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The principle of moduli flexibility for real algebraic manifolds

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Abstract. Given a real closed field R, we define a real algebraic manifold as an irreducible nonsingular algebraic subset of some R^n. This paper deals with deformations of real algebraic manifolds. The main purpose is to prove rigorously the reasonableness of the following principle, which is in sharp contrast with the compact complex case: "The algebraic structure of every real algebraic manifold of positive dimension can be deformed by an arbitrarily large number of effective parameters".

1. Introduction and main result

1.1. Introduction. This paper deals with deformations of real algebraic structures on Nash manifolds. Our main purpose is to show that, in positive dimension, such deformations can depend on an arbitrarily large number of effective parameters. Our point of view is purely real.

In order to specify the meaning we give to "purely real" and hence to place our results in the correct setting, we need to recall some basic facts concerning the notion of deformation.

The notion of deformation originated in complex analytic geometry. Deformations of the complex structure of a compact complex analytic manifold have been studied since the time of Riemann. Let B be a domain of $\mathbb{C}^b$ containing 0. A set $\mathscr{M} = \{M_t\}_{t\in B}$ of compact complex analytic manifolds, parametrized by B, is said to be a *complex analytic family of compact complex analytic manifolds* if there exist a complex analytic manifold $\mathcal{M}$ and a proper holomorphic map $\pi : \mathcal{M} \to B$ such that $M_t \simeq \pi^{-1}(t)$ for each $t \in B$. Here $M_t \simeq \pi^{-1}(t)$ means that M_t is biholomorphically isomorphic to $\pi^{-1}(t)$. Suppose π has the above properties and hence $\mathscr{M}$ is a complex analytic family. As a differentiable map, π is locally trivial. More precisely, for each $t \in B$ and for each open ball U of $\mathbb{C}^b$ centered at t and contained in B, there exists a differentiable map $\pi' : \pi^{-1}(U) \to \pi^{-1}(t)$

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such that, denoting by $\pi_| : \pi^{-1}(U) \to U$ the restriction of π, the map $\pi_| \times \pi' : \pi^{-1}(U) \to U \times \pi^{-1}(t)$ is a diffeomorphism. It follows that M_t is diffeomorphic to M_s for each $t, s \in B$. In this sense, $\mathscr{M}$ can be considered as a family of complex structures on the compact differentiable manifold underlying M_0, holomorphically dependent on $t \in B$.

Fix a compact complex analytic manifold M. The complex analytic family $\mathscr{M}$ is called a *deformation* of M if $M \simeq M_0$.

In the context of deformations, a first basic problem is as follows:

PROBLEM. *How many effective parameters can a deformation of M depend on?*

Riemann himself treated this problem in the one-dimensional case, proving that the number of effective parameters on which a deformation of his surfaces of genus $g \geq 2$ can depend is $\leq 3g-3$ and hence it is finite. Later, in the famous papers [34], Kodaira and Spencer discovered that a key tool to understand the nature of this problem is the first cohomology complex vector space $H^1(M, \Theta_M)$ of M with coefficients in its sheaf Θ_M of germs of holomorphic vector fields. Their starting idea was to consider M as a finite number of polydiscs glued together via certain biholomorphic identifications and to interpret the deformations of M as holomorphic variations of such identifications. Quite naturally, these variations can be represented by the elements of $H^1(M, \Theta_M)$, called *infinitesimal deformations* of M. The compactness of M is not only crucial to defining the notion of infinitesimal deformation, but it also ensures that the complex dimension of $H^1(M, \Theta_M)$, which we denote by h_M, is finite. With regard to the preceding problem, the importance of $H^1(M, \Theta_M)$ was fully clarified by Kuranishi [37]. He proved that there exist a complex analytic subset K of an open neighborhood of 0 in $\mathbb{C}^{h_M}$ and a holomorphic map $\theta : \mathcal{K} \to K$ from a complex analytic space $\mathcal{K}$ to K such that $0 \in K$, $\theta^{-1}(s)$ is a compact complex analytic manifold for each $s \in K$, $\theta^{-1}(0) \simeq M$ and, for each deformation $\{M_t\}_{t \in B}$ of M with B sufficiently small around 0, there exists a holomorphic map $\varphi : B \to K$ such that $\varphi(0) = 0$ and $M_t \simeq \theta^{-1}(\varphi(t))$ for each $t \in B$.

A philosophical consequence of this completeness result is that the complex analytic structure of every compact complex analytic manifold can be deformed by an at most finite number of effective parameters.

The latter assertion can be made precise in several ways, depending on the meaning we give to the term "effective parameters". Kodaira and Spencer themselves furnished in [34] a definition of effectively parametrized family $\{M_t\}_{t \in B}$ of compact complex analytic manifolds by requiring that each linear map $T_t(B) \to H^1(M_t, \Theta_{M_t})$, sending tangent vectors of B at t to the corresponding infinitesimal deformations of M_t, is injective (see also [33]). However, the aforementioned completeness theorem of Kuranishi suggests different, and more elementary, ways of defining the notion of effectiveness of parameters. Let $\mathscr{M} = \{M_t\}_{t\in B}$ be a deformation of M, parametrized by a domain B of some $\mathbb{C}^b$. We can say that $\mathscr{M}$ is *perfectly* parametrized if $M_t \not\simeq M_s$ for each $t, s \in B$ with $t \neq s$, where $M_t \not\simeq M_s$ means that M_t is not biholomorphically isomorphic to M_s. A weaker notion of effectiveness of parameters is as follows: we can say that $\mathscr{M}$ is *almost perfectly* parametrized if, for each $t \in B$, the set $\{s \in B \mid M_s \simeq M_t\}$ is finite. Evidently, if $\mathscr{M}$ is almost perfectly parametrized, then Kuranishi's result implies that $b \leq h_M$. In the next subsection, the above notions of perfectly parametrized deformation and of almost perfectly parametrized deformation will be suitably reformulated in our real setting.

It is important to remark that, in contrast to the compact case, the deformations of noncompact complex analytic manifolds can depend on an arbitrarily large number of effective parameters. We remind the reader of the case of bounded domains of $\mathbb{C}^n$, where $n \geq 2$, for which a natural parameter space is the set of their boundaries with the Hausdorff distance (see [23, p. 289]).

In the setting of complex algebraic geometry, the notion of deformation has a different, more algebraic, nature. In fact, it is deeply connected with the moduli problem, that is, the problem of finding spaces, called moduli spaces, that classify, up to complex biregular isomorphism, all the (irreducible) projective complex algebraic manifolds with assigned numerical invariants or additional structures, like polarizations. In any case, since the projective complex algebraic case is a particular case of the compact complex analytic one, we can assert again that the complex algebraic structure of every projective complex algebraic manifold can be deformed by an at most finite number of effective parameters.

In complex algebraic geometry, a real manifold is usually defined as a pair (X, σ) in which X is a projective complex algebraic manifold and $\sigma : X \to X$ is an anti-holomorphic involution. The real part of (X, σ) is the fixed point set of σ. We will call (X, σ) a *real-complex* algebraic manifold and σ a *real-complex* algebraic structure on X. A complex biregular isomorphism between real-complex algebraic manifolds is said to be a *real isomorphism* if it is invariant under anti-holomorphic involutions. By focusing on the notions of real-complex algebraic manifold and of real isomorphism, the moduli problem specializes to the real moduli problem. A very important formulation of the latter problem is as follows: find spaces, called real moduli spaces, that classify all real isomorphic classes of real-complex algebraic manifolds (X, σ) with assigned numerical invariants by considering all the possible topological types of the real parts of the (X, σ)'s. This is an old question considered first by Klein and Weichold in the case of real-complex curves of fixed genus, and later by Comessatti in the cases of real-complex surfaces and of real-complex abelian varieties. Presently, the real moduli problem for real-complex curves of fixed genus is very well understood. In higher dimensions, the case of real-complex surfaces represents a very active and fascinating field of research. For these topics, we refer the reader to the following selection of titles and to the references mentioned therein: [5, 9, 10, 11, 12, 13, 15, 16, 19, 20, 21, 22, 28, 29, 30, 36, 39, 42, 43].

In the real-complex setting, we do not know a general result which ensures that the real-complex algebraic structure of every real-complex algebraic manifold can be deformed by an at most finite number of effective parameters. However, it is evident that, if the real isomorphic class of a given real-complex algebraic manifold (X,σ) belongs to a (coarse) real moduli space $\mathcal{R}$, then the maximum number of effective parameters of the corresponding deformations of (X,σ) is less than or equal to the real dimension of $\mathcal{R}$. An important example is the one of real-complex curves (see [28, 39]). Furthermore, there are quite rigid examples: the complex projective line $\mathbb{P}^1(\mathbb{C})$, viewed as a differentiable surface, admits only two real-complex algebraic structures and, for each nonnegative even integer n, the differentiable $2n$-manifold $\mathbb{P}^n(\mathbb{C})$ admits only one (see Exercises 1.10 and 1.12 of [35]).

In this paper, we treat the notion of deformation of algebraic structures from the point of view of *purely real* algebraic geometry; that is, of the real algebraic geometry systematically studied, as an independent discipline, in the foundational book [6] and also in the books [1] and [4]. As far as we know, this is the first time that such a treatment has been done.

Given a real closed field R, by a *real algebraic set* we simply mean the set of solutions in some R^n of a polynomial system with coefficients in R. If a real algebraic set is irreducible and nonsingular, then we call it a *real algebraic manifold*.

Roughly speaking, from our point of view, a real-complex algebraic manifold is a real algebraic manifold for which one of its nonsingular projective complexifications has been fixed. This makes a real-complex algebraic manifold rigid. Changing the point of view, one can say that a real algebraic manifold is the germ of the real part of a real-complex algebraic manifold. This makes a real algebraic manifold flexible.

As previously stated, the main aim of this paper is to make rigorous the following informal principle (see Theorem 1.3 below), which is in sharp contrast with the compact complex analytic, projective complex algebraic, and real-complex algebraic cases.

Principle of real moduli flexibility. *The algebraic structure of every real algebraic manifold of positive dimension can be deformed by an arbitrarily large number of effective parameters.*

Real algebraic geometers are accustomed to the plasticity of real algebraic objects. The above principle confirms and extends this plasticity. In this sense, we can say that the principle was expected. The same is true in the context of complex analytic and algebraic geometry. In fact, by considering only the germs of the real parts of real-complex algebraic manifolds, we lose the "complex" compactness and hence there are no obstructions to the validity of the principle.

1.2. Main result. *Let us fix a real closed field R.* As we have just said, a real algebraic manifold is an irreducible nonsingular real algebraic set. Let X be a real algebraic manifold. An irreducible nonsingular Zariski closed subset of X is called a *real algebraic submanifold* of X. Given another real algebraic manifold Y, we say that X and Y are *birationally isomorphic*, and we write $X \sim Y$, if there exists a biregular isomorphism from a Zariski dense Zariski open subset of X to a Zariski dense Zariski open subset of Y. If such a biregular isomorphism does not exist, then we say that X and Y are birationally nonisomorphic and we write $X \nsim Y$. In what follows, we will use basic concepts and facts from real semi–algebraic and Nash geometry. Our standard reference for these topics is [6] (see also [4, 41]).

Let us introduce the definition of algebraic real-deformation of a real algebraic manifold.

Definition 1.1. Let V be a real algebraic manifold and let $\pi : \mathcal{V} \to R^b$ be a surjective regular map from a real algebraic manifold $\mathcal{V}$ to R^b. We say that π is an *algebraic real-deformation of V (parametrized by R^b)* if π is a submersion, $\pi^{-1}(y)$ is irreducible (and hence it is a real algebraic submanifold of $\mathcal{V}$) for each $y \in R^b$ and there exists a regular map $\pi' : \mathcal{V} \to V$ such that the map $\pi \times \pi' : \mathcal{V} \to R^b \times V$ is a Nash isomorphism and the restriction of π' to $\pi^{-1}(0)$ is a biregular isomorphism.

The reader can observe that an algebraic real-deformation of a real algebraic manifold V, parametrized by R^b, can be interpreted as a family of real algebraic structures on the Nash manifold underlying V, algebraically dependent on $y \in R^b$, which coincides with the real algebraic structure of V itself on $y = 0$.

Suppose now that $\pi : \mathcal{V} \to R^b$ is an algebraic real-deformation of the real algebraic manifold V. Define the map $\rho_b : R^b \times R^b \to R^b$ and the subset $\mathcal{S}_\pi$ of $R^b \times R^b$ by setting $\rho_b(y, z) := y$ and

$$\mathcal{S}_\pi := \{(y, z) \in R^b \times R^b \mid \pi^{-1}(y) \sim \pi^{-1}(z)\}. \tag{1.1}$$

Definition 1.2. We say that π is *perfectly parametrized by R^b* if $\pi^{-1}(y) \nsim \pi^{-1}(z)$ for each $y, z \in R^b$ with $y \neq z$. Moreover, we say that π is *almost perfectly parametrized by R^b* if there exists a semialgebraic subset $\mathcal{T}$

of $R^b \times R^b$ such that $\mathcal{T}$ contains $\mathcal{S}_\pi$ and the restriction of ρ_b to $\mathcal{T}$ has finite fibers.

