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A problem with almost everywhere equality

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Abstract. A topological space Y is said to have (AEEP) if the following condition is satisfied: Whenever $(X, \mathfrak{M})$ is a measurable space and $f, g\colon X \to Y$ are two measurable functions, then the set $\Delta(f, g) = \{x \in X\colon f(x) = g(x)\}$ is a member of $\mathfrak{M}$. It is shown that a metrizable space Y has (AEEP) iff the cardinality of Y is not greater than $2^{\aleph_0}$.

1. Introduction. In several aspects of mathematics dealing with measurability (such as measure theory, descriptive set theory, stochastic processes, ergodic theory, study of L^p-spaces, etc.) the idea of identifying functions which are equal almost everywhere is quite natural. The experience gained from real-valued functions may lead to the false conclusion that the set on which two measurable functions (which take values in a common arbitrary topological space) coincide is always measurable. It is well-known and quite easy to prove that this happens when the functions take values in a space with countable base, or when they have separable images lying in a metrizable space. However, it is not true in general. The reason for this is that $\mathfrak{B}(Y) \otimes \mathfrak{B}(Y)$ differs (in general) from $\mathfrak{B}(Y \times Y)$, where $\mathfrak{B}(Y)$ is the σ-algebra of all Borel subsets of a topological space Y.

Let us say that a topological space Y has the *almost everywhere equality property* (briefly, (AEEP)) if Y satisfies the condition stated in the abstract. It is easily seen (see Lemma 2.1 below) that Y has (AEEP) iff the diagonal of Y belongs to $\mathfrak{B}(Y) \otimes \mathfrak{B}(Y)$. So, it may turn out that Y has (AEEP) but still $\mathfrak{B}(Y) \otimes \mathfrak{B}(Y) \neq \mathfrak{B}(Y \times Y)$. The aim of this short note is to prove that a metrizable space Y has (AEEP) iff $\operatorname{card}(Y) \leq 2^{\aleph_0}$. Thus, among metrizable spaces only those whose topological weight is not greater than $2^{\aleph_0}$ have (AEEP). It may seem surprising that not only separable spaces appear in this characterization.

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Nonseparable metric spaces are widely investigated in functional analysis and operator theory. In fact, the Banach algebra of all bounded linear operators acting on a separable Banach space is usually nonseparable. Also infinite-dimensional von Neumann algebras are nonseparable. Another example of a nonseparable Banach space is the space ℓ_∞ of all bounded sequences, widely explored in the geometry of Banach spaces. So, our result may find applications in investigations of those spaces.

Notation and terminology. For every topological space Y, $\mathfrak{B}(Y)$ stands for the σ-algebra of all Borel subsets of Y. That is, $\mathfrak{B}(Y)$ is the smallest σ-algebra containing all open sets. Whenever $(\Omega,\mathfrak{M})$ and $(\Lambda,\mathfrak{N})$ are two measurable spaces, a function $f\colon(\Omega,\mathfrak{M})\to(\Lambda,\mathfrak{N})$ is said to be *measurable* iff $f^{-1}(B)\in\mathfrak{M}$ for each $B\in\mathfrak{N}$. Furthermore, $\mathfrak{M}\otimes\mathfrak{N}$ denotes the product σ-algebra of $\mathfrak{M}$ and $\mathfrak{N}$, i.e. $\mathfrak{M}\otimes\mathfrak{N}$ is the smallest σ-algebra on $\Omega\times\Lambda$ which contains all sets of the form $A\times B$ with $A\in\mathfrak{M}$ and $B\in\mathfrak{N}$. If $g\colon(\Omega,\mathfrak{M})\to Y$ where Y is a topological space, then g is measurable if $g^{-1}(U)\in\mathfrak{M}$ for any open set $U\subset Y$ or, equivalently, if $g\colon(\Omega,\mathfrak{M})\to(Y,\mathfrak{B}(Y))$ is measurable. For two functions $u,v\colon D\to E$, $\Delta(u,v)$ stands for the set $\{x\in D\colon u(x)=v(x)\}$. Additionally, for every set E, Δ_E denotes the diagonal of E, i.e. $\Delta_E=\{(x,x)\colon x\in E\}$. Finally, $\mathrm{card}(E)$ is the cardinality of E.

