear differential equation

$$f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f = 0, \quad a_0 \neq 0, \tag{2.1}$$

where $a_0, a_1, \ldots, a_{n-1}$ are rational functions.

The following lemma is essential to the proof of Theorem 1.1.

Lemma 2.5 ([11]). *Let f be a meromorphic solution of* (2.1), *and let α_j denote the degree of a_j at infinity, $j = 0, 1, \ldots, n-1$. Then*

$$\rho(f) \le 1 + \max_{j=0,1,\ldots,n-1} \frac{\alpha_j}{n-j}.$$

Lemma 2.6. *Let f and α be meromorphic functions with $\rho(\alpha) < \rho(f)$. Then there exists a set $I = \{r_n\}_{n=1}^{\infty}$ such that $r_n \to \infty$ and $T(r_n, \alpha) = o(T(r_n, f))$ as $n \to \infty$.*

Proof. By the definition of the order, for any $\varepsilon > 0$, there exists a set $I = \{r_n\}_{n=1}^{\infty}$ ($r_n \to \infty$ as $n \to \infty$) satisfying

$$T(r_n, \alpha) \le r_n^{\rho(\alpha)+\varepsilon}, \qquad T(r_n, f) \ge r_n^{\rho(f)-\varepsilon}.$$

Take $0 < \varepsilon < (\rho(f) - \rho(\alpha))/2$, that is, $\rho(\alpha) - \rho(f) + 2\varepsilon < 0$. Then

$$\lim_{n\to\infty} \frac{T(r_n, \alpha)}{T(r_n, f)} \le \lim_{n\to\infty} \frac{r_n^{\rho(\alpha)+\varepsilon}}{r_n^{\rho(f)-\varepsilon}} \le \lim_{n\to\infty} r_n^{\rho(\alpha)-\rho(f)+2\varepsilon} = 0,$$

which implies that $T(r_n, \alpha) = o(T(r_n, f))$ as $n \to \infty$.

In the case of Lemma 2.6, we say that α is a *small function* of f on I and write $T(r, \alpha) = S(r, f)$ ($r \in I$).

3. Proof of Theorem 1.3. In the proof, we use some ideas of [7]. For the convenience of the reader, we present the proof in detail.

Let $H = f - \alpha$. Then we have

$$(1)\ H = 0 \Rightarrow H' = \alpha - \alpha', \qquad (2)\ H' = \alpha - \alpha' \Rightarrow H'' = \alpha - \alpha''.$$

Put $\beta = \alpha - \alpha' = P_1 e^Q$ and $\gamma = \alpha - \alpha'' = P_2 e^Q$, where P_1 ($\ne 0$) and P_2 are polynomials.

Define $F = H/\beta$. We distinguish two cases.

Case 1: *F is of finite order.* Then $f = F\beta + \alpha$ is of finite order as well.

Case 2: *F is of infinite order.* By Lemma 2.2, there exist $w_n \to \infty$ such that for every $N > 0$, if n is sufficiently large,

$$F^{\sharp}(w_n) > |w_n|^N. \tag{3.1}$$

Next, we construct a family of holomorphic functions.

Obviously, $\beta = P_1 e^Q$ has only finitely many zeros, so there exists $r > 0$ such that $F(z)$ is analytic in $D = \{z : |z| \ge r\}$. Since $w_n \to \infty$ as $n \to \infty$, we may assume $|w_n| \ge r + 1$ for all n. Define $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) = F(w_n + z) = \frac{H(w_n + z)}{\beta(w_n + z)}.$$

Noting that $|w_n| \ge r + 1$ for all n, we have, for each $z \in D_1$,

$$|w_n + z| \ge |w_n| - |z| \ge r,$$

so $w_n + z \in D$ for each $z \in D_1$. As $F(z)$ is analytic in D, $F_n(z) = F(w_n + z)$ is analytic in D_1. Thus, we have constructed a family $(F_n)_n$ of holomorphic functions.

Now, fix $z \in D_1$. If $F_n(z) = 0$, then $H(w_n + z) = 0$. It is clear from assumption (1) that $H'(w_n + z) = \beta(w_n + z)$. Hence (for n large enough)

$$|F_n'(z)| = \left| \frac{H'(w_n + z)}{\beta(w_n + z)} - \frac{H(w_n + z)}{\beta(w_n + z)} \frac{\beta'(w_n + z)}{\beta(w_n + z)} \right| = 1. \tag{3.2}$$

In what follows, we prove that $(F_n)_n$ is normal at $z = 0$.