The reader can observe that, if π is almost perfectly parametrized by R^b, then there exists an integer L such that, for each $y \in R^b$, the cardinality of the set $\{z \in R^b \mid \pi^{-1}(y) \sim \pi^{-1}(z)\}$ is $\leq L$.

The main result of this paper is as follows.

Theorem 1.3. *Every real algebraic manifold V of positive dimension has the following property: for each nonnegative integer b, there exists an algebraic real-deformation of V almost perfectly parametrized by R^b.*

We do not know if, in the above statement, one can omit "almost". However, we conjecture that this is the case.

A version of the latter result was conjectured in point (i) of Remark 1.10 of [24]. In [17], Coste and Shiota proved that every connected affine Nash manifold over R is Nash isomorphic to a real algebraic manifold. By combining this result with Theorem 1.3, we immediately deduce that, given any connected affine Nash manifold N over R of positive dimension, the set of birationally nonisomorphic real algebraic manifolds which are Nash isomorphic to N has the same cardinality as R. In this way, we have rediscovered Corollary 2 of [25] (see also [2, 8]).

The remainder of the paper is organized as follows. In Subsection 1.3, we present the idea of the proof of Theorem 1.3. Section 2 deals with three preparatory results we use in Section 3 to prove Theorem 1.3, first in the bounded case and then in the unbouded one.

1.3. Idea of the proof of the main result. The proof of Theorem 1.3 we give in Section 3 is quite long and technical. However, the ideas used are easy to describe.

Let $V \subset R^n$ be a real algebraic manifold of positive dimension and let b be a nonnegative integer.

First, suppose that V is bounded, that is, contained in some open ball of R^n. Let $C = R[\sqrt{-1}]$ be the algebraic closure of R. Identify R^n with the subset of $\mathbb{P}^n(R)$ consisting of points $[x_0, x_1, \ldots, x_n]$ with $x_0 \neq 0$, and identity $\mathbb{P}^n(R)$ with the fixed point set of the complex conjugation of $\mathbb{P}^n(C)$. Since V is bounded, up to biregular isomorphism, we may suppose that the Zariski closure Z of V in $\mathbb{P}^n(C)$ is nonsingular.

Fix an odd integer $d \geq 3$ and a positive integer k. Consider the family $\{P_\alpha\}_{\alpha \in C^K}$ of all polynomials in $C[X_0, X] = C[X_0, X_1, \ldots, X_n]$ homogeneous of degree kd, parametrized by their coefficients $\alpha \in C^K$, where $K := \binom{n+kd}{n}$. Denote by α_0 the element of R^K such that $P_{\alpha_0}(X_0, X) = X_0^{kd}$ and by A the nonempty Zariski open subset of C^K consisting of all nonnull elements α such that P_α is irreducible and its vanishing set D_α in $\mathbb{P}^n(C)$ is nonsingular and transverse to Z. Evidently, α_0 is not an element of A. For each $\alpha \in C^K$, we perform the simple d-cyclic covering $\pi_\alpha : Z_\alpha \to Z$ of Z branched along $Z \cap D_\alpha$, obtaining an "algebraic" family $\{Z_\alpha\}_{\alpha \in C^K}$ of projective complex algebraic varieties. Such a family is "singular over $\alpha = \alpha_0$", but it becomes a true (flat) algebraic family of projective complex algebraic manifolds over $\alpha \in A$. Let V_α be the real part of each Z_α. Since $V \cap D_{\alpha_0} = \emptyset$, there exists an open neighborhood U of α_0 in R^K such that, for each $\alpha \in U$, $V \cap D_\alpha = \emptyset$ or, equivalently, P_α does not vanish on V. Since the integer d is odd and $V \subset R^n$, we see that, for each $\alpha \in U$, V_α is the graph in $V \times R$ of the Nash function $g_\alpha : V \to R$ sending x to $\sqrt[d]{P_\alpha(1,x)}$, and hence the restriction $\pi_\alpha(R)$ of π_α from V_α to V is a Nash isomorphism. Moreover, since g_{α_0} is constantly equal to 1, $\pi_{\alpha_0}(R)$ is a biregular isomorphism.

Choosing k sufficiently large, we may suppose that K is arbitrarily large and, for each $\alpha \in A$, the canonical complex line bundle ω_{Z_α} of Z_α is ample. Thanks to the latter property of the ω_{Z_α}'s with $\alpha \in A$, we can use the theory of coarse moduli spaces to distinguish the complex biregular (birational indeed) isomorphic classes of the Z_α's with $\alpha \in A$, via a "classifying" complex regular map $u : A \to \mathcal{M}$ from A to a suitable quasi-projective complex algebraic variety $\mathcal{M}$. Now, since α_0 is an accumulation point of $A \cap R^K$ in R^K with respect to the euclidean topology, our strategy to complete the proof consists in costructing a regular map $\phi : R^b \to R^K$ such that $\phi(0) = \alpha_0$, $\phi(R^b \setminus \{0\}) \subset (U \cap A) \setminus \{\alpha_0\}$ and the family $\mathscr{V}_\phi := \{V_{\phi(y)}\}_{y \in R^b}$ is an algebraic real-deformation of V almost perfectly parametrized by R^b.

Two main difficulties arise. The first is to prove that the complex codimension of each fiber of u is sufficiently large (at least b) and hence that it is possible to choose ϕ "transverse" to each fiber of u. The second is to understand if it is possible to choose ϕ in such a way that the corresponding family $\mathscr{V}_\phi$ is "nonsingular over $y = 0$". The second difficulty is a fake (although not trivial) problem, while the first is a delicate problem. The latter can be solved by a careful procedure in which one must repeat the construction concerning the simple d-cyclic coverings twice.

Finally, if V is unbounded, then one can reduce to the bounded case by using the algebraic Alexandrov compactification and Hironaka's desingularization theorem.

2. Preparatory results

2.1. Terminology for complex objects. Denote by $C = R[s]/(s^2+1)$ the algebraic closure of R. By a *(quasi-)projective complex algebraic manifold* we mean an irreducible and nonsingular (quasi-)projective complex algebraic subvariety of some $\mathbb{P}^k(C)$. Let $\Omega \subset \mathbb{P}^k(C)$ be an irreducible quasi-projective complex algebraic variety. We denote by $\dim_C \Omega$ the complex dimension of Ω. Let Θ be another irreducible quasi-projective complex algebraic variety and let $\varphi : \Omega \to \Theta$ be a complex regular map. We call φ a *complex biregular embedding* if $\varphi(\Omega)$ is Zariski closed in Θ and the restriction of φ from Ω to $\varphi(\Omega)$ is a complex biregular isomorphism. If Ω and Θ are nonsingular, φ is a surjective submersion and $\varphi^{-1}(y) \subset \mathbb{P}^k(C)$ is a projective complex algebraic manifold for each $y \in \Theta$, then we call φ an *algebraic family of projective complex algebraic manifolds.*

Let $n, m \in \mathbb{N}$. Denote by $\sigma_n : \mathbb{P}^n(C) \to \mathbb{P}^n(C)$ the complex conjugation involution of $\mathbb{P}^n(C)$ and identify $\mathbb{P}^n(R)$ with the fixed point set of σ_n. Let S be a subset of $\mathbb{P}^n(C)$. The set S is said to be *defined over* R if $\sigma_n(S) = S$ and its real part $S(R)$ is defined as the intersection $S \cap \mathbb{P}^n(R)$. Let T be a subset of $\mathbb{P}^m(C)$ and let $f : S \to T$ be a map from S to T. We say that f is defined over R if both sets S and T are defined over R and $\sigma_m(f(x)) = f(\sigma_n(x))$ for each $x \in S$. The real part $f(R) : S(R) \to T(R)$ of f is defined as the restriction of f from $S(R)$ to $T(R)$.

Let $V \subset R^n$ be a real algebraic manifold. Suppose V is bounded, that is, contained in an open ball of R^n. Given a projective complex algebraic manifold Z, we say that Z is a *nonsingular complexification* of V if it is defined over R and its real part $Z(R)$ is biregularly isomorphic to V.

2.2. A special embedding of $\mathbb{P}^n(C)$. Let $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$, let $x = (x_1, \ldots, x_n)$ be the coordinates of C^n, let $(x_0, x) = (x_0, x_1, \ldots, x_n)$ be the coordinates of C^{n+1} and let $[x_0, x] = [x_0, x_1, \ldots, x_n]$ be the corresponding homogeneous coordinates of $\mathbb{P}^n(C)$. For each $j \in \{0, 1, \ldots, n\}$, define $U_{n,j} := \{[x_0, x] \in \mathbb{P}^n(C) \mid x_j \neq 0\}$.

In what follows, we identify C^n with $U_{n,0}$ via the coordinate chart sending x to $[1, x]$. Under this identification, R^n coincides with the Zariski open subset $U_{n,0}(R)$ of $\mathbb{P}^n(R)$. Furthermore, given $m \in \mathbb{N}$, we identify $C^n \times C^m$ with C^{n+m} and hence $R^n \times R^m$ with R^{n+m} in the natural way.

Define $|x|_n := (\sum_{j=1}^n x_j^2)^{1/2}$ for each $x = (x_1, \ldots, x_n) \in R^n$ and $S^{n-1} := \{x \in R^n \mid |x|_n = 1\}$.

Fix $h \in \mathbb{N}^*$. Let $\mathbb{N}_h^{n+1}$ be the subset of $\mathbb{N}^{n+1}$ consisting of all elements $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ such that $\sum_{j=0}^n \alpha_j = h$. As is easy to verify, the cardinality of $\mathbb{N}_h^{n+1}$ is equal to $\binom{n+h}{n}$. Denote by $v(n)$ the cardinality $\binom{n+2}{n} = (n+2)(n+1)/2$ of $\mathbb{N}_2^{n+1}$. Let $\mathrm{x} = (x_\alpha)_{\alpha \in \mathbb{N}_2^{n+1}}$ be the coordinates of $C^{v(n)}$ ordered in some fixed way, let $[\mathrm{x}]$ be the corresponding homogeneous coordinates of $\mathbb{P}^{v(n)-1}(C)$ and let $\nu_n : C^{n+1} \to C^{v(n)}$ be the map sending (x_0, x) to $((x_0, x)^\alpha)_{\alpha \in \mathbb{N}_2^{n+1}}$, where $(x_0, x)^\alpha$ is equal to $x_0^{\alpha_0} \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ if $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ (here $x_j^{\alpha_j} = 1$ if $\alpha_j = 0$). The map $\mathfrak{V}_n : \mathbb{P}^n(C) \to \mathbb{P}^{v(n)-1}(C)$ sending $[x_0, x]$ to $[\nu_n(x_0, x)]$ is the classical double Veronese embedding of $\mathbb{P}^n(C)$.

Let us define a variant $\mathfrak{V}_n^* : \mathbb{P}^n(C) \to \mathbb{P}^{v(n)}(C) = \mathbb{P}(C \times C^{v(n)})$ of $\mathfrak{V}_n$ by setting

$$\mathfrak{V}_n^*([x_0, x]) := [x_0^2 + x_1^2 + \cdots + x_n^2, \nu_n(x_0, x)]. \tag{2.1}$$

Evidently, $\mathfrak{V}_n^*$ is a complex biregular embedding defined over R and

$$\mathfrak{V}_n^*(R)(\mathbb{P}^n(R)) \text{ is a real algebraic submanifold of } R^{v(n)}. \tag{2.2}$$

LEMMA 2.1. *Let V be a bounded real algebraic submanifold of R^n. Then there exists a projective complex algebraic manifold $Z \subset \mathbb{P}^m(C)$ such that Z is a nonsingular complexification of V and $Z(R)$ is contained in R^m.*

Proof. By Hironaka's desingularization theorem [27], there exists a real algebraic submanifold V' of some $\mathbb{P}^\ell(R)$ such that V' is biregularly isomorphic to V and the Zariski closure of V' in $\mathbb{P}^\ell(C)$ is nonsingular (see Proposition 2.5 of [7] for details). However, it may happen that V' is not contained in R^ℓ, that is, $V' \cap \{[x_0, x_1, \ldots, x_\ell] \in \mathbb{P}^\ell(R) \mid x_0 = 0\} \neq \emptyset$. In any case, by (2.2), the Zariski closure Z of $\mathfrak{V}_\ell^*(R)(V')$ in $\mathbb{P}^{v(\ell)}(C)$ has all the desired properties. ∎

2.3. The complex line bundles $\mathcal{O}_Z(h)$. For each $i, j \in \{0, 1, \ldots, n\}$, define the complex regular function $g_{ij} : U_{n,j} \cap U_{n,i} \to C \setminus \{0\}$ by setting $g_{ij}([x_0, x]) := x_j/x_i$. Let $\mathcal{O}_{\mathbb{P}^n(C)}(h)$ be the complex line bundle over $\mathbb{P}^n(C)$ defined by the cocycle $\{(g_{ij})^h\}_{i,j\in\{0,1,\ldots,n\}}$, let $\pi_{n,h} : \mathcal{L}_{\mathbb{P}^n(C)}(h) \to \mathbb{P}^n(C)$ be the bundle projection of $\mathcal{O}_{\mathbb{P}^n(C)}(h)$ from its total space $\mathcal{L}_{\mathbb{P}^n(C)}(h)$ onto its base $\mathbb{P}^n(C)$ and, for each $j \in \{0, 1, \ldots, n\}$, let $\varphi_{n,h,j} : U_{n,j} \times C \to (\pi_{n,h})^{-1}(U_{n,j})$ be the corresponding trivialization chart over $U_{n,j}$. Up to bundle isomorphism, $\mathcal{O}_{\mathbb{P}^n(C)}(h)$ coincides with $\mathcal{O}_{\mathbb{P}^n(C)}(1)^{\otimes h}$.