2. The result.

We begin with a simple

LEMMA 2.1. *For a topological space Y the following conditions are equivalent:*

(i) *Y has* (AEEP),
(ii) $\Delta_Y\in\mathfrak{B}(Y)\otimes\mathfrak{B}(Y)$.

Proof. If $f,g\colon(\Omega,\mathfrak{M})\to Y$ are two measurable functions, then

$$h\colon(\Omega,\mathfrak{M})\ni\omega\mapsto(f(\omega),g(\omega))\in(Y\times Y,\mathfrak{B}(Y)\otimes\mathfrak{B}(Y))$$

is measurable as well and thus $\Delta(f,g)=h^{-1}(\Delta_Y)$ is a member of $\mathfrak{M}$. This shows that (i) follows from (ii). To see the converse, notice that the natural projections $p_j\colon(Y\times Y,\mathfrak{B}(Y)\otimes\mathfrak{B}(Y))\ni(y_1,y_2)\mapsto y_j\in Y$ $(j=1,2)$ are measurable and that $\Delta(p_1,p_2)=\Delta_Y$, which finishes the proof. ∎

LEMMA 2.2. *For an arbitrary topological space Y, every member F of $\mathfrak{B}(Y)\otimes\mathfrak{B}(Y)$ may be written in the form*

$$F=\bigcup_{t\in[0,1]}(A_t\times B_t) \tag{2.1}$$

where $A_t,B_t\in\mathfrak{B}(Y)$ $(t\in[0,1])$.

Proof. Let $\mathcal{A}$ be the family of all subsets of $Y\times Y$ which are finite unions of sets of the form $A\times B$ with $A,B\in\mathfrak{B}(Y)$. It is easily seen that $\mathcal{A}$ is an algebra of subsets of $Y \times Y$. Hence, by the monotone class theorem (see e.g. Theorem 1.3 of [3]), $\mathfrak{B}(Y) \otimes \mathfrak{B}(Y)$ is the smallest family $\mathcal{F}$ such that $\mathcal{A} \subset \mathcal{F}$ and

$$F_1, F_2, \ldots \in \mathcal{F} \;\Rightarrow\; \bigcap_{n=1}^{\infty} F_n, \bigcup_{n=1}^{\infty} F_n \in \mathcal{F}. \tag{2.2}$$

Now let $\mathcal{F}$ consist of all sets F which may be written in the form (2.1) (with $A_t, B_t \in \mathfrak{B}(Y)$). Since $\emptyset \in \mathcal{F}$, we have $\mathcal{A} \subset \mathcal{F}$. So, it remains to check that $\mathcal{F}$ satisfies (2.2). Let $F_j = \bigcup_{t\in[0,1]} A_t^{(j)} \times B_t^{(j)}$ for $j \in \mathbb{N}$. It is clear that $\bigcup_{j\in\mathbb{N}} F_j \in \mathcal{F}$. Finally, put $\Lambda = [0,1]^{\mathbb{N}}$ and observe that

$$\bigcap_{j\in\mathbb{N}} F_j = \bigcup_{\xi\in\Lambda} \Big[\Big(\bigcap_{j\in\mathbb{N}} A_{\xi(j)}^{(j)}\Big) \times \Big(\bigcap_{j\in\mathbb{N}} B_{\xi(j)}^{(j)}\Big)\Big],$$

which finishes the proof since there is a bijection between Λ and $[0,1]$. ∎

As an immediate consequence of Lemmas 2.1 and 2.2, we obtain

COROLLARY 2.3. *If a topological space Y has* (AEEP), *then*

$$\operatorname{card}(Y) \le 2^{\aleph_0}.$$

Now we want to prove the converse of Corollary 2.3 for metrizable Y. Recall that the *topological cone* $C(Y)$ over a topological space Y is the set $(Y \times (0,1]) \cup \{\omega_Y\}$ equipped with the topology such that $\{\omega_Y\}$ is closed in $C(Y)$, the topology of $Y \times (0,1]$ inherited from $C(Y)$ coincides with the product one, and the sets $(Y \times (0,t)) \cup \{\omega_Y\}$ with $t \in (0,1)$ form a basis of open neighbourhoods of ω_Y in $C(Y)$. The next result is elementary and we omit its proof.