Otherwise, by Lemma 2.1, passing to an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequences $(z_n)_n$ and $(\rho_n)_n$ such that $|z_n| < r < 1$, $\rho_n \to 0$ and

$$g_n(\zeta) = \rho_n^{-1} F_n(z_n + \rho_n \zeta) = \rho_n^{-1} \frac{H(w_n + z_n + \rho_n \zeta)}{\beta(w_n + z_n + \rho_n \zeta)} \to g(\zeta) \tag{3.3}$$

locally uniformly in $\mathbb{C}$, where g is a non-constant entire function of order at most 1. Moreover, $g^\sharp(\zeta) \le g^\sharp(0) = 2$ for all $\zeta \in \mathbb{C}$ and

$$\rho_n \le \frac{M}{F_n^\sharp(0)} = \frac{M}{F^\sharp(w_n)} \tag{3.4}$$

for a positive number M. From (3.1) and (3.4), we deduce that, for every $N > 0$, if n is sufficiently large,

$$\rho_n \le M |w_n|^{-N}. \tag{3.5}$$

Differentiating (3.3), we have

$$\begin{aligned} g_n'(\zeta) &= \frac{H'(w_n + z_n + \rho_n \zeta)}{\beta(w_n + z_n + \rho_n \zeta)} - \frac{H(w_n + z_n + \rho_n \zeta)}{\beta(w_n + z_n + \rho_n \zeta)} \frac{\beta'(w_n + z_n + \rho_n \zeta)}{\beta(w_n + z_n + \rho_n \zeta)} \\ &= \frac{H'(w_n + z_n + \rho_n \zeta)}{\beta(w_n + z_n + \rho_n \zeta)} - \rho_n g_n(\zeta) \frac{\beta'(w_n + z_n + \rho_n \zeta)}{\beta(w_n + z_n + \rho_n \zeta)} \to g'(\zeta). \end{aligned} \tag{3.6}$$

From (3.5), we deduce that

$$\rho_n g_n(\zeta) \frac{\beta'(w_n + z_n + \rho_n \zeta)}{\beta(w_n + z_n + \rho_n \zeta)} = \rho_n g_n(\zeta) \frac{P_3(w_n + z_n + \rho_n \zeta)}{P_1(w_n + z_n + \rho_n \zeta)} \to 0, \tag{3.7}$$

where P_3 is a polynomial.

Combining (3.6) and (3.7) yields

$$\frac{H'(w_n + z_n + \rho_n \zeta)}{\beta(w_n + z_n + \rho_n \zeta)} \to g'(\zeta). \tag{3.8}$$

In a similar way, we can obtain

$$\rho_n \frac{H''(w_n + z_n + \rho_n \zeta)}{\beta(w_n + z_n + \rho_n \zeta)} \to g''(\zeta). \tag{3.9}$$

In the following, we will prove:

(I) $g(\zeta) = 0 \Rightarrow g'(\zeta) = 1$,
(II) $g'(\zeta) = 1 \Rightarrow g''(\zeta) = 0$.

For (I), suppose that $g(\zeta_0) = 0$. Then by Hurwitz's theorem and (3.3), there exist $\zeta_n \to \zeta_0$ such that (for n sufficiently large)

$$g_n(\zeta_n) = \rho_n^{-1} \frac{H(w_n + z_n + \rho_n \zeta_n)}{\beta(w_n + z_n + \rho_n \zeta_n)} = 0.$$

Thus $H(w_n + z_n + \rho_n \zeta_n) = 0$ and

$$H'(w_n + z_n + \rho_n \zeta_n) = \beta(w_n + z_n + \rho_n \zeta_n).$$

By (3.8), we derive that

$$g'(\zeta_0) = \lim_{n\to\infty} \frac{H'(w_n + z_n + \rho_n \zeta_n)}{\beta(w_n + z_n + \rho_n \zeta_n)} = 1,$$

which implies that $g(\zeta) = 0 \Rightarrow g'(\zeta) = 0$.