Let $H^0(\mathbb{P}^n(C), \mathcal{O}_{\mathbb{P}^n(C)}(h))$ be the complex vector space of all complex regular sections of $\mathcal{O}_{\mathbb{P}^n(C)}(h)$ and let $C[X_0, X]_h = C[X_0, X_1, \ldots, X_n]_h$ be the complex vector space of all homogeneous polynomials in $C[X_0, X] = C[X_0, X_1, \ldots, X_n]$ of degree h. Each polynomial $P \in C[X_0, X]_h$ determines uniquely an element $\sigma_{n,h}(P)$ of $H^0(\mathbb{P}^n(C), \mathcal{O}_{\mathbb{P}^n(C)}(h))$ such that

$$\sigma_{n,h}(P)([x_0, x]) = \varphi_{n,h,j}([x_0, x], P(x_0/x_j, x/x_j)) \quad \text{if } [x_0, x] \in U_{n,j} \tag{2.3}$$

for some $j \in \{0, 1, \ldots, n\}$. Vice versa, each element of $H^0(\mathbb{P}^n(C), \mathcal{O}_{\mathbb{P}^n(C)}(h))$ is of the form $\sigma_{n,h}(P)$ for some P in $C[X_0, X]_h$. In other words, the map $\sigma_{n,h} : C[X_0, X]_h \to H^0(\mathbb{P}^n(C), \mathcal{O}_{\mathbb{P}^n(C)}(h))$ sending P to $\sigma_{n,h}(P)$ is a complex vector isomorphism. In particular, the complex dimension of $H^0(\mathbb{P}^n(C), \mathcal{O}_{\mathbb{P}^n(C)}(h))$ is equal to $\binom{n+h}{n}$.

Let $Z \subset \mathbb{P}^n(C)$ be a projective complex algebraic manifold. Denote by $\mathcal{O}_Z(h)$ the complex line bundle over Z obtained restricting $\mathcal{O}_{\mathbb{P}^n(C)}(h)$. Let $r := \dim_C Z$ and let $\deg(Z)$ be the degree of Z in $\mathbb{P}^n(C)$. Recall that, for each $\ell \in \mathbb{Z}$ and for each $i \in \mathbb{N}$, the cohomology complex vector space $H^i(Z, \mathcal{O}_Z(\ell))$ has finite dimension and is null if $i > r$. This fact allows us to define the numerical function $\chi_{\mathcal{O}_Z(1)} : \mathbb{Z} \to \mathbb{Z}$ by setting $\chi_{\mathcal{O}_Z(1)}(\ell)$ equal to the sum $\sum_{i=0}^{r}(-1)^i \dim_C H^i(Z, \mathcal{O}_Z(\ell))$. It is well-known that there exists a (unique) polynomial $\mathcal{H}_Z$ in one indeterminate with coefficients in $\mathbb{Q}$ of degree r such that $\chi_{\mathcal{O}_Z(1)}(\ell) = \mathcal{H}_Z(\ell)$ for each $\ell \in \mathbb{Z}$ (see Proposition I.7.3 and Exercise III.5.2 of [26]). Furthermore, the leading coefficient of $\mathcal{H}_Z$ is equal to $\deg(Z)/r! > 0$. The polynomial $\mathcal{H}_Z$ is called the Hilbert polynomial of $\mathcal{O}_Z$ with respect to $\mathcal{O}_Z(1)$. For short, we call $\mathcal{H}_Z$ the *Hilbert polynomial of* Z. We denote by ω_Z the canonical complex line bundle of Z.

In the next lemma, we collect some well-known facts concerning the bundles $\mathcal{O}_Z(h)$.

LEMMA 2.2. *Let $Z \subset \mathbb{P}^n(C)$ be a projective complex algebraic manifold of complex dimension r, let δ be its degree* $\deg(Z)$ *and let $\kappa := (r+1)(\delta-1)$. Then:*

- (i) *$\omega_Z \otimes \mathcal{O}_Z(h)$ is ample for each $h \geq r+2$.*
- (ii) *$H^i(Z, \mathcal{O}_Z(h)) = 0$ for each $i \geq 1$ and for each $h \geq \kappa$.*
- (iii) *The natural restriction map $H^0(\mathbb{P}^n(C), \mathcal{O}_{\mathbb{P}^n(C)}(h)) \to H^0(Z, \mathcal{O}_Z(h))$ is surjective for each $h \geq \kappa$.*
- (iv) *$\mathcal{H}_Z(h) = \dim_C H^0(Z, \mathcal{O}_Z(h))$ for each $h \geq \kappa$.*

Proof. Example 1.5.35 of [38] contains (i). Point (ii) is an immediate consequence of Mumford's bound for the Cartan–Serre–Grothendieck theorem (see Theorem 1.2.6, Definitions 1.8.1 and 1.8.28, and Example 1.8.48 of [38] for details). Points (iii) and (iv) follow at once from (ii). ∎

2.4. Restrictions with finite fibers. Let $R^+ := \{t \in R \mid t > 0\}$. Given $n \in \mathbb{N}^*$, $x \in R^n$ and $\varepsilon \in R^+$, we denote by $B_n(x, \varepsilon)$ the open ball of R^n centered at x with radius ε.

We conclude this section by proving a useful result.

LEMMA 2.3. *Let $m, b \in \mathbb{N}^*$ with $m \geq b+2$, let $\varepsilon \in R^+$, let Ω be a nonempty proper Zariski open subset of C^m, let $w \in R^m \setminus \Omega$ and let $u : \Omega \to \mathcal{M}$ be a complex regular map from Ω to a quasi-projective complex algebraic variety $\mathcal{M}$ such that*

$$\dim_C u^{-1}(u(y)) \leq m - b - 2 \quad \textit{for each } y \in \Omega.$$

Then there exists a regular map $\phi : R^b \to R^m$ with the following properties:

- (i) *$\phi(0) = w$, $\phi(R^b \setminus \{0\}) \subset \Omega(R)$ and $\phi(R^b) \subset B_m(w, \varepsilon)$.*
- (ii) *All the fibers of the map $u_\phi : R^b \setminus \{0\} \to \mathcal{M}$ defined by setting $u_\phi(y) := u(\phi(y))$ are finite.*

Proof. We subdivide the proof into two steps.

STEP I. We begin by proving that, up to restricting Ω, there exists a complex polynomial embedding $g : C^{b+2} \to C^m$ defined over R such that

$g(0)=w$, $g(C^{b+2})\cap\Omega\neq\emptyset$ and the set $g(C^{b+2})\cap u^{-1}(u(y))$ is finite for each $y\in g(C^{b+2})\cap\Omega$.

For simplicity, we may assume that $w=0$. Up to replacing Ω with a smaller nonempty Zariski open subset of C^m and $\mathcal{M}$ with one of its Zariski locally closed subsets, we may suppose that $\Omega\cap(\{0\}\times C^{m-1})=\emptyset$, $\mathcal{M}$ is nonsingular and u is surjective. Furthermore, by using Sard's lemma, we may also suppose that u is submersive. In this way, for each $p\in\mathcal{M}$, $u^{-1}(p)$ is a nonempty nonsingular Zariski closed subset of Ω of dimension $m-c$ for some $c\in\{b+2,\dots,m\}$.

If $c=m$, then it suffices to choose a complex linear injection $g: C^{b+2}\to C^m$ defined over R such that $g(C^{b+2})\cap\Omega\neq\emptyset$.

Let $c<m$. Since $\Omega\cap(\{0\}\times C^{m-1})=\emptyset$, we can define the map $\chi:\Omega\to C^{m-1}$ by $\chi(y_1,\dots,y_m):=(y_2,\dots,y_m)\cdot y_1^{-1}$. Observe that, given $y\in\Omega$, $(1,\chi(y))$ is the unique intersection point between the complex vector line of C^m through y and the affine hyperplane $\{1\}\times C^{m-1}$ of C^m. It follows that, for each $p\in\mathcal{M}$, $\dim_C\chi(u^{-1}(p))$ is equal to either $m-c$ or $m-c-1$.

Suppose that there exists $q\in\mathcal{M}$ such that $\dim_C\chi(u^{-1}(q))=m-c$. Thanks to Bertini's theorem, there exists a complex vector subspace H of C^m of dimension c defined over R which intersects $u^{-1}(q)$ transversely in at least one point. Hence q is a regular value of the restriction $u^*:\Omega\cap H\to\mathcal{M}$ of u to $\Omega\cap H$. In this way, applying Sard's lemma to u^*, we obtain a nonempty Zariski open subset $\mathcal{N}$ of $\mathcal{M}$ such that, for each $p\in\mathcal{N}$, H intersects $u^{-1}(p)$ transversely in a nonempty finite set. Now, replacing Ω with $u^{-1}(\mathcal{N})$, one can obtain g as in the case $c=m$.

Finally, suppose that $\dim_C\chi(u^{-1}(p))=m-c-1$ for each $p\in\mathcal{M}$. Proceeding as above, we obtain a complex vector subspace H' of C^m of dimension $c+1$ defined over R and a nonempty Zariski open subset $\mathcal{N}'$ of $\mathcal{M}$ such that, for each $p\in\mathcal{N}'$, H' intersects $u^{-1}(p)$ transversely in C^m. It follows that, for each $p\in\mathcal{N}'$, $H'\cap u^{-1}(p)$ is equal to the nonempty intersection between Ω and a finite union of complex vector lines of C^m. Let us restrict Ω by setting $\Omega:=u^{-1}(\mathcal{N}')$. By applying a complex linear automorphism of $C^m=C^{c+1}\times C^{m-c-1}$ defined over R, we may suppose that H' is equal to $C^{c+1}\times\{0\}$ and that $(C^{b+3}\times\{0\})\cap\Omega\neq\emptyset$. Bearing in mind the latter fact, if we define $g:C^{b+2}\to C^m$ by $g(z_1,\dots,z_{b+2}):=(z_1,\dots,z_{b+2},az_1^2,0,\dots,2)$ with $a\in R$ sufficiently general, then $g(C^{b+2})\cap\Omega\neq\emptyset$ and $g(C^{b+2})\cap u^{-1}(u(y))$ is finite for each $y\in g(C^{b+2})\cap\Omega$, as desired.

STEP II. Let g be as above, let $\varepsilon'\in R^+$ be such that $g(B_{b+2}(0,\varepsilon'))\subset B_m(0,\varepsilon)$ and let E be the proper real algebraic subset of R^{b+2} defined by setting $E:=R^{b+2}\setminus g^{-1}(\Omega)$. Observe that $0\in E$. Since E is proper, its tangent cone at 0 is not the whole R^{b+2}. In this way, we may suppose that the vector $e_{b+2}=(0,\dots,0,1)$ of R^{b+2} is not contained in such a tangent cone.

It follows that, for some sufficiently small $\varepsilon'' \in R^+$, the "cusp embedding" $\phi^* : B_{b+1}(0, \varepsilon'') \to B_{b+2}(0, \varepsilon')$ sending $(w_1, \ldots, w_{b+1})$ to $(w_1^3, \ldots, w_{b+1}^3, w_1^2 + \cdots + w_{b+1}^2)$, is a well-defined injective regular map such that $\phi^*(0) = 0$ and $\phi^*(B_{b+1}(0, \varepsilon'')\setminus\{0\}) \cap E = \emptyset$, that is, $\phi^*(B_{b+1}(0, \varepsilon'')\setminus\{0\}) \subset g^{-1}(\Omega)$. Up to a homothety of R^{b+1}, we may suppose that $\varepsilon'' = 2$. Let S be the sphere of R^{b+1} with center $e_{b+1} = (0, \ldots, 0, 1)$ and radius 1, let $S' := S \setminus \{2e_{b+1}\}$ and let $\phi^{**} : R^b \to S'$ be the inverse stereographic projection, which sends $y \in R^b$ to the unique intersection point between S' and the affine line of $R^{b+1} = R^b \times R$ containing $2e_{b+1}$ and $(y, 0)$. By construction, the map $\phi : R^b \to R^m$ sending y to $g(\phi^*(\phi^{**}(y)))$ is a well-defined regular map with the desired properties. ∎

3. Proof of Theorem 1.3

3.1. Proof in the bounded case. In this subsection, given a bounded real algebraic manifold V of positive dimension and an arbitrarily large integer b, we construct, in a rather explicit way, an algebraic real-deformation of V almost perfectly parametrized by R^b.

