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On the relationship between hyperbolic and cone-hyperbolic structures in metric spaces

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Abstract. We give necessary and sufficient conditions for topological hyperbolicity of a homeomorphism of a metric space, restricted to a given compact invariant set. These conditions are related to the existence of an appropriate finite covering of this set and a corresponding cone-hyperbolic graph-directed iterated function system.

1. Introduction. As has already been confirmed by the works of several authors, the notion of a cone-field provides a useful tool for the study of hyperbolic systems, both from analytical (see, e.g., [4]) and (strictly) numerical (see, e.g., [2]) point of view. In particular, using this notion Newhouse [4] gave necessary and sufficient conditions for the existence of a hyperbolic splitting for a diffeomorphism on a compact invariant subset of a given smooth manifold.

In [3] Kułaga and Tabor constructed a global metric analogue of a cone-field while in [9] Struski, Tabor and Kułaga defined and studied its local version. Both these generalizations go beyond differential structure (by admitting small Lipschitz disturbances) and are well adapted for rigorous verification with the use of computers [9]. Moreover, they guarantee expansivity of the system, which, when combined with shadowing (also called the pseudo-orbit tracing property), characterizes a hyperbolic structure in topological (metric) terms [5, 7]. Nevertheless, the existence of a metric cone-field does not imply the shadowing property [9].

In this paper we explore these ideas and obtain an analogue of the result of [4] mentioned above, which makes a connection between the concepts established in [5, 6, 7] and [3]. Specifically, we give necessary and sufficient

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conditions for the topological hyperbolicity of a homeomorphism of a metric space, restricted to a given compact invariant set. These conditions are related to the existence of an appropriate finite covering of the set and a corresponding cone-hyperbolic graph-directed iterated function system (this idea is, in a sense, similar to the construction of a Markov partition, see Remark 3.2).

2. Preliminaries

2.1. Hyperbolicity in metric spaces. In this section we recall some results on hyperbolicity in compact metric spaces from the works of Ombach [5], Ruelle [6] and Sakai [7, 8]. We begin by establishing the relevant terminology.

Let (K, d) be a compact metric space and $f\colon K \to K$ be a homeomorphism, considered as a discrete dynamical system on K, where the *orbit* of a point $x \in K$ is the sequence $(x_n)_{n=-\infty}^{\infty} \subset K$ defined as follows: $x_0 = x$ and $x_{n+1} = f(x_n)$ for $n \in \mathbb{Z}$.

Let $\delta \geq 0$ be a constant. A sequence $(y_n)_{n=l}^{r} \subset K$, where $l, r \in \mathbb{Z} \cup \{-\infty, \infty\}$ and $l \leq 0 < r$, is called a *δ-pseudo-orbit* of f if

$$d(f(y_n), y_{n+1}) \leq \delta \quad \text{for every } n \in \{l, \ldots, r-1\}.$$

Note that a 0-pseudo-orbit of f is simply a segment of a genuine orbit.

We say that f has the *shadowing* property if for every $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following condition: given a δ-pseudo-orbit $\xi = (y_n)_{n=l}^{r}$ we can find a point $x \in K$ whose orbit *ε-traces* ξ, i.e.,

$$d(f^n(x), y_n) \leq \varepsilon \quad \text{for every } n \in \{l, \ldots, r\}.$$

Let us note that (see, e.g., [1]) the "full" shadowing property is guaranteed by the condition of tracing only finite pseudo-orbits (i.e., $l, r \in \mathbb{Z}$) by finite segments of orbits.

We call f *expansive* if there exists $e > 0$ (an *expansive constant*) with the following property: there are no two different points $x, y \in K$ whose orbits are e-close, i.e.,

$$\max_{n \in \mathbb{Z}} d(f^n(x), f^n(y)) \leq e \;\Rightarrow\; x = y.$$

It can be easily seen (see, e.g., [1]) that both the properties above are independent of the choice of a compatible metric on K.