To prove (II), suppose that $g'(\eta_0) = 1$. We know $g' \not\equiv 1$, since otherwise $g^\sharp(0) \le 1 < 2$, a contradiction. Hence by (3.8) and Hurwitz's theorem, there exist $\eta_n \to \eta_0$ such that (for n sufficiently large)

$$H'(w_n + z_n + \rho_n \eta_n) = \beta(w_n + z_n + \rho_n \eta_n).$$

It is obvious from (2) that $H''(w_n + z_n + \rho_n \eta_n) = \gamma(w_n + z_n + \rho_n \eta_n)$. Then

$$\begin{aligned} g''(\eta_0) &= \lim_{n\to\infty} \rho_n \frac{H''(w_n + z_n + \rho_n \eta_n)}{\beta(w_n + z_n + \rho_n \eta_n)} = \lim_{n\to\infty} \rho_n \frac{\gamma(w_n + z_n + \rho_n \eta_n)}{\beta(w_n + z_n + \rho_n \eta_n)} \\ &= \lim_{n\to\infty} \rho_n \frac{P_2(w_n + z_n + \rho_n \eta_n)}{P_1(w_n + z_n + \rho_n \eta_n)} = 0, \end{aligned}$$

which yields (II).

From Lemma 2.3, it is easy to deduce that $g(\zeta) = \zeta - b_0$, where b_0 is a constant; then $g^\sharp(0) \le 1 < 2$, which is also a contradiction.

All the foregoing discussion shows that $(F_n)_n$ is normal at $z = 0$.

On the other hand, it follows from $F_n^\sharp(0) = F^\sharp(w_n) \to \infty$ as $n \to \infty$ and Marty's criterion that $(F_n)_n$ is not normal at $z = 0$, a contradiction. Hence, Case 2 cannot occur.

This completes the proof of Theorem 1.3.

4. Proof of Theorem 1.1. If $\deg Q = 0$, then α reduces to a polynomial. Therefore, by Theorem E, we obtain the desired result.

In the following, we suppose that $\deg Q \ge 1$.

From Theorem 1.3, we know that f is of finite order. Let $\beta = \alpha - \alpha'$ and $F = f - \alpha$. By assumption, we have

(I) $F(z) = 0 \Rightarrow F'(z) = \beta(z)$, (II) $F'(z) = \beta(z) \Rightarrow F''(z) = \beta(z) + \beta'(z)$.

Put

$$\phi = \frac{\beta F'' - (\beta+\beta')F'}{F}. \tag{4.1}$$

It follows from Lemma 2.6 that α, β are small functions of f and F on I, where $I = \{r_n\}_{n=1}^{\infty}$ is as in Lemma 2.6.

In the following, we assume that $r \in I$. If $T(r,g) = o(T(r,f))$ on I, for brevity we omit I and just say that g is a small function of f and $T(r,g) = S(r,f)$.

If $\phi = 0$, then $\beta F'' - (\beta+\beta')F' = 0$. Integrating this yields

$$F'(z) = A\beta(z)e^z = A(\alpha(z)-\alpha'(z))e^z = A(P(z)-P(z)Q'(z)-P'(z))e^{Q(z)+z},$$

where A is a non-zero constant. From the form of F', we deduce that

$$\deg Q = \rho(\alpha) < \rho(f) = \rho(F) = \rho(F') = \deg(Q(z)+z), \tag{4.2}$$

which implies that Q is a constant, a contradiction.

Now suppose that $\phi \not\equiv 0$. By the lemma of logarithmic derivative, we have $m(r,\phi) = S(r,F)$. From assumption (II), it is easy to deduce that the simple zeros of F are not poles of ϕ. And by (I), F has only finitely many multiple zeros, that is, $N_{(2}(r,1/F) = O(\log r) = S(r,F)$. Noting that all poles of ϕ come from zeros of F, from the above discussion we get $N(r,\phi) \le N_{(2}(r,1/F) = S(r,F)$. Thus, $T(r,\phi) = m(r,\phi) + N(r,\phi) = S(r,F)$, which means that ϕ is a small function of F.

Rewrite (4.1) as

$$F = \frac{\beta}{\phi}F'' - \frac{\beta+\beta'}{\phi}F'. \tag{4.3}$$

By differentiating (4.3), we have

$$F' = \left(\frac{\beta}{\phi}\right)'F'' + \frac{\beta}{\phi}F''' - \left(\frac{\beta+\beta'}{\phi}\right)'F' - \frac{\beta+\beta'}{\phi}F'', \tag{4.4}$$

which implies that

$$\left[1 + \left(\frac{\beta+\beta'}{\phi}\right)'\right]F' = \left[\left(\frac{\beta}{\phi}\right)' - \frac{\beta+\beta'}{\phi}\right]F'' + \frac{\beta}{\phi}F'''. \tag{4.5}$$