Here is a detailed reformulation of Theorem 1.3:

THEOREM 3.1. *Let V be a bounded real algebraic manifold of positive dimension, let $b \in \mathbb{N}^*$ and let d be an odd integer ≥ 3. Then there exist $M \in \mathbb{N}^*$, a real algebraic submanifold $\mathcal{V}$ of $R^b \times V \times R^M \times R \times R \times R$ and regular maps $\phi_1 : R^b \to R^M$, $\phi_2 : R^b \to R$, $G_1 : R^b \times V \to R$ and $G_2 : R^b \times V \times R \to R$ with the following properties:*

(i) *$G_1(R^b \times V) \subset (0, 2)$ and $G_2(R^b \times V \times (0, 2)) \subset R^+$.*
(ii) *$G_1(0, x) = 1$ and $G_2(0, x, 1) = 1$ for each $x \in V$.*
(iii) *$\mathcal{V}$ is equal to the subset of $R^b \times V \times R^M \times R \times R \times R$ consisting of points $(y, x, \mathfrak{a}, s, t, v)$ such that $\mathfrak{a} = \phi_1(y)$, $s = \phi_2(y)$, $t^d = G_1(y, x)$ and $v^d = G_2(y, x, t)$. In particular, $\mathcal{V}$ is the graph of a Nash map from $R^b \times V$ to $R^M \times R \times R \times R$.*
(iv) *The projection $\pi : \mathcal{V} \to R^b$ sending $(y, x, \mathfrak{a}, s, t, v)$ to y is an algebraic real-deformation of V almost perfectly parametrized by R^b.*

Our proof of this result requires nine steps. In order to make the reading of the proof easier, we first give an informal description of these steps. Some notations used here do not coincide with the ones employed in the postponed rigorous proof.

Steps of the proof. Step I. First, by Lemma 2.1, we embed V into some R^n in such a way that its Zariski closure Z in $\mathbb{P}^n(C)$ is nonsingular. Then, by using point (i) of Lemma 2.2, we choose an integer $k_1 \in \mathbb{N}^*$ so large that $\omega_Z \otimes \mathcal{O}_Z(k_1(d-1))$ is ample.

Step II. By Bertini's theorem, there exist a homogeneous polynomial $\mathrm{P} \in C[X_0, X]_{k_1 d} = C[X_0, X_1, \ldots, X_n]_{k_1 d}$ with coefficients in R and a finite subset F of C defined over R and containing 0 such that: if P_s denotes the homogeneous polynomial $X_0^{k_1 d} + s\mathrm{P}(X_0, X)$ and D_s is the vanishing set of P_s in $\mathbb{P}^n(C)$ for each $s \in C$, then P_s is irreducible and D_s is a nonsingular divisor of $\mathbb{P}^n(C)$ transverse to Z for each $s \in C \setminus F$. Define $S := C \setminus F$. Observe that the divisor D_0 of $\mathbb{P}^n(C)$ is equal to the hyperplane at infinity $\{X_0 = 0\}$ of C^n in $\mathbb{P}^n(C)$ counted with multiplicity $k_1 d > 1$. Now, for each $s \in C$, we perform the simple d-cyclic covering $\pi_{n,k_1,s} : Z(s) \to Z$ of Z branched along $Z \cap \mathrm{D}_s$. It follows that $\{Z(s)\}_{s \in C}$ is an "algebraic" family of projective complex algebraic varieties, singular over $s = 0$, which turns out to be a true algebraic family of projective complex algebraic manifolds over S. A crucial point here is that the above-mentioned singularity over $s = 0$ disappears in the real setting. In fact, since $V \cap \mathrm{D}_0 = \emptyset$, there exists an open neighborhood U of 0 in R such that, for each $s \in U$, $V \cap \mathrm{D}_s = \emptyset$ or, equivalently, P_s does not vanish on V. Since d is odd and $V \subset R^n$, we infer that, over $U \times V$, the real algebraic variety $\mathrm{Z} := \bigcup_{s \in R}(\{s\} \times Z(s)(R))$ is nonsingular and coincides with the graph of the Nash function $g : U \times V \to R$ sending (s, x) to $\sqrt[d]{\mathrm{P}_s(1, x)}$. In particular, it follows that $g(0, x) = 1$ for each $x \in V$ and hence the real part $\pi_{n,k_1,0}(R) : Z(0)(R) \to V$ of $\pi_{n,k_1,0}$ is a biregular isomorphism.

Finally, since $\omega_Z \otimes \mathcal{O}_Z(k_1(d-1))$ is ample, it follows that $\omega_{Z(s)}$ is ample for each $s \in S$.

Step III. By using the complex biregular embeddings $\mathfrak{V}_n^*$ defined in (2.1), we embed each $Z(s)$ in a suitable $\mathbb{P}^m(C)$ in such a way that $Z(s)(R)$ is contained in R^m. Each variety $Z(s)$ is now renamed as $Z'(s)$.

Step IV. Since S is connected with respect to the euclidean topology and $\{Z'(s)\}_{s \in S}$ is an algebraic family of projective complex algebraic manifolds with ample canonical complex line bundles, it is known that all the $Z'(s)$'s have the same Hilbert polynomial. Combining this fact with Lemma 2.2, we infer at once the existence of an integer $k_2 \in \mathbb{N}^*$ so large that, for each $s \in S$, the following statements hold:

$$\omega_{Z'(s)} \otimes \mathcal{O}_{Z'(s)}(k_2(d-1)) \text{ is ample,} \tag{3.1}$$

$$\dim_C H^0(Z'(s), \mathcal{O}_{Z'(s)}(k_2 d)) \geq b + 3, \tag{3.2}$$

and the natural restriction map

$$H^0(\mathbb{P}^m(C), \mathcal{O}_{\mathbb{P}^m(C)}(k_2 d)) \to H^0(Z'(s), \mathcal{O}_{Z'(s)}(k_2 d)) \tag{3.3}$$

is surjective.

Step V. We use again the argument of Step II. Let $M := \binom{m+k_2 d}{m}$. Identify each polynomial in $C[W_0, W_1, \ldots, W_m]_{k_2 d}$ with the M-uple in C^M

formed by its coefficients, ordered in some fixed way. Given $\mathfrak{a} \in C^M$, denote by $Q_{\mathfrak{a}}$ the polynomial in $C[W_0, W_1, \ldots, W_m]_{k_2 d}$ corresponding to $\mathfrak{a}$ via such an identification and by $E_{\mathfrak{a}}$ the vanishing set of $Q_{\mathfrak{a}}$ in $\mathbb{P}^m(C)$. Let E be the element of C^M such that Q_{E} is equal to $W_0^{k_2 d}$. There exists a nonempty Zariski open subset Ω of $(C^M \setminus \{0\}) \times S$ defined over R such that, for each $(\mathfrak{a}, s) \in \Omega$, $Q_{\mathfrak{a}}$ is irreducible and $E_{\mathfrak{a}}$ is nonsingular and transverse to $Z'(s)$ in $\mathbb{P}^m(C)$. Evidently, Ω does not contain $(\mathrm{E}, 0)$.

Now, for each $(\mathfrak{a}, s) \in C^M \times C$, we perform the simple d-cyclic covering $\pi_{m,k_2,\mathfrak{a},s} : Z'(\mathfrak{a}, s) \to Z'(s)$ of $Z'(s)$ branched along $Z'(s) \cap E_{\mathfrak{a}}$. It follows that $\{Z'(\mathfrak{a}, s)\}_{(\mathfrak{a},s) \in C^M \times C}$ is an "algebraic" family of projective complex algebraic varieties, singular over $(\mathfrak{a}, s) = (\mathrm{E}, 0)$, which turns out to be a true algebraic family of projective complex algebraic manifolds over Ω. As above, the real part $\{Z'(\mathfrak{a}, s)(R)\}_{(\mathfrak{a},s) \in R^M \times R}$ of such a family has good properties over $(\mathrm{E}, 0)$: the real part $\pi_{m,k_2,\mathrm{E},0}(R) : Z'(\mathrm{E}, 0)(R) \to Z'(0)(R)$ of $\pi_{m,k_2,\mathrm{E},0}$ is a biregular isomorphism (in particular $Z'(\mathrm{E}, 0)(R)$ is biregularly isomorphic to V) and there exists an open neighborhood U' of $(\mathrm{E}, 0)$ in $R^M \times R$ such that, over $U' \times V$, the real algebraic variety $\mathrm{Z}' := \bigcup_{(\mathfrak{a},s) \in R^M \times R} (\{(\mathfrak{a}, s)\} \times Z'(\mathfrak{a}, s)(R))$ is nonsingular and coincides with the graph of a Nash map. Furthermore, (3.1) ensures that $\{Z'(\mathfrak{a}, s)\}_{(\mathfrak{a},s) \in \Omega}$ is an algebraic family of projective complex algebraic manifolds with ample canonical complex line bundles. This allows us to apply the theory of coarse moduli spaces, obtaining a complex regular map $u : \Omega \to \mathcal{M}$ from Ω to a quasi-projective complex algebraic variety $\mathcal{M}$, which is able to distinguish the complex birational classes of the $Z'(\mathfrak{a}, s)$'s: for each $(\mathfrak{a}, s), (\mathfrak{a}', s') \in \Omega$, $Z'(\mathfrak{a}, s)$ is complex birationally isomorphic to $Z'(\mathfrak{a}', s')$ if and only if $u(\mathfrak{a}, s) = u(\mathfrak{a}', s')$.

Step VI. In this step, we prove that

$$\sup_{(\mathfrak{a},s) \in \Omega} \dim_C u^{-1}(u(\mathfrak{a}, s)) \leq M - b - 1 = (M + 1) - b - 2. \tag{3.4}$$

This uniform upper bound for the complex dimension of the fibers of u is the reason why we have performed the construction concerning simple d-cyclic coverings twice, first in Step II and then in Step V.

Let $(\mathfrak{a}, s) \in \Omega$ and let $s' \in S$. Define $\mathrm{U}_{s'}$ as the subset of C^M consisting of all points $\mathfrak{a}'$ such that $(\mathfrak{a}', s') \in \Omega$ and there exists a complex biregular isomorphism $\varphi_{\mathfrak{a}'}$ from $Z'(\mathfrak{a}, s)$ to $Z'(\mathfrak{a}', s')$. Since $u^{-1}(u(\mathfrak{a}, s)) \cap (C^M \times \{s'\}) = \mathrm{U}_{s'} \times \{s'\}$, s' varies in S and $\dim_C S = 1$, in order to prove (3.4), it suffices to see that

$$\dim_C \mathrm{U}_{s'} \leq M - b - 2. \tag{3.5}$$

Let us prove the latter inequality. Consider the complex linear map $\mathrm{r} : C^M = H^0(\mathbb{P}^m(C), \mathcal{O}_{\mathbb{P}^m(C)}(k_2 d)) \to H^0(Z'(s'), \mathcal{O}_{Z'(s')}(k_2 d))$ sending $\mathfrak{a}'$ to the restriction of the section $\sigma_{m,k_2 d}(Q_{\mathfrak{a}'})$ to $Z'(s')$ (see (2.3) for the definition of σ_{m,k_2d}). If $\mathfrak{a}' \in \mathrm{U}_{s'}$, then $(\mathfrak{a}', s') \in \Omega$ and hence $\mathrm{r}(\mathfrak{a}') \neq 0$. Furthermore, if $\mathfrak{a}''$ is another element of $\mathrm{U}_{s'}$, then $\mathrm{r}(\mathfrak{a}')$ is proportional to $\mathrm{r}(\mathfrak{a}'')$ if and only if $Z'(s') \cap E_{\mathfrak{a}'} = Z'(s') \cap E_{\mathfrak{a}''}$. Thanks to (3.2) and (3.3), the fibers of r have complex dimension $\leq M - b - 3$. In this way, in order to prove (3.5), it suffices to show that $\mathrm{r}(\mathrm{U}_{s'})$ is contained in a finite union of complex vector lines of $H^0(Z'(s'), \mathcal{O}_{Z'(s')}(k_2d))$. Suppose this is false. Then there exists a sequence $\{\mathfrak{a}_i\}_{i\in\mathbb{N}}$ in $\mathrm{U}_{s'}$ such that $\mathrm{r}(\mathfrak{a}_i)$ is not proportional to $\mathrm{r}(\mathfrak{a}_j)$ (or, equivalently, $Z'(s') \cap E_{\mathfrak{a}_i} \neq Z'(s') \cap E_{\mathfrak{a}_j}$) for each $i, j \in \mathbb{N}$ with $i \neq j$. For each $i \in \mathbb{N}$, the composition map $f_i := \pi_{m,k_2,\mathfrak{a}_i,s'} \circ \varphi_{\mathfrak{a}_i} : Z'(\mathfrak{a}, s) \to Z'(s')$ is a surjective complex regular map having $Z'(s') \cap E_{\mathfrak{a}_i}$ as branched locus. Since $Z'(s') \cap E_{\mathfrak{a}_i} \neq Z'(s') \cap E_{\mathfrak{a}_j}$ if $i \neq j$, the maps f_i and f_j are different and hence the set $\{f_i\}_{i\in\mathbb{N}}$ is infinite. This is impossible by a classical finiteness result, because $Z'(s')$ is of general type (see [32]). Inequality (3.4) is proved.