If f is an expansive homeomorphism with the shadowing property, we say that f is *topologically hyperbolic.*

The next concept originally comes from Ruelle [6], who considered it, in terms of thermodynamic formalism, as a generalization of Axiom A for diffeomorphisms. Afterwards, this notion was introduced into topological dynamics, e.g., in the works of Ombach [5] and Sakai [7].

We say that (K, f) is a *Smale space* if the following conditions hold:

(LPS) (*local product structure*) there exists $\eta > 0$ and a continuous map

$$[\cdot,\cdot]\colon \{(x,y) \in K \times K : d(x,y) < \eta\} \to K$$

satisfying

$$[x,x] = x, \qquad [[x,y],z] = [x,z], \qquad [x,[y,z]] = [x,z]$$

and

$$f([x,y]) = [f(x), f(y)]$$

for all $x, y, z \in K$ such that both sides of these relations are defined,

(HS) (*hyperbolic structure*) there exist $\nu > 0$ and $\lambda \in (0,1)$ such that for all $x \in K$ and $n \geq 0$ we have

$$\begin{aligned} d(f^n(y), f^n(z)) &\leq \lambda^n d(y,z) \qquad \text{if } y, z \in V_\nu^s(x), \\ d(f^n(y), f^n(z)) &\geq \lambda^{-n} d(y,z) \quad \text{if } y, z \in V_\nu^u(x), \end{aligned}$$

where

$$\begin{aligned} V_\nu^s(x) &= \{y \in K : y = [x,y],\ d(x,y) \leq \nu\}, \\ V_\nu^u(x) &= \{y \in K : y = [y,x],\ d(x,y) \leq \nu\}. \end{aligned}$$

In [6] Ruelle extended the above definition by introducing the following extra condition, simultaneously asking whether it is a consequence of (LPS) and (HS) after changing the metric on K to some compatible one. This question was positively answered by Sakai [8].

(LC) (*L-condition*) There exists a constant $L > 0$ such that

$$\max\{d(x,[x,y]), d(y,[x,y])\} \leq L d(x,y)$$

for all $x, y \in K$ with $d(x,y) < \eta$.

We finish this section with a statement which combines the notions discussed above and gives an important tool for the proof of our main result.

THEOREM 2.1. *The following conditions are equivalent:*

(i) (K, f) *is a Smale space with respect to some compatible metric,*
(ii) (K, f) *is a Smale space satisfying condition* (LC) *with respect to some compatible metric,*
(iii) f *is topologically hyperbolic.*

Proof. This follows immediately from [7, Corollary], [8, Theorem 1] and the following simple observation (see [6, (7.4)–(7.6)]):

$$V_\nu^s(x) = W_\nu^s(x) \quad \text{and} \quad V_\nu^u(x) = W_\nu^u(x)$$

provided that $\nu > 0$ is sufficiently small, where $W^s_\nu(x)$ and $W^u_\nu(x)$ denote the *local stable* and *unstable manifolds* of $x \in K$, i.e,

$$W^s_\nu(x) = \{y \in K : d(f^n(x), f^n(y)) \le \nu \text{ for all } n \ge 0\},$$
$$W^u_\nu(x) = \{y \in K : d(f^n(x), f^n(y)) \le \nu \text{ for all } n \le 0\}. \blacksquare$$

2.2. Cone-fields in metric spaces. In this section we recall and adapt to our needs the relevant notions and results from the paper of Kułaga and Tabor [3].

Let C be a compact subset of a metric space (X, d). A pair of functions $c_s, c_u \colon C \times C \to [0, \infty)$ is called a *cone-field* if there exists a constant $D > 0$ such that

$$\frac{1}{D} d(x, y) \le c(x, y) \le D d(x, y) \quad \text{for all } x, y \in C,$$

where $c(x, y) = \max\{c_s(x, y), c_u(x, y)\}$. In that case C is called a *cone-set* in X and the sets

$$C^s = \{(x, y) \in C \times C : c_s(x, y) \ge c_u(x, y)\},$$
$$C^u = \{(x, y) \in C \times C : c_s(x, y) \le c_u(x, y)\}$$

are called the *stable* and *unstable cones*, respectively. We will keep the above notations c_s, c_u and c for all cone-sets, regardless of their names; we hope this will not confuse the readers.