LEMMA 2.4. *If $u\colon X \to Y$ is a continuous function between topological spaces X and Y, then the function $\widehat{u}\colon C(X) \to C(Y)$ given by $\widehat{u}((x,t)) = (u(x),t)$ and $\widehat{u}(\omega_X) = \omega_Y$ is continuous as well.*

An important example of a topological cone is the so-called *hedgehog* space ([1, Example 4.1.5]). The hedgehog $J(\mathfrak{m})$ of spininess $\mathfrak{m} \ge \aleph_0$ is the topological cone over a discrete space of cardinality $\mathfrak{m}$. Its importance is justified by the following result of Kowalsky [2] (see also [1, Theorem 4.4.9]; it is known that $[J(\mathfrak{m})]^{\aleph_0}$ with infinite $\mathfrak{m}$ is homeomorphic to the Hilbert space of Hilbert space dimension $\mathfrak{m}$; see [4, 5] or Remark in Exercise 4.4.K of [1]).

THEOREM 2.5. *Every metrizable space of topological weight not greater than $\mathfrak{m}$ (where $\mathfrak{m} \ge \aleph_0$) is homeomorphic to a subset of $[J(\mathfrak{m})]^{\aleph_0}$.*

As a consequence of Theorem 2.5, we obtain the following result, which may be interesting in itself.

Proposition 2.6. *Every metrizable space* Y *with* $\operatorname{card}(Y) \le 2^{\aleph_0}$ *admits a continuous one-to-one function from* Y *into a separable metrizable space.*

Proof. Let $D = [0,1]$ and $T = [0,1]$ be equipped with, respectively, the discrete and the natural topology. By Lemma 2.4, the function $C(D) \ni z \mapsto z \in C(T)$ is continuous. This means that there is a one-to-one continuous function $u\colon J(2^{\aleph_0}) \to W$ where W is a separable metrizable space. But then the function $[J(2^{\aleph_0})]^{\aleph_0} \ni (x_n)_{n=1}^{\infty} \mapsto (u(x_n))_{n=1}^{\infty} \in W^{\aleph_0}$ is continuous and one-to-one as well. Now it suffices to apply Theorem 2.5. ∎

We are now able to prove the main result of the paper.

Theorem 2.7. *For a metrizable space* Y *the following conditions are equivalent:*

(i) Y *has* (AEEP),
(ii) *the topological weight of* Y *is not greater than* $2^{\aleph_0}$,
(iii) $\operatorname{card}(Y) \le 2^{\aleph_0}$.

Proof. The equivalence of (ii) and (iii) follows from $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. So, thanks to Lemma 2.1 and Corollary 2.3, we only have to show that $\Delta_Y \in \mathfrak{B}(Y) \otimes \mathfrak{B}(Y)$ provided $\operatorname{card}(Y) \le 2^{\aleph_0}$.

Assume (iii) is satisfied. We infer from Proposition 2.6 that there is a separable metrizable space X and a continuous one-to-one function $u\colon Y \to X$. Then $v = u \times u\colon (Y \times Y, \mathfrak{B}(Y) \otimes \mathfrak{B}(Y)) \to (X \times X, \mathfrak{B}(X) \otimes \mathfrak{B}(X))$ is measurable ($v(x,y) = (u(x), u(y))$ for $x, y \in Y$). Since X is separable, $\mathfrak{B}(X) \otimes \mathfrak{B}(X) = \mathfrak{B}(X \times X)$ and therefore $\Delta_Y = v^{-1}(\Delta_X) \in \mathfrak{B}(Y) \otimes \mathfrak{B}(Y)$ and we are done. ∎

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