Step VII. We describe explicitly each real algebraic variety $Z'(\mathfrak{a}, s)(R)$ as follows. Since $Z(R) = V \subset R^n$ (see Step I) and $Z'(s)(R) \subset R^m$ for each $s \in R$ (see Step III), up to biregular isomorphism, we can write

$$Z'(\mathfrak{a}, s) = \{(x, t, v) \in V \times R \times R \mid t^d = 1 + s\mathrm{P}(1, x),\ v^d = Q_{\mathfrak{a}}(1, \psi_1(x, t))\}$$

for each $(\mathfrak{a}, s) \in R^M \times R$, where P is the homogeneous polynomial considered in Step II and $\psi_1 : R^n \times R \to R^m$ is a certain biregular embedding.

Step VIII. Thanks to (3.4), we can apply Lemma 2.3 obtaining a regular map $\phi = (\phi_1, \phi_2) : R^b \to R^M \times R$ such that $\phi(0) = (\mathrm{E}, 0)$, $\phi(R^b \setminus \{0\}) \subset \Omega(R)$, the map $u_\phi : R^b \to \mathcal{M}$ sending y to $u(\phi(y))$ has finite fibers and the family $\{Z'(\phi(y))(R)\}_{y\in R^b}$ satisfies points (i)–(iii) of Theorem 3.1 with maps G_1 and G_2 defined as follows: $G_1(y, x) := 1 + \phi_2(y)\mathrm{P}(1, x)$ and $G_2(y, x, t) := Q_{\phi_1(y)}(1, \psi_1(x, t))$.

Step IX. By combining the property of u stated at the end of Step V, the above–mentioned properties of ϕ and the fact that each birational isomorphism between real algebraic manifolds extends to a complex birational isomorphism between anyway fixed complexifications of those manifolds, it follows easily that $\{Z'(\phi(y))(R)\}_{y\in R^b}$ is an algebraic real-deformation of V almost perfectly parametrized by R^b, completing the proof.

Given a set S, denote by $\mathrm{id}_S : S \to S$ the identity map on S.

Proof of Theorem 3.1. Suppose that V is a bounded real algebraic submanifold of some R^n of positive dimension r. As outlined above, the proof is divided into nine steps.

Step I. Thanks to Lemma 2.1, we may suppose that the Zariski closure Z of V in $\mathbb{P}^n(C)$ is nonsingular. In particular,

$$Z(R) = V \text{ is contained in } R^n. \tag{3.6}$$

By Lemma 2.2(i), $\omega_Z \otimes \mathcal{O}_Z(h)$ is ample for each $h \geq r+2$. Choose $k_1 \in \mathbb{N}^*$ in such a way that $k_1(d-1) \geq r+2$. In this way, we find that

$$\omega_Z \otimes \mathcal{O}_Z(k_1(d-1)) \text{ is ample.} \tag{3.7}$$

Step II. Let $N = \binom{n+k_1 d}{n}$ be the cardinality of $\mathbb{N}^{n+1}_{k_1 d}$ and let ω be a bijective map from $\{1, \ldots, N\}$ to $\mathbb{N}^{n+1}_{k_1 d}$ such that $\omega(1) = (k_1 d, 0, \ldots, 0)$. For each $a = (a_1, \ldots, a_N) \in C^N$, define the polynomial $P_a(X_0, X)$ in $C[X_0, X]_{k_1 d} = C[X_0, X_1, \ldots, X_n]_{k_1 d}$ by setting

$$P_a(X_0, X) := \sum_{j=1}^{N} a_j (X_0, X)^{\omega(j)},$$

where $(X_0, X)^{\omega(j)}$ denotes the monomial $X_0^{\alpha_0} \cdot X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ if $\omega(j)$ is equal to $(\alpha_0, \alpha_1, \ldots, \alpha_n)$. Identify C^N with $C[X_0, X]_{k_1 d}$ via the complex vector isomorphism sending a to P_a. Observe that, if $\mathrm{e} = (1, 0, \ldots, 0)$ denotes the first element of the canonical base of C^N, then

$$P_{\mathrm{e}}(X_0, X) \text{ is equal to } X_0^{k_1 d}. \tag{3.8}$$

Let A be the subset of $C^N \setminus \{0\}$ consisting of all elements a such that the polynomial P_a is irreducible and its vanishing set D_a in $\mathbb{P}^n(C)$ is a nonsingular divisor of $\mathbb{P}^n(C)$ which intersects Z transversely; that is, $D_a \cap Z \neq \emptyset$ and D_a is transverse to Z in $\mathbb{P}^n(C)$. Since $r \geq 1$, Bertini's theorem ensures that A is a nonempty Zariski open subset of C^N. Evidently, e does not belong to A. Choose a point v in $A(R)$. Define the complex affine embedding $L : C \to C^N$ by setting

$$L(s) := \mathrm{e} + s(\mathrm{v} - \mathrm{e}) \tag{3.9}$$

and $S := L^{-1}(A)$. Observe that S is equal to C with a finite set (containing the origin) removed.

Denote by τ the tautological section of $(\pi_{n,k_1})^*(\mathcal{O}_{\mathbb{P}^n(C)}(k_1))$ and by τ^d the complex regular section $\tau^{\otimes d}$ of $(\pi_{n,k_1})^*(\mathcal{O}_{\mathbb{P}^n(C)}(k_1 d))$ (recall that $\pi_{n,k_1} : \mathcal{L}_{\mathbb{P}^n(C)}(k_1) \to \mathbb{P}^n(C)$ denotes the bundle projection of $\mathcal{O}_{\mathbb{P}^n(C)}(k_1)$). Let $\mathcal{Z}$ be the complex algebraic subvariety of $C \times \mathcal{L}_{\mathbb{P}^n(C)}(k_1)$ consisting of points (s, ν) such that

$$\pi_{n,k_1}(\nu) \in Z \quad \text{and} \quad \tau^d(\nu) = (\pi_{n,k_1})^*(\sigma_{n,k_1 d}(P_{L(s)}))(\nu) \tag{3.10}$$

(see (2.3) for the definition of $\sigma_{n,k_1 d}$), let $\mathit{\Pi} : \mathcal{Z} \to C$ be the projection sending (s, ν) to s and, for each $s \in C$, let $Z(s)$ be the complex algebraic subvariety of $\mathcal{L}_{\mathbb{P}^n(C)}(k_1)$ such that $\mathit{\Pi}^{-1}(s) = \{s\} \times Z(s)$. Observe that, for each $s \in S$, $D_{L(s)}$ intersects Z transversely in $\mathbb{P}^n(C)$ and hence $Z(s)$ is nonsingular and the restriction $\pi_{n,k_1,s} : Z(s) \to Z$ of π_{n,k_1} from $Z(s)$ to Z is a simple d-cyclic covering (see [3, Section I.17] or [14] for basic results concerning simple d-cyclic coverings). It is immediate to verify that $\mathit{\Pi}^{-1}(S)$

is nonsingular and Zariski closed in $S \times \mathcal{L}_{\mathbb{P}^n(C)}(k_1)$, and the restriction of Π from $\Pi^{-1}(S)$ to S is a submersion.

Let $s \in S$. Lemma 17.1(iii) of [3, p. 55] asserts that $\omega_{Z(s)}$ is bundle isomorphic to $(\pi_{n,k_1,s})^*(\omega_Z \otimes \mathcal{O}_Z(k_1(d-1)))$. On the other hand, by construction, $\pi_{n,k_1,s}$ is a finite complex regular map and, by (3.7), $\omega_Z \otimes \mathcal{O}_Z(k_1(d-1))$ is ample. In this way, by applying Proposition 1.2.13 of [38] (or Theorem 1.19 of [18]), we infer that

$$\omega_{Z(s)} \text{ is ample for each } s \in S. \tag{3.11}$$

Step III. Let $\mathcal{P} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^n(C)}(k_1) \otimes \mathcal{O}_{\mathbb{P}^n(C)})$ be the $\mathbb{P}^1(C)$-bundle over $\mathbb{P}^n(C)$ obtained by performing the projective closure of each fiber of $\mathcal{O}_{\mathbb{P}^n(C)}(k_1)$, let $\overline{\mathcal{L}}_{\mathbb{P}^n(C)}(k_1)$ be the total space of $\mathcal{P}$ and let $\eta : \mathcal{L}_{\mathbb{P}^n(C)}(k_1) \to \overline{\mathcal{L}}_{\mathbb{P}^n(C)}(k_1)$ be the inclusion map. The equation $\tau^d = (\pi_{n,k_1})^*(\sigma_{n,k_1 d}(P_{L(s)}))$ in definition (3.10) of $\mathcal{Z}$ ensures that $(\mathrm{id}_C \times \eta)(\mathcal{Z})$ is Zariski closed in $C \times \overline{\mathcal{L}}_{\mathbb{P}^n(C)}(k_1)$. It is well-known that the total space $\overline{\mathcal{L}}_{\mathbb{P}^n(C)}(k_1)$ of $\mathcal{P}$ is a projective complex algebraic manifold defined over R. Fix a complex biregular embedding $\varsigma : \overline{\mathcal{L}}_{\mathbb{P}^n(C)}(k_1) \to \mathbb{P}^\ell(C)$ defined over R of $\overline{\mathcal{L}}_{\mathbb{P}^n(C)}(k_1)$ into some $\mathbb{P}^\ell(C)$. Let $m := \binom{\ell+2}{\ell}$ and let $\mathfrak{V}^*_\ell : \mathbb{P}^\ell(C) \to \mathbb{P}^m(C)$ be the complex biregular embedding defined in (2.1). Define the map $\psi : \mathcal{L}_{\mathbb{P}^n(C)}(k_1) \to \mathbb{P}^m(C)$ by setting

$$\psi := \mathfrak{V}^*_\ell \circ \varsigma \circ \eta, \tag{3.12}$$

the map $\Psi : \mathcal{Z} \to C \times \mathbb{P}^m(C)$ as the restriction of $\mathrm{id}_C \times \psi : C \times \mathcal{L}_{\mathbb{P}^n(C)}(k_1) \to C \times \mathbb{P}^m(C)$ to $\mathcal{Z}$ and the subset $\mathcal{Z}'$ of $C \times \mathbb{P}^m(C)$ as $\Psi(\mathcal{Z})$. Since $(\mathrm{id}_C \times \eta)(\mathcal{Z})$ is Zariski closed in $C \times \overline{\mathcal{L}}_{\mathbb{P}^n(C)}(k_1)$ and $\mathcal{Z}' = (\mathrm{id}_C \times (\mathfrak{V}^*_\ell \circ \varsigma))((\mathrm{id}_C \times \eta)(\mathcal{Z}))$, it follows that $\mathcal{Z}'$ is Zariski closed in $C \times \mathbb{P}^m(C)$ and hence Ψ is a complex biregular embedding defined over R.