First, we assume that $1 + \left(\frac{\beta+\beta'}{\phi}\right)' = 0$. Then the above equation implies

$$\left[\left(\frac{\beta}{\phi}\right)' - \frac{\beta+\beta'}{\phi}\right]F'' + \frac{\beta}{\phi}F''' = 0.$$

Rewrite this as

$$\frac{F'''}{F''} = 1 + \frac{\beta'}{\beta} - \frac{\left(\frac{\beta}{\phi}\right)'}{\frac{\beta}{\phi}}. \tag{4.6}$$

By integrating, we derive that

$$F'' = B\phi e^z,$$

where B is a non-zero constant. Noting that $\deg Q \geq 1$, we have $\rho(e^z) \leq \rho(\beta) < \rho(f) = \rho(F)$. Thus, by Lemma 2.6, e^z is a small function of f and F, that is, $T(r, e^z) = S(r, F)$. Then, it follows from the form of F'' that

$$T(r, F'') \leq T(r, e^z) + T(r, \phi) + S(r, F) = S(r, F) = S(r, F''),$$

a contradiction.

Next, we assume that $1 + \left(\frac{\beta+\beta'}{\phi}\right)' \neq 0$. Rewrite (4.5) as

$$\left[1 + \left(\frac{\beta+\beta'}{\phi}\right)'\right][F' - \beta] + \left[1 + \left(\frac{\beta+\beta'}{\phi}\right)'\right]\beta$$
$$= \left[\left(\frac{\beta}{\phi}\right)' - \frac{\beta+\beta'}{\phi}\right][F'' - \beta'] + \left[\left(\frac{\beta}{\phi}\right)' - \frac{\beta+\beta'}{\phi}\right]\beta' + \frac{\beta}{\phi}[F''' - \beta''] + \frac{\beta}{\phi}\beta''.$$

Define

$$A_1 = 1 + \left(\frac{\beta+\beta'}{\phi}\right)', \quad A_2 = \left(\frac{\beta}{\phi}\right)' - \frac{\beta+\beta'}{\phi},$$
$$A_3 = \frac{\beta}{\phi}, \quad A_4 = \left[1 + \left(\frac{\beta+\beta'}{\phi}\right)'\right]\beta - \left[\left(\frac{\beta}{\phi}\right)' - \frac{\beta+\beta'}{\phi}\right]\beta' - \frac{\beta}{\phi}\beta''.$$

Obviously, A_i $(i = 1, \ldots, 4)$ are small functions of F. Then we can rewrite the above equation as

$$A_4 = A_2[F'' - \beta'] + A_3[F''' - \beta''] - A_1[F' - \beta]. \tag{4.7}$$

We consider two cases.

Case 1: $A_4 = 0$. A routine calculation leads to

$$2\beta'\phi + \phi^2 - \beta\phi' = 0.$$

Furthermore, we have $\left(\frac{\beta^2}{\phi}\right)' = -\beta$.

Put $K' = \beta$; then $K'' = \beta'$, $K''' = \beta''$, where K is a primitive function of β. Thus,

$$\phi = -\frac{K'^2}{K}. \tag{4.8}$$

Observing that $K' = \beta = \alpha - \alpha' = P_1 e^Q$, where P_1 is a polynomial, we deduce that $K = P_2 e^Q + C$, where P_2 is a polynomial and C is a constant. We claim that $C = 0$. Indeed, assume $C \neq 0$. We have

$$-\frac{K'^2}{K} = \phi = \frac{K'F'' - (K' + K'')F'}{F}. \tag{4.9}$$

Thus, by the left side of (4.9),

$$(4.10)\quad T(r,\phi)=T\left(r,-\frac{K'^2}{K}\right)=T\left(r,-\frac{(P_1)^2e^{2Q}}{P_2e^Q+C}\right)=2T(r,e^Q)+S(r,e^Q),$$

while by the right side of (4.9) and Lemma 2.4,

(4.11)

$$\begin{aligned}T(r,\phi)&=m\left(r,\frac{K'F''-(K'+K'')F'}{F}\right)+N\left(r,\frac{K'F''-(K'+K'')F'}{F}\right)\\&=m\left(r,\frac{K'[F''-(1+\frac{K''}{K'})F']}{F}\right)+N(r,\phi)\\&\le m\left(r,\frac{K'[F''-(1+\frac{K''}{K'})F']}{F}\right)+O(\log r)\\&\le m(r,K')+m\left(r,\frac{F''}{F}\right)+m\left(r,\frac{F'}{F}\right)+m\left(r,1+\frac{K''}{K'}\right)+O(\log r)\\&=T(r,e^Q)+O(\log r).\end{aligned}$$

Comparing (4.10) and (4.11), we have $T(r,e^Q)\le S(r,e^Q)+O(\log r)$, a contradiction.