Moreover, thanks to property (2.2) of $\mathfrak{V}^*_\ell$, we know that

$$\mathcal{Z}'(R) \text{ is contained in } R \times R^m. \tag{3.13}$$

For each $s \in C$, define the complex algebraic subvariety $Z'(s)$ of $\mathbb{P}^m(C)$ by setting

$$Z'(s) := \psi(Z(s)).$$

Let $\Pi' : \mathcal{Z}' \to C$ be the projection sending (s, q) to s. Observe that $(\Pi')^{-1}(s) = \{s\} \times Z'(s)$ for each $s \in C$, $Z'(s)$ is nonsingular for each $s \in S$ and the restriction $\Pi^* : \mathcal{Z}^* \to S$ of Π' from $\mathcal{Z}^* := (\Pi')^{-1}(S)$ to S is an algebraic family of projective complex algebraic manifolds. By definition, for each $s \in S$, $Z'(s)$ is complex biregularly isomorphic to $Z(s)$. In this way, by (3.11), we infer that

$$\omega_{Z'(s)} \text{ is ample for each } s \in S. \tag{3.14}$$

STEP IV. For each $s \in S$, let $\mathcal{H}_s$ be the Hilbert polynomial of $Z'(s)$ and let δ_s be the degree of $Z'(s)$ in $\mathbb{P}^m(C)$. Since Π^* is an algebraic family of projective complex algebraic manifolds with ample canonical complex line bundles and S is connected with respect to the euclidean topology, the Hilbert polynomial $\mathcal{H}_s$ (and hence δ_s) does not depend on $s \in S$. Define $\mathcal{H} := \mathcal{H}_s$ and $\delta := \delta_s$ for some (and hence for all) $s \in S$. Let $\kappa := (r+1)(\delta-1)$. Since $\deg(\mathcal{H}) = r > 0$ and the leading coefficient of $\mathcal{H}$ is $\delta/r! > 0$, there exists $h_0 \in \mathbb{N}^*$ such that $h_0 \geq \kappa$ and $\mathcal{H}(h) \geq b+3$ for each $h \geq h_0$. By Lemma 2.2(iv), it follows that

$$\dim_C H^0(Z'(s), \mathcal{O}_{Z'(s)}(h)) = \mathcal{H}_s(h) = \mathcal{H}(h) \geq b+3$$

for each $s \in S$ and for each $h \geq h_0$. Point (i) of Lemma 2.2 implies that $\omega_{Z'(s)} \otimes \mathcal{O}_{Z'(s)}(h)$ is ample for each $h \geq r+2$ and for each $s \in S$. Moreover, thanks to point (iii) of the same lemma, the natural restriction map

$$\rho_{s,h} : H^0(\mathbb{P}^m(C), \mathcal{O}_{\mathbb{P}^m(C)}(h)) \to H^0(Z'(s), \mathcal{O}_{Z'(s)}(h))$$

is surjective for each $h \geq \kappa$ and for each $s \in S$. Choose $k_2 \in \mathbb{N}^*$ in such a way that $k_2(d-1) \geq r+2$ and $k_2 d \geq h_0 (\geq \kappa)$. We have just proved that

(3.15) $\qquad \omega_{Z'(s)} \otimes \mathcal{O}_{Z'(s)}(k_2(d-1))$ is ample for each $s \in S$,

(3.16) $\qquad \dim_C H^0(Z'(s), \mathcal{O}_{Z'(s)}(k_2 d)) \geq b+3 \quad$ for each $s \in S$,

(3.17) $\qquad \rho_{s,k_2 d}$ is surjective for each $s \in S$.

STEP V. Let $M = \binom{m+k_2 d}{m}$ be the cardinality of $\mathbb{N}^{m+1}_{k_2 d}$ and let ξ be a bijective map from $\{1, \ldots, M\}$ to $\mathbb{N}^{m+1}_{k_2 d}$ such that $\xi(1) = (k_2 d, 0, \ldots, 0)$, and for each $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_M) \in C^M$ define the polynomial $Q_{\mathfrak{a}}(W_0, W)$ in $C[W_0, W]_{k_2 d} = C[W_0, W_1, \ldots, W_m]_{k_2 d}$ by setting

$$Q_{\mathfrak{a}}(W_0, W) := \sum_{j=1}^{M} \mathfrak{a}_j (W_0, W)^{\xi(j)},$$

where $(W_0, W)^{\xi(j)}$ denotes the monomial $W_0^{\alpha_0} \cdot W_1^{\alpha_1} \cdots W_m^{\alpha_m}$ if $\xi(j)$ is equal to $(\alpha_0, \alpha_1, \ldots, \alpha_m)$. Identify C^M with $C[W_0, W]_{k_2 d}$ via the complex vector isomorphism sending $\mathfrak{a}$ to $Q_{\mathfrak{a}}$. Observe that, if $\mathrm{E} = (1, 0, \ldots, 0)$ denotes the first element of the canonical base of C^M, then

(3.18) $\qquad Q_{\mathrm{E}}(W_0, W)$ is equal to $W_0^{k_2 d}$.

Let B be the nonempty Zariski open subset of $C^M \setminus \{0\}$ consisting of all elements $\mathfrak{a}$ such that the polynomial $Q_{\mathfrak{a}}$ is irreducible and its vanishing set $E_{\mathfrak{a}}$ in $\mathbb{P}^m(C)$ is nonsingular. Evidently, E does not belong to B. Define the complex algebraic subvariety Γ of $B \times \mathcal{Z}^* \subset B \times C \times \mathbb{P}^m(C)$ and the complex regular map $\Lambda : \Gamma \to B \times S$ by setting

$$\Gamma := \{(\mathfrak{a}, s, [w_0, w]) = (\mathfrak{a}, (s, [w_0, w_1, \ldots, w_m])) \in B \times \mathcal{Z}^* \mid Q_{\mathfrak{a}}(w_0, w) = 0\}$$

and

$$\Lambda(\mathfrak{a}, s, [w_0, w]) := (\mathfrak{a}, s),$$

respectively. By direct computations, which are standard in the context of Bertini-type theorems (see [31]), one can easily verify that Γ is nonsingular and a point $(\mathfrak{a}, s)$ in $B \times S$ is a regular value of Λ if and only if the nonsingular divisor $E_{\mathfrak{a}}$ of $\mathbb{P}^m(C)$ is transverse to $Z'(s)$. By applying Sard's lemma to Λ, we obtain a nonempty Zariski open subset Ω of $C^M \times C$ contained in $B \times S$ such that, for each $(\mathfrak{a}, s) \in \Omega$, $E_{\mathfrak{a}}$ is transverse to $Z'(s)$ in $\mathbb{P}^m(C)$. Observe that Ω does not contain $(\mathrm{E}, 0)$.

Denote by Υ the tautological section of $(\pi_{m,k_2})^*(\mathcal{O}_{\mathbb{P}^m(C)}(k_2))$ and by Υ^d the complex regular section $\Upsilon^{\otimes d}$ of $(\pi_{m,k_2})^*(\mathcal{O}_{\mathbb{P}^m(C)}(k_2 d))$. Let $\mathscr{Z}$ be the complex algebraic subvariety of $\mathscr{C} := C^M \times C \times \mathcal{L}_{\mathbb{P}^m(C)}(k_2)$ consisting of all points $(\mathfrak{a}, s, \upsilon)$ such that

$$(3.19) \qquad (s, \pi_{m,k_2}(\upsilon)) \in \mathcal{Z}' \quad \text{and} \quad \Upsilon^d(\upsilon) = (\pi_{m,k_2})^*(\sigma_{m,k_2 d}(Q_{\mathfrak{a}}))(\upsilon),$$

let $\mathscr{P} : \mathscr{Z} \to C^M \times C$ be the projection sending $(\mathfrak{a}, s, \upsilon)$ to $(\mathfrak{a}, s)$, and for each $(\mathfrak{a}, s) \in C^M \times C$ let $Z'(\mathfrak{a}, s)$ be the complex algebraic subvariety of $\mathcal{L}_{\mathbb{P}^m(C)}(k_2)$ such that $\mathscr{P}^{-1}((\mathfrak{a}, s)) = \{(\mathfrak{a}, s)\} \times Z'(\mathfrak{a}, s)$. For each $(\mathfrak{a}, s) \in \Omega$, $E_{\mathfrak{a}}$ intersects $Z'(s)$ transversely in $\mathbb{P}^m(C)$, so $Z'(\mathfrak{a}, s)$ is a projective complex algebraic manifold and the restriction $\pi_{m,k_2,\mathfrak{a},s} : Z'(\mathfrak{a}, s) \to Z'(s)$ of π_{m,k_2} is a simple d-cyclic covering of $Z'(s)$. Moreover, the restriction $\mathscr{P}^* : \mathscr{Z}^* \to \Omega$ of $\mathscr{P}$ from $\mathscr{Z}^* := \mathscr{P}^{-1}(\Omega)$ to Ω is an algebraic family of projective complex algebraic manifolds.

Let $(\mathfrak{a}, s) \in \Omega$. Thanks to (3.15), we know that $\omega_{Z'(s)} \otimes \mathcal{O}_{Z'(s)}(k_2(d-1))$ is ample. Bearing in mind this fact, we can apply Lemma 17.1(iii) of [3, p. 55] and Proposition 1.2.13 of [38] to $\pi_{m,k_2,\mathfrak{a},s}$. We find that $\omega_{Z'(\mathfrak{a},s)}$ is ample for each $(\mathfrak{a}, s) \in \Omega$. It follows that $\mathscr{P}^*$ is an algebraic family of projective complex algebraic manifolds with ample canonical complex line bundles. Since Ω is connected with respect to the euclidean topology, the Hilbert polynomial $\mathscr{H}_{\mathfrak{a},s}$ of $\mathcal{O}_{Z'(\mathfrak{a},s)}$ with respect to $\omega_{Z'(\mathfrak{a},s)}$ does not depend on $(\mathfrak{a}, s) \in \Omega$. Define $\mathscr{H} := \mathscr{H}_{\mathfrak{a},s}$ for some (and hence for all) $(\mathfrak{a}, s)$ in Ω. Let $\mathcal{M}$ be the coarse moduli space of all canonically polarized r-dimensional projective complex algebraic manifolds with ample canonical complex line bundle and with corresponding Hilbert polynomial equal to $\mathscr{H}$. By Theorem 1.11 of [44], $\mathcal{M}$ exists and is a quasi-projective complex algebraic variety. In this way, there exists a complex regular map $u : \Omega \to \mathcal{M}$ such that, for each $(\mathfrak{a}, s), (\mathfrak{a}', s') \in \Omega$, $u(\mathfrak{a}, s) = u(\mathfrak{a}', s')$ if and only if $Z'(\mathfrak{a}, s)$ is complex biregularly isomorphic to $Z'(\mathfrak{a}', s')$.

We remind the reader that two projective complex algebraic manifolds with ample canonical complex line bundles are complex biregularly isomorphic if and only if they are complex birationally isomorphic (see [18, p. 170]).

It follows that

$$\text{(3.20)} \qquad \begin{aligned}&\text{for each } (\mathfrak{a},s),(\mathfrak{a}',s')\in\Omega,\ u(\mathfrak{a},s)=u(\mathfrak{a}',s') \text{ if and only if}\\ &\qquad Z'(\mathfrak{a},s) \text{ is complex birationally isomorphic to } Z'(\mathfrak{a}',s').\end{aligned}$$

Step VI. Now we give an upper bound for the complex dimension of the fibers of u.

Let $\theta:\Omega\to S$ be the projection sending $(\mathfrak{a},s)$ to s, let $\lambda:\Omega\to B$ be the projection sending $(\mathfrak{a},s)$ to $\mathfrak{a}$, and for each $s\in S$ let Ω_s be the Zariski open subset of C^M contained in B defined by setting $\Omega_s:=\lambda(\theta^{-1}(s))$.

Fix $(\mathfrak{a},s)\in\Omega$ and $s'\in S$. By (3.14), $\omega_{Z'(s')}$ is ample and hence $Z'(s')$ is of general type. Denote by $\mathrm{H}_{s'}$ the set $H^0(Z'(s'),\mathcal{O}_{Z'(s')}(k_2d))\setminus\{0\}$, by $\mathbb{P}\mathrm{H}_{s'}$ the projectivization of $H^0(Z'(s'),\mathcal{O}_{Z'(s')}(k_2d))$, by $\mu:\mathrm{H}_{s'}\to\mathbb{P}\mathrm{H}_{s'}$ the corresponding natural projection, by

$$\rho':H^0(\mathbb{P}^m(C),\mathcal{O}_{\mathbb{P}^m(C)}(k_2d))\to H^0(Z'(s'),\mathcal{O}_{Z'(s')}(k_2d))$$

the natural restriction map, by K the set $H^0(\mathbb{P}^m(C),\mathcal{O}_{\mathbb{P}^m(C)}(k_2d))\setminus\ker(\rho')$, by $\rho:\mathrm{K}\to\mathrm{H}_{s'}$ the restriction of ρ' from K to $\mathrm{H}_{s'}$ and by $\mathrm{U}_{s'}$ the set $\lambda(u^{-1}(u(\mathfrak{a},s))\cap\theta^{-1}(s'))$. Observe that

$$\mathrm{U}_{s'}=\{\mathfrak{a}'\in\Omega_{s'}\mid Z'(\mathfrak{a}',s') \text{ is complex biregularly isomorphic to } Z'(\mathfrak{a},s)\}.$$

Since $\sigma_{m,k_2d}(\mathrm{U}_{s'})\subset\sigma_{m,k_2d}(\Omega_{s'})\subset\mathrm{K}$, we can define the map $\sigma:\mathrm{U}_{s'}\to\mathrm{K}$ as the restriction of σ_{m,k_2d} from $\mathrm{U}_{s'}$ to K. Consider the sequence of maps

$$\mathrm{U}_{s'}\xrightarrow{\sigma}\mathrm{K}\xrightarrow{\rho}\mathrm{H}_{s'}\xrightarrow{\mu}\mathbb{P}\mathrm{H}_{s'}$$

and two points $\mathfrak{a}'$ and $\mathfrak{a}''$ in $\mathrm{U}_{s'}$. Evidently, $(\mu\circ\rho\circ\sigma)(\mathfrak{a}')\neq(\mu\circ\rho\circ\sigma)(\mathfrak{a}'')$ is equivalent to saying that the restricted sections $\sigma_{m,k_2d}(Q_{\mathfrak{a}'})|_{Z'(s')}$ and $\sigma_{m,k_2d}(Q_{\mathfrak{a}''})|_{Z'(s')}$ are not proportional. On the other hand, the latter condition is equivalent to saying that the zero locus $Z'(s')\cap E_{\mathfrak{a}'}$ of $\sigma_{m,k_2d}(Q_{\mathfrak{a}'})|_{Z'(s')}$ and the zero locus $Z'(s')\cap E_{\mathfrak{a}''}$ of $\sigma_{m,k_2d}(Q_{\mathfrak{a}''})|_{Z'(s')}$ are different as subsets of $Z'(s')$. In this way, we have

$$\text{(3.21)} \qquad (\mu\circ\rho\circ\sigma)(\mathfrak{a}')\neq(\mu\circ\rho\circ\sigma)(\mathfrak{a}'')\ \Leftrightarrow\ Z'(s')\cap E_{\mathfrak{a}'}\neq Z'(s')\cap E_{\mathfrak{a}''}.$$

Let us prove that

$$\text{(3.22)} \qquad (\mu\circ\rho\circ\sigma)(\mathrm{U}_{s'}) \text{ is finite.}$$

Suppose this is false. Let $\{\mathfrak{a}_i\}_{i\in\mathbb{N}}$ be a subset of $\mathrm{U}_{s'}$ such that $(\mu\circ\rho\circ\sigma)(\mathfrak{a}_i)\neq(\mu\circ\rho\circ\sigma)(\mathfrak{a}_j)$ for each $i,j\in\mathbb{N}$ with $i\neq j$. By definition of $\mathrm{U}_{s'}$, for each $i\in\mathbb{N}$, there exists a complex biregular isomorphism $\varphi_i:Z'(\mathfrak{a},s)\to Z'(\mathfrak{a}_i,s')$ and hence the composition map $f_i:=\pi_{m,k_2,\mathfrak{a}_i,s'}\circ\varphi_i:Z'(\mathfrak{a},s)\to Z'(s')$ is a surjective complex regular map having $Z'(s')\cap E_{\mathfrak{a}_i}$ as branched locus. Given $i,j\in\mathbb{N}$ with $i\neq j$, by (3.21), we know that the sets $Z'(s')\cap E_{\mathfrak{a}_i}$ and $Z'(s')\cap E_{\mathfrak{a}_j}$ are different and hence the maps f_i and f_j are different as well. In this way, we have found an infinite set $\{f_i\}_{i\in\mathbb{N}}$ of surjective complex regular maps from $Z'(\mathfrak{a}, s)$ to the projective complex algebraic manifold $Z'(s')$ of general type. This is impossible by a classical finiteness theorem of S. Kobayashi and T. Ochiai [32]. By combining (3.22) and (3.17), we see that $\mathrm{U}_{s'}$ is contained in the union of a finite number of complex vector subspaces of C^M of complex dimension $M - \dim_C H^0(Z'(s'), \mathcal{O}_{Z'(s')}(k_2 d)) + 1$. Thanks to (3.16), we infer that $\dim_C(u^{-1}(u(\mathfrak{a}, s)) \cap \theta^{-1}(s')) \leq M - b - 2$ for each $(\mathfrak{a}, s) \in \Omega$ and for each $s' \in S$. Since $\dim_C S = 1$, we have

$$\dim_C u^{-1}(u(\mathfrak{a}, s)) \leq M - b - 1 = (M + 1) - b - 2 \tag{3.23}$$

for each $(\mathfrak{a}, s) \in \Omega$.

Step VII. Let us give an explicit description of the real part of $\mathscr{P} : \mathscr{Z} \to C^M \times C$ via the trivialization charts $\varphi_{n,k_1,0}$ of $\mathcal{O}_{\mathbb{P}^n(C)}(k_1)$ and $\varphi_{m,k_2,0}$ of $\mathcal{O}_{\mathbb{P}^m(C)}(k_2)$.

Let $x = (x_1, \dots, x_n)$ be the coordinates of C^n and let $\chi_1 : C^n \to U_{n,0}$ be the coordinate chart sending x to $[1, x]$. Consider the trivialization chart $\varphi_{n,k_1,0} : U_{n,0} \times C \to (\pi_{n,k_1})^{-1}(U_{n,0})$ of $\mathcal{O}_{\mathbb{P}^n(C)}(k_1)$ over $U_{n,0}$. Define the complex biregular isomorphism $g_1 : C^n \times C \to (\pi_{n,k_1})^{-1}(U_{n,0})$ and the subset $\mathbf{Z}$ of $C \times C^n \times C$ by setting $g_1 := \varphi_{n,k_1,0} \circ (\chi_1 \times \mathrm{id}_C)$ and $\mathbf{Z} := (\mathrm{id}_C \times g_1)^{-1}(\mathcal{Z})$, respectively ($\mathcal{Z}$ was defined in (3.10)). Thanks to (3.8), (3.9) and (3.10), we have

$$\mathbf{Z} = \{(s, x, t) \in C \times (Z \cap C^n) \times C \mid t^d = 1 + s(P_{\mathrm{v}}(1, x) - 1)\}.$$

Let $P(X)$ be the polynomial in $R[X] = R[X_1, \dots, X_n]$ such that $P(x) = P_{\mathrm{v}}(1, x) - 1$ for each $x \in R^n$. By (3.6), it follows that

$$\mathbf{Z}(R) = \{(s, x, t) \in R \times V \times R \mid t^d = 1 + sP(x)\}. \tag{3.24}$$

Let $w = (w_1, \dots, w_m)$ be the coordinates of C^m, let $\chi_2 : C^m \to U_{m,0}$ be the coordinate chart sending w to $[1, w]$, let $\varphi_{m,k_2,0} : U_{m,0} \times C \to (\pi_{m,k_2})^{-1}(U_{m,0})$ be the trivialization chart of $\mathcal{O}_{\mathbb{P}^m(C)}(k_2)$ over $U_{m,0}$, let $g_2 : C^m \times C \to (\pi_{m,k_2})^{-1}(U_{m,0})$ be the complex biregular isomorphism defined by setting $\varphi_{m,k_2,0} \circ (\chi_2 \times \mathrm{id}_C)$ and let $\mathfrak{Z} := (\mathrm{id}_{C^M} \times \mathrm{id}_C \times g_2)^{-1}(\mathscr{Z})$. Thanks to (3.18) and (3.19), we have

$$\mathfrak{Z} = \{(\mathfrak{a}, (s, w), v) \in C^M \times (\mathcal{Z}' \cap (C \times C^m)) \times C \mid v^d = Q_{\mathfrak{a}}(1, w)\}$$

($\mathcal{Z}'$ was defined in Step III). Thanks to (3.13), we obtain

$$\mathfrak{Z}(R) = \{(\mathfrak{a}, (s, w), v) \in R^M \times \mathcal{Z}'(R) \times R \mid v^d = Q_{\mathfrak{a}}(1, w)\}. \tag{3.25}$$

For each $(\mathfrak{a}, s) \in R^M \times R$, let $\mathfrak{Z}(R)(\mathfrak{a}, s)$ be the real algebraic subset of $R^m \times R$ such that $\mathfrak{Z}(R) \cap (\{(\mathfrak{a}, s)\} \times R^m \times R) = \{(\mathfrak{a}, s)\} \times \mathfrak{Z}(R)(\mathfrak{a}, s)$. By construction, it follows that

$$\mathfrak{Z}(R)(\mathfrak{a}, s) \text{ is biregularly isomorphic to } Z'(\mathfrak{a}, s)(R) \tag{3.26}$$

for each $(\mathfrak{a}, s) \in R^M \times R$. Define the regular map $\psi_1 : R^n \times R \to R^m$ by

$$\psi_1(x,t) := \psi(g_1(x,t)),$$

where $\psi : \mathcal{L}_{\mathbb{P}^n(C)}(k_1) \to \mathbb{P}^m(C)$ is the embedding defined in (3.12). Observe that the restriction $\varPsi_1 : \mathbf{Z}(R) \to \mathcal{Z}'(R)$ of $\mathrm{id}_R \times \psi_1 : R \times (R^n \times R) \to R \times R^m$ from $\mathbf{Z}(R)$ to $\mathcal{Z}'(R)$ is a well-defined biregular isomorphism. Define $\mathbf{V}$ as the real algebraic subset of $R^M \times (R \times V \times R) \times R$ consisting of all points $(\mathfrak{a}, (s, x, t), v)$ such that

$$t^d = 1 + sP(x) \quad \text{and} \quad v^d = Q_{\mathfrak{a}}(1, \psi_1(x,t)).$$

Thanks to (3.24) and to (3.25), we infer that the restriction of $\mathrm{id}_{R^M} \times \varPsi_1 \times \mathrm{id}_R$ from $\mathbf{V}$ to $\mathfrak{Z}(R)$ is a biregular isomorphism. For each $(\mathfrak{a}, s) \in R^M \times R$, let $\mathbf{V}(\mathfrak{a}, s)$ be the real algebraic subset of $V \times R \times R$ such that $\mathbf{V} \cap (\{(\mathfrak{a}, s)\} \times V \times R \times R) = \{(\mathfrak{a}, s)\} \times \mathbf{V}(\mathfrak{a}, s)$. By (3.26), it follows that

$$\mathbf{V}(\mathfrak{a}, s) \text{ is biregularly isomorphic to } Z'(\mathfrak{a}, s)(R) \tag{3.27}$$

for each $(\mathfrak{a}, s) \in R^M \times R$.

Step VIII. For each $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m) \in \mathbb{N}^{m+1}_{k_2 d}$, denote by $\mathrm{M}_\alpha : R^m \to R$ the polynomial function sending $w = (w_1, \ldots, w_m)$ to $(1, w)^\alpha = w_1^{\alpha_1} \cdots w_m^{\alpha_m}$ (here $w_j^{\alpha_j} = 1$ if $\alpha_j = 0$). Since V is bounded in R^n, there exists $\beta \in R^+$ such that

$$P(V) \subset (-\beta, \beta) \quad \text{and} \quad (\mathrm{M}_\alpha \circ \psi_1)(V \times [0,2]) \subset (-\beta, \beta) \tag{3.28}$$

for each $\alpha \in \mathbb{N}^{m+1}_{k_2 d}$.