Thus, the claim is true: $K=P_2e^Q$. It is easy to deduce that $K''=P_3e^Q$, where P_3 is a polynomial. Furthermore,

$$(4.12)\quad \deg(P_1)=\deg(P_2)+\deg(Q'),\qquad \deg(P_3)=\deg(P_2)+2\deg(Q').$$

From (4.9), we derive that

$$(4.13)\qquad F''+R_1F'+R_2F=0,$$

where

$$R_1=-\left(1+\frac{K''}{K'}\right)=-\left(1+\frac{P_3}{P_1}\right),\qquad R_2=\frac{K'}{K}=\frac{P_1}{P_2}$$

are rational functions with $\deg(R_1)=\deg(R_2)=\deg(Q')$.

It follows from Lemma 2.5 that

$$\begin{aligned}\rho(f)&\le 1+\max\{\deg(R_1),\deg(R_2)/2\}=1+\deg(Q')\\&=\deg(Q)=\rho(\alpha)<\rho(f),\end{aligned}$$

a contradiction. Thus, this case is impossible.

Case 2. $A_4\ne 0$. Then

$$(4.14)\qquad \frac{A_4}{F'-\beta}=A_2\frac{F''-\beta'}{F'-\beta}+A_3\frac{F'''-\beta''}{F'-\beta}-A_1.$$

Thus, by the lemma of logarithmic derivative, we obtain

$$(4.15)\qquad m\left(r,\frac{1}{F'-\beta}\right)\le m\left(r,\frac{A_4}{F'-\beta}\right)+m\left(r,\frac{1}{A_4}\right)$$
$$\le m\left(r,A_2\frac{F''-\beta'}{F'-\beta}+A_3\frac{F'''-\beta''}{F'-\beta}-A_1\right)+S(r,F)\le S(r,F).$$

Then

$$(4.16)\qquad N\left(r,\frac{1}{F'-\beta}\right)=T(r,F'-\beta)+S(r,F)=T(r,F')+S(r,F).$$

Next we will prove $N\big(r,\frac{1}{F'-\beta}\big)=N\big(r,\frac{1}{F}\big)+S(r,F)$.

Denote by $N(r,\beta;F'\mid F\neq 0)$ the counting function of those 0-points of $F'-\beta$, counted with multiplicity, which are not 0-points of F; and denote by $N(r,\beta;F'\mid F=0)$ the counting function of the remaining 0-points of $F'-\beta$.

Suppose z_0 is a zero of $F'-\beta$ of multiplicity m, and not a zero of F. By (4.1), z_0 is also a zero of ϕ. Moreover, it follows from the fact $F'=\beta\Rightarrow F''=\beta-\beta'$ that $F'-\beta$ has finitely many multiple zeros, which means $N_{(2}\big(r,\frac{1}{F'-\beta}\big)=O(\log\ r)=S(r,F)$. Therefore,

$$(4.17)\qquad N(r,\beta;F'\mid F\neq 0)\le N\left(r,\frac{1}{\phi}\right)+N_{(2}\left(r,\frac{1}{F'-\beta}\right)=S(r,F).$$

Furthermore, by (II), we have

$$(4.18)\qquad N\left(r,\frac{1}{F'-\beta}\right)=N(r,\beta;F'\mid F\neq 0)+N(r,\beta;F'\mid F=0)$$
$$\le N\left(r,\frac{1}{F}\right)+N_{(2}\left(r,\frac{1}{F'-\beta}\right)+S(r,F)$$
$$=N\left(r,\frac{1}{F}\right)+S(r,F).$$

On the other hand, from (I), we obtain $N_{(2}(r,1/F)=O(\log r)=S(r,F)$. Moreover, (I) implies

$$(4.19)\qquad N\left(r,\frac{1}{F}\right)\le N\left(r,\frac{1}{F'-\beta}\right)+N_{(2}\left(r,\frac{1}{F}\right)=N\left(r,\frac{1}{F'-\beta}\right)+S(r,F).$$

Combining (4.18) and (4.19) yields

$$N\left(r,\frac{1}{F'-\beta}\right)=N\left(r,\frac{1}{F}\right)+S(r,F).$$

as desired.