Define $\varepsilon := 1/(2 + M\beta)$ (recall that $M = \binom{m+k_2 d}{m}$). Thanks to (3.23) and to Lemma 2.3, there exist a regular map $\phi_1 = (\phi_{1,1}, \phi_{1,2}, \ldots, \phi_{1,M}) : R^b \to R^M$ and a regular function $\phi_2 : R^b \to R$ such that the regular map $\phi := (\phi_1, \phi_2) : R^b \to R^M \times R$ has the following properties:

$$\phi_1(0) = \mathrm{E} \quad \text{and} \quad \phi_2(0) = 0, \tag{3.29}$$

$\phi(R^b \setminus \{0\}) \subset \varOmega(R)$, $\phi(R^b) \subset B_{M+1}((\mathrm{E}, 0), \varepsilon)$ and the map $u_\phi : R^b \setminus \{0\} \to \mathcal{M}$ defined by setting $u_\phi(y) := u(\phi(y))$ has finite fibers. Since $\mathrm{E} = (1, 0, \ldots, 0) \in R^M$, the inclusion $\phi(R^b) \subset B_{M+1}((\mathrm{E}, 0), \varepsilon)$ implies that

$$\phi_{1,1}(R^b) \subset (1-\varepsilon, 1+\varepsilon) \quad \text{and} \quad \phi_{1,j}(R^b) \subset (-\varepsilon, \varepsilon) \tag{3.30}$$

for each $j \in \{2, \ldots, M\}$, and

$$\phi_2(R^b) \subset (-\varepsilon, \varepsilon).$$

By combining the latter inclusion, the first part of (3.28) and the obvious inequality $\varepsilon < 1/\beta$, we infer that

$$0 < 1 + \phi_2(y) P(x) < 2 \quad \text{for each } (y, x) \in R^b \times V. \tag{3.31}$$

Let $\phi_3 : R^b \times V \to R$ be the function sending (y,x) to $\sqrt[d]{1+\phi_2(y)P(x)}$. Inequalities (3.31) imply that ϕ_3 is a Nash function and

$$0 < \phi_3(y,x) < 2 \quad \text{for each } (y,x) \in R^b \times V. \tag{3.32}$$

Moreover, by the second part of (3.29), we have

$$\phi_3(0,x) = 1 \quad \text{for each } x \in V. \tag{3.33}$$

Observe that $\varepsilon < 1/(1+M\beta)$ or, equivalently, $1-\varepsilon-\varepsilon M\beta > 0$. By combining the latter inequality with (3.30) and with the second part of (3.28), it follows that

$$\begin{aligned} Q_{\phi_1(y)}(1,\psi_1(x,t)) &= \phi_{1,1}(y) + \sum_{j=2}^{M} \phi_{1,j}(y) \cdot \mathrm{M}_{\xi(j)}(\psi_1(x,t)) \\ &> 1-\varepsilon-\varepsilon M\beta > 0 \end{aligned} \tag{3.34}$$

for each $(y,x,t) \in R^b \times V \times (0,2)$. Define the function $\phi_4 : R^b \times V \to R$ by setting

$$\phi_4(y,x) := \sqrt[d]{Q_{\phi_1(y)}(1,\psi_1(x,\phi_3(y,x)))}.$$

Thanks to (3.32) and (3.34), we infer that ϕ_4 is a Nash function. Furthermore, (3.18) and the first part of (3.29) imply that

$$\phi_4(0,x) = 1 \quad \text{for each } x \in V. \tag{3.35}$$

Let $\mathbf{p} : \mathbf{V} \to R^M \times R$ be the projection sending $(\mathfrak{a},s,x,t,v)$ to $(\mathfrak{a},s)$, let $\mathscr{R} := R^b \times R^M \times R \times V \times R \times R$ and let $\mathcal{V}$ be the fiber product between $\phi = (\phi_1,\phi_2)$ and $\mathbf{p}$. The set $\mathcal{V}$ consists of all points $(y,\mathfrak{a},s,x,t,v)$ of $\mathscr{R}$ satisfying the following conditions:

$$\mathfrak{a} = \phi_1(y), \quad s = \phi_2(y), \quad t^d = 1+sP(x), \quad v^d = Q_{\mathfrak{a}}(1,\psi_1(x,t)). \tag{3.36}$$

Denote by $\pi : \mathcal{V} \to R^b$ and by $\pi' : \mathcal{V} \to V$ the projections sending $(y,\mathfrak{a},s,x,t,v)$ to y and to x, respectively. Thanks to (3.27), we know that

$$V_y := \pi^{-1}(y) \text{ is biregularly isomorphic to } Z'(\phi(y))(R) \tag{3.37}$$

for each $y \in R^b$.

By rearranging the coordinates, we may suppose that $\mathscr{R}$ is equal to the real algebraic submanifold $R^b \times V \times R^M \times R \times R \times R$ of $R^b \times R^n \times R^M \times R \times R \times R$. Denote by $\mathrm{F} : R^b \times R^n \times R^M \times R \times R \times R \to R^M \times R \times R \times R$ the regular map sending $(y,x,\mathfrak{a},s,t,v)$ to

$$\big(\mathfrak{a}-\phi_1(y), s-\phi_2(y), t^d-1-\phi_2(y)P(x), v^d-Q_{\phi_1(y)}(1,\psi_1(x,t))\big).$$

By (3.36), $\mathcal{V}$ is equal to $\mathscr{R} \cap \mathrm{F}^{-1}((0,0,0,0))$. Let $p_0 := (y_0,x_0,\mathfrak{a}_0,s_0,t_0,v_0) \in \mathcal{V}$ and let J_0 be the $(M+3)\times(M+3)$-matrix with coefficients in R obtained by extracting the last $M+3$ columns from the jacobian matrix of F valued at p_0. Thanks to (3.32) and (3.34), we see that t_0 and v_0 belong to R^+ and hence they are nonnull. Since $\det(J_0) = d^2 t_0^{d-1} v_0^{d-1} \neq 0$, we infer that p_0 is a nonsingular point of $\mathcal{V}$. It follows that $\mathcal{V}$ is a real algebraic submanifold of $\mathscr{R}$. Define $G_1 : R^b \times V \to R$ and $G_2 : R^b \times V \times R \to R$ by setting $G_1(y,x) := 1 + \phi_2(y)P(x)$ and $G_2(y,x,t) := Q_{\phi_1(y)}(1, \psi_1(x,t))$, respectively. Then (3.31) and (3.34) imply (i), (3.18) and (3.29) imply (ii), while (iii) follows from (3.36).

Step IX. We complete the proof by showing that π also satisfies (iv). Thanks to the explicit description of $\mathcal{V}$ given in (3.36), we see that $\mathcal{V}$ is the graph in $\mathscr{R}$ of the Nash map $\Phi : R^b \times V \to R^M \times R \times R \times R$ defined by setting $\Phi(y,x) := (\phi_1(y), \phi_2(y), \phi_3(y,x), \phi_4(y,x))$. In other words, $\pi \times \pi' : \mathcal{V} \to R^b \times V$ is a Nash isomorphism. Furthermore, (3.29), (3.33) and (3.35) imply that $\Phi(0,x) = (\mathrm{E}, 0, 1, 1)$ for each $x \in V$ and hence the restriction of π' to V_0 is a biregular isomorphism. This proves that π is an algebraic real-deformation of V.

It remains to show that π is almost perfectly parametrized by R^b. We have to prove the existence of a semialgebraic subset $\mathcal{T}$ of $R^b \times R^b$ containing $\mathcal{S}_\pi$ such that the projection $\rho_{\mathcal{T}} : \mathcal{T} \to R^b$ sending (y,y') to y has finite fibers. Denote by T the subset of R^b consisting of points y such that V_y is birationally isomorphic to V (or, equivalently, to V_0) and by R^{2b}_* the subset $(R^b \setminus \{0\}) \times (R^b \setminus \{0\})$ of $R^b \times R^b$. Define

$$\mathcal{T}_1 := \{(y,y') \in R^{2b}_* \mid u_\phi(y) = u_\phi(y')\} \ \text{ and } \mathcal{T} := \mathcal{T}_1 \cup (T \times \{0\}) \cup (\{0\} \times T).$$

Let $(y,y') \in \mathcal{S}_\pi \cap R^{2b}_*$. By definition of $\mathcal{S}_\pi$ (see (1.1)), V_y and $V_{y'}$ are birationally isomorphic and hence, by (3.37), there exists a biregular isomorphism from a Zariski dense Zariski open subset of $Z'(\phi(y))(R)$ to a Zariski dense Zariski open subset of $Z'(\phi(y'))(R)$. Such a biregular isomorphism extends to a complex birational isomorphism from $Z'(\phi(y))$ to $Z'(\phi(y'))$. In this way, by (3.20), $u_\phi(y)$ is equal to $u_\phi(y')$, that is, $(y,y') \in \mathcal{T}_1$. It follows that $\mathcal{S}_\pi \subset \mathcal{T}$. Since all the fibers of u_ϕ are finite, we infer that T is finite as well. In particular, the subset $\mathcal{T}$ of $R^b \times R^b$ is semialgebraic and $\rho_{\mathcal{T}}$ has finite fibers. ∎

Remark 3.2. Given $\ell \in \mathbb{N}^*$, the algebraic real-deformation $\pi : \mathcal{V} \to R^b$ of V almost perfectly parametrized by R^b just constructed in the preceding proof can be choosen with the following additional property:

(v) *There exists $c \in \mathbb{N}^*$ with $c \geq \ell$ such that, for each $y \in R^b \setminus \{0\}$, $\pi^{-1}(y)$ admits a nonsingular complexification Z_y with ample canonical complex line bundle ω_{Z_y} and $\omega^r_{Z_y} = c$, where $r = \dim V$.*

Let us re-examine a part of the construction of π. Fix $y \in R^b \setminus \{0\}$. The fiber $\pi^{-1}(y)$ is biregularly isomorphic to the real part of some $Z'(\mathfrak{a}, s)$. Up to complex biregular isomorphism, the projective complex algebraic manifold

$Z_2 := Z'(\mathfrak{a}, s)$ is obtained as follows: first, one perfoms the simple d-cyclic covering $\pi_{n,k_1,s} : Z'(s) \to Z$ branched along a nonsingular divisor of Z and then the simple d-cyclic covering $\pi_{m,k_2,\mathfrak{a},s} : Z_2 \to Z'(s)$ branched along a nonsingular divisor of $Z'(s)$. Here, d is a fixed odd integer ≥ 3, k_1 is an integer $\geq (r+2)/(d-1)$ and k_2 is a positive integer so large that ω_{Z_2} is ample.

Choose $k_1 := r + 2$. For brevity, write $Z_1 := Z'(s)$, $\pi_1 := \pi_{n,k_1,s}$ and $\pi_2 := \pi_{m,k_2,\mathfrak{a},s}$.

In the remainder of this remark, we abuse notations by confusing complex line bundles with their first Chern classes.

Let us compute the self-intersection number $\omega_{Z_2}^r$ of ω_{Z_2}. Since π_1 and π_2 are finite complex regular maps of degree d, the homomorphisms $\mathbb{Z} \cong H^{2r}(Z;\mathbb{Z}) \to H^{2r}(Z_1;\mathbb{Z}) \cong \mathbb{Z}$ induced by π_1 and $\mathbb{Z} \cong H^{2r}(Z_1;\mathbb{Z}) \to H^{2r}(Z_2;\mathbb{Z}) \cong \mathbb{Z}$ induced by π_2 coincide with the multiplication by d. Since $\omega_{Z_1} \cong \pi_1^*\big(\omega_Z \otimes \mathcal{O}_Z(k_1(d-1))\big)$ and $\omega_{Z_2} \cong \pi_2^*\big(\omega_{Z_1} \otimes \mathcal{O}_{Z_1}(k_2(d-1))\big)$, we have

$$\begin{aligned}\omega_{Z_2}^r &= d\big(\omega_{Z_1} + (k_2(d-1))\mathcal{O}_{Z_1}(1)\big)^r \\ &= d\left(\sum_{i=0}^{r}\binom{r}{i}(k_2(d-1))^{r-i}\omega_{Z_1}^i \cdot \mathcal{O}_{Z_1}(1)^{r-i}\right)\end{aligned}$$

and

$$\omega_{Z_1}^i \cdot \mathcal{O}_{Z_1}(1)^{r-i} = d\big(\omega_Z + (k_1(d-1))\mathcal{O}_Z(1)\big)^i \cdot \mathcal{O}_Z(1)^{r-i}$$

for each $i \in \{0, 1, \dots, r\}$. In this way, bearing in mind that $\mathcal{O}_Z(1)^r$ is equal to the degree $\deg(Z)$ of Z, there exists a polynomial U in one indeterminate of degree $< r$ with integer coefficients depending only on Z such that $\omega_{Z_2}^r = \deg(Z)d^2(d-1)^r k_2^r + \mathrm{U}(k_2)$. It follows that $\omega_{Z_2}^r$ can be made arbitrarily large by choosing k_2 sufficiently large. This proves (v).

3.2. Proof in the unbounded case. Let V be an unbounded real algebraic submanifold of some R^n of positive dimension. We remind the reader that the Alexandrov compactification of V can be made algebraic (see Lemma 2.6.2 of [1] and [6, pp. 76–77]). More precisely, there exist a bounded real algebraic subset $\dot{V}$ of some R^m, a point $p \in \dot{V}$ and a biregular isomorphism from V to $\dot{V} \setminus \{p\}$. Identify V with $\dot{V} \setminus \{p\}$ via such a biregular isomorphism. Observe that V is contained in the nonsingular locus of $\dot{V}$. By Hironaka's desingularization theorem, there exist a bounded real algebraic submanifold V^* of some R^m and a regular map $\varrho : V^* \to \dot{V}$ such that the restriction of ϱ from $\varrho^{-1}(V)$ to V is a biregular isomorphism. Identify V with $\varrho^{-1}(V)$ via ϱ. Let $b \in \mathbb{N}^*$. By Theorem 1.3, there exists an algebraic real-deformation $\pi^* : \mathcal{V}^* \to R^b$ of V^* almost perfectly parametrized by R^b. Let $\pi' : \mathcal{V}^* \to V^*$ be a regular map such that the restriction of π' to $(\pi^*)^{-1}(0)$

is a biregular isomorphism and $\pi^* \times \pi' : \mathcal{V}^* \to R^b \times V^*$ is a Nash isomorphism. Define $\mathcal{V} := (\pi^* \times \pi')^{-1}(R^b \times V)$ and $\pi : \mathcal{V} \to R^b$ as the restriction of π^* to $\mathcal{V}$. It is evident that π is an algebraic real-deformation of V almost perfectly parametrized by R^b. The proof is complete.

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