Rewrite (4.1) as

$$(4.20)\qquad F=\frac{\beta F''-(\beta+\beta')F'}{\phi}.$$

Then

$$(4.21)\quad \begin{aligned} T(r,F) = m(r,F) &= m\left(r, \frac{\beta F'' - (\beta+\beta')F'}{\phi}\right) \\ &= m\left(r, \frac{F'[\beta\frac{F''}{F'} - (\beta+\beta')]}{\phi}\right) \\ &\le m(r,F') + S\left(r, \frac{\beta}{\phi}\frac{F''}{F'}\right) + m\left(r, \frac{\beta+\beta'}{\phi}\right) + O(1) \\ &= m(r,F') + S(r,F) = T(r,F') + S(r,F) \le T(r,F) + S(r,F), \end{aligned}$$

which implies that

$$(4.22)\qquad T(r,F') = T(r,F) + S(r,F).$$

Furthermore, the above discussion yields

$$(4.23)\quad \begin{aligned} N\left(r,\frac{1}{F}\right) + m\left(r,\frac{1}{F}\right) &= T(r,F) + S(r,F) = T(r,F') + S(r,F) \\ &= T(r,F'-\beta) + S(r,F) = T\left(r,\frac{1}{F'-\beta}\right) + S(r,F) \\ &= m\left(r,\frac{1}{F'-\beta}\right) + N\left(r,\frac{1}{F'-\beta}\right) + S(r,F) \\ &= N\left(r,\frac{1}{F'-\beta}\right) + S(r,F) = N\left(r,\frac{1}{F}\right) + S(r,F), \end{aligned}$$

which indicates that $m(r,1/F) = S(r,F)$.

Define

$$(4.24)\qquad \varphi = \frac{F'-\beta}{F}.$$

If $\varphi = 0$, then $F' = \beta$, a contradiction. Thus, $\varphi \ne 0$. By (I) and the lemma of logarithmic derivative, it is easy to see that $N(r,\varphi) = S(r,F)$ and $m(r,\varphi) \le m(r,F'/F) + m(r,\beta) + m(r,1/F) + O(1) = S(r,F)$. Thus,

$$(4.25)\qquad T(r,\varphi) = m(r,\varphi) + N(r,\varphi) = S(r,F).$$

Rewrite (4.24) as

$$(4.26)\qquad F' = \varphi F + \beta.$$

By differentiating (4.26), we have

$$(4.27)\qquad F'' = \varphi' F + \varphi F' + \beta' = (\varphi' + \varphi^2)F + \beta' + \varphi\beta.$$

Assume that c_0 is a zero of F, hence of $F'' - (\beta+\beta')$. Substituting c_0 into (4.27) yields $\beta(c_0)(1-\varphi(c_0)) = 0$.

If $\beta(1-\varphi)\neq 0$, then by (4.16), we derive that

$$T(r,F') = N\left(r,\frac{1}{F'-\beta}\right) + S(r,F) = N\left(r,\frac{1}{F}\right) + S(r,F)$$
$$\leq N\left(r,\frac{1}{\beta(1-\varphi)}\right) + S(r,F) = T(r,\beta(1-\varphi)) + S(r,F) = S(r,F),$$

a contradiction. Hence $\beta(1-\varphi)=0$, so obviously $1=\varphi$. Thus, from (4.26), we have $F'-\beta=F$, that is, $f=f'$.

This completes the proof of the theorem.

5. Supplement to Theorem 1.1. In Remark 2, we claim that if $\deg P\leq 2$, then case (b) cannot occur. Indeed, suppose that it can. Let $\beta=\alpha-\alpha'$ and $F=f-\alpha$. Noting that α reduces to a polynomial in case (b), we have $\deg\beta=\deg\alpha=\deg P$. By assumption,

(I) $F(z)=0\Rightarrow F'(z)=\beta(z)$, (II) $F'(z)=\beta(z)\Rightarrow F''(z)=\beta(z)+\beta'(z)$.

From case (b),

$$F'(z) = A\beta(z)e^z, \tag{5.1}$$

where A is a non-zero constant. Integrating (5.1) yields

$$F(z) = A\kappa(z)e^z + c, \tag{5.2}$$

where κ is a polynomial with

$$\deg\kappa = \deg P \quad \text{and} \quad \kappa+\kappa'=\beta. \tag{5.3}$$

Suppose that $c\neq 0$. Then, from (I), we have

$$A\kappa(z)e^z + c = 0 \Rightarrow \beta(z)(Ae^z-1)=0,$$

a contradiction. Thus $c=0$ and

$$F(z) = A\kappa(z)e^z. \tag{5.4}$$

Differentiating (5.4) twice yields

$$F'(z) = A[\kappa(z)+\kappa'(z)]e^z = A\beta(z)e^z, \tag{5.5}$$

$$F''(z) = A[\kappa(z)+2\kappa'(z)+\kappa''(z)]e^z = A[\beta(z)+\beta'(z)]e^z. \tag{5.6}$$

We consider three cases.

Case 1: $\deg P=0$. Then α is a constant and it follows from Theorem A that $f=f'$.

Case 2: $\deg P=1$. Then $\deg\kappa=\deg P=1$. Assume that $\kappa(z)=Bz+C$, where $B\neq 0$ and C are constants. By (5.3), we have $\beta(z)=Bz+B+C$. Substituting $\kappa(z)=Bz+C$ into (5.5) and (5.6) yields

$$F'(z) = A[Bz+B+C]e^z, \tag{5.7}$$

$$F''(z) = A[Bz+2B+C]e^z. \tag{5.8}$$

Observing that $z = -C/B$ is a zero of F and (I), we deduce that $z = -C/B$ is also a zero of $F' - \beta$. Putting $z = -C/B$ into $F' - \beta = 0$, we deduce that $Ae^{-C/B} = 1$. Similarly, $z = -(B+C)/B$ is a zero of $F' - \beta$ and $F'' - (\beta + \beta')$. Putting $z = -\frac{B+C}{B}$ into $F'' - (\beta + \beta')$, we obtain $Ae^{-1-C/B} = 1$. By the two formulas, we deduce that $e^{-1} = 1$, a contradiction.

Case 3: $\deg P = 2$. Then $\deg \kappa = \deg P = 2$. Assume that $\kappa(z) = az^2 + bz + c$, where $a \neq 0$, b, c are constants. Substituting $\kappa(z) = az^2 + bz + c$ into (5.4)–(5.6) yields

$$F(z) = A[az^2 + bz + c]e^z, \tag{5.9}$$

$$F'(z) = A[az^2 + (2a+b)z + b + c]e^z, \tag{5.10}$$

$$F''(z) = A[az^2 + (4a+b)z + 2a + 2b + c]e^z. \tag{5.11}$$

We consider two subcases.

Subcase 1: *κ has two distinct zeros z_i $(i = 1, 2)$*. Then $z_{1,2} = (-b \pm \sqrt{b^2 - 4ac})/(2a)$ and z_i $(i = 1, 2)$ is a simple zero of κ. Thus, $\kappa'(z_i) \neq 0$ and $\beta(z_i) = \kappa'(z_i) + \kappa(z_i) \neq 0$ $(i = 1,\ 2)$.

So, it follows from (I) that $Ae^{z_i} = 1$ $(i = 1, 2)$. Putting the form of z_i into $Ae^{z_i} = 1$ $(i = 1, 2)$, we easily deduce that

$$b^2 - 4ac = -4a^2k^2\pi^2, \tag{5.12}$$

where $k \neq 0$ is an integer.

From (5.10), we have

$$F'(z) - \beta(z) = [az^2 + (2a+b)z + b + c][Ae^z - 1]. \tag{5.13}$$

We know that $\beta(z) = az^2 + (2a+b)z + b + c$ has two distinct simple zeros. In fact, by (5.12), we have

$$\varDelta_1 = (2a+b)^2 - 4a(b+c) = 4a^2 + b^2 - 4ac = 4a^2[1 - k^2\pi^2] \neq 0,$$

thus, $\beta(z)$ has two distinct simple zeros $z_{3,4} = \frac{-(2a+b) \pm \sqrt{4a^2+b^2-4ac}}{2a}$. Obviously, $\beta'(z_i) \neq 0$ and $\beta'(z_i) + \beta(z_i) \neq 0$ $(i = 3, 4)$. Then (II) yields $Ae^{z_i} = 1$ $(i = 3, 4)$. As above, we deduce that

$$\varDelta_1 = 4a^2(1 - k^2\pi^2) = -4a^2m^2\pi^2, \tag{5.14}$$

where $m \neq 0$ is an integer. This implies that $(k^2 - m^2)\pi^2 = 1$, which is impossible.

Subcase 2: *κ has a double zero z_5*. Similarly to the above discussion, we have

$$b^2 = 4ac. \tag{5.15}$$

Noting that $\beta(z) = az^2 + (2a+b)z + b + c$, from (5.15), we have $\varDelta_2 = (2a+b)^2 - 4a(b+c) = 4a^2 + b^2 - 4ac = 4a^2 \neq 0$. Thus, $\beta(z)$ has two distinct simple zeros $z_6 = -b/(2a) - 2$ and $z_7 = -b/(2a)$. As in the last argument of Subcase 1, we deduce that

$$Ae^{-b/(2a)-2} = 1 \quad \text{and} \quad Ae^{-b/(2a)} = 1.$$

From the two formules, we derive that $e^{-2} = 1$, a contradiction.

Acknowledgements. We are grateful to the reviewers for their helpful comments and suggestions. The research was supported by the NSFC Tianyuan Mathematics Youth Fund (Nos. 11026146 and 11126327), the NSF of Guangdong Province (No. 9452902001003278) and the Fundamental Research Funds for the Central Universities (Nos. 12CX04080A and 10CX04038A).

References

[1] C. A. Berenstein and R. Gay, *Complex Variables*, Springer, New York, 1991.
[2] R. Brück, *On entire functions which share one value CM with their first derivatives*, Results Math. 30 (1996), 21–24.
[3] J. M. Chang, *Uniqueness of entire functions*, J. Math. Anal. Appl. 288 (2003), 97–111.
[4] J. M. Chang and M. L. Fang, *Uniqueness of entire functions and fixed points*, Kodai Math. J. 25 (2002), 309–320.
[5] J. M. Chang, M. L. Fang and L. Zalcman, *Normal families of holomorphic functions*, Illinois J. Math. 48 (2004), 319–337.
[6] J. Clunie and W. K. Hayman, *The spherical derivative of integral and meromorphic functions*, Comment. Math. Helv. 40 (1966), 117–148.
[7] J. Grahl and C. Meng, *Entire functions sharing a polynomial with their derivatives and normal families*, Analysis (Munich) 28 (2008), 51–61.
[8] G. G. Gundersen and L. Z. Yang, *Entire functions that share one value with one or two of their derivatives*, J. Math. Anal. Appl. 223 (1998), 88–95.
[9] G. Jank, E. Mues und L. Volkmann, *Meromorphe Funktionen, die mit ihrer ersten und zweiten Ableitung einen endlichen Wert teilen*, Complex Variables Theory Appl. 6 (1986), 51–71.
[10] I. Lahiri and G. K. Ghosh, *Uniqueness of entire functions and their derivatives*, Ann. Polon. Math. 96 (2009), 239–246.
[11] L. Laine, *Nevanlinna Theory and Complex Differential Equations*, de Gruyter, Berlin, 1993.
[12] X. J. Liu, S. Nevo and X. C. Pang, *On the kth derivative of meromorphic functions with zeros of multiplicity at least $k+1$*, J. Math. Anal. Appl. 348 (2008), 516–529.
[13] F. Lü, J. F. Xu and A. Chen, *Entire functions sharing polynomials with their first derivatives*, Arch. Math. (Basel) 92 (2009), 593–601.
[14] F. Lü and H. X. Yi, *On the uniqueness problems of meromorphic functions and their linear differential polynomials*, J. Math. Anal. Appl. 362 (2010), 301–312.
[15] D. Minda, *Yosida functions*, in: Lectures on Complex Analysis (Xian, 1987), C. T. Chuang (ed.), World Sci., Singapore, 1988, 197–213.
[16] X. C. Pang and L. Zalcman, *Normal families and shared values*, Bull. London Math. Soc. 32 (2000), 325–331.

[17] L. A. Rubel and C. C. Yang, *Values shared by an entire function and its derivative*, in: Complex Analysis, Lecture Notes in Math. 599, Springer, Berlin, 1976, 101–103.
[18] J. Schiff, *Normal Families*, Springer, 1993.
[19] J. P. Wang, *Entire functions that share a polynomial with their derivatives*, J. Math. Anal. Appl. 320 (2006), 703–717.
[20] H. X. Yi and C. C. Yang, *Uniqueness Theory of Meromorphic Functions*, Sci. Press, Beijing, 1995.

Feng Lü
College of Science
China University of Petroleum
Qingdao 266555, Shandong, China
E-mail: lvfeng18@gmail.com

Junfeng Xu
Department of Mathematics
Wuyi University
Jiangmen, Guangdong, 529020, P.R. China
E-mail: xujunf@gmail.com

Received 18.5.2011
and in final form 28.6.2011 (2451)