Let C_1 and C_2 be cone-sets in X and $f \colon C_1 \rightharpoonup C_2$ be a *partial map*, i.e., the domain of f, denoted by $\operatorname{dom} f$, is a nonempty subset of C_1, not assumed to be equal to C_1. We define the constants

$$|f|_s = \inf\{R \in [0, \infty] : c(f(x), f(y)) \le R c(x, y) \quad \text{for } x, y \in \operatorname{dom} f,\ (f(x), f(y)) \in C^s_2\},$$
$$\langle f \rangle_u = \sup\{R \in [0, \infty] : c(f(x), f(y)) \ge R c(x, y) \quad \text{for } x, y \in \operatorname{dom} f,\ (x, y) \in C^u_1\},$$

which we call the *s-contraction rate* and the *u-expansion rate* of f, respectively. We say that f is *cone-hyperbolic* if

$$|f|_s < 1 < \langle f \rangle_u.$$

For the proof of our main result we need the following auxiliary lemma that is an immediate consequence of [3, Corollary 2.1].

Lemma 2.2. *Consider a sequence $(f_i)_{i=l}^{r-1}$, where $l < 0 < r$ and each f_i is a cone-hyperbolic map between some cone-sets C_i and C_{i+1} in X. Let $(x_i)_{i=l}^r$ and $(y_i)_{i=l}^r$ be sequences of points in X such that*

$$x_i, y_i \in \operatorname{dom} f_i,\ x_{i+1} = f_i(x_i),\ y_{i+1} = f_i(y_i) \quad \textit{for } i \in \{l, \dots, r-1\}.$$

Then

$$c(x_0, y_0) \le \max(\langle f_{r-1} \rangle_u^{-1} \cdots \langle f_0 \rangle_u^{-1} c(x_r, y_r), |f_{-1}|_s \cdots |f_l|_s c(x_l, y_l)).$$

We end this section by recalling several other notions that are necessary for the statement of our main result.

A (*directed*) *graph* G is a pair (V, E), where V is a finite set of *vertices*, and E is a set of *edges*, i.e., ordered pairs of vertices. If $e = (v, w)$ is an edge in G, we denote the vertices v and w by $i(e)$ and $t(e)$, respectively. By a *directed path* in G we mean a finite sequence $(e_1, \dots, e_n)$ of edges such that $t(e_k) = i(e_{k+1})$ for all $k \in \{1, \dots, n-1\}$.

For a given graph $G = (V, E)$ we define an *iterated function system directed by* G (or, briefly, a G-*directed system*) as a triple $(G, \{C_v\}_{v\in V}, \{f_e\}_{e\in E})$, where each C_v is a cone-set in X and each $f_e \colon C_{i(e)} \rightharpoonup C_{t(e)}$ is a (partial) map with a closed graph. We call this system *cone-hyperbolic* if all the maps f_e are cone-hyperbolic with respect to some compatible metric on the subspace $\bigcup_{v\in V} C_v \subset X$.

3. Main result. In this section we state and prove the main result of the paper, which compares two approaches to the concept of hyperbolic structure in metric spaces (presented in the previous sections). First we formulate an auxiliary definition.

Let X be a metric space and $f \colon X \to X$ be a continuous map. Assume that $K \subset X$ is a compact invariant set (i.e., $f(K) = K$) and $\{B_v\}_{v\in V}$ is a finite covering of K by nonempty sets open in K. Consider the corresponding graph $G = (V, E)$, where

$$E = \{(v, w) \in V \times V : B_v \cap f^{-1}(B_w) \neq \emptyset\}.$$

We say that a G-directed system $(G, \{C_v\}_{v\in V}, \{f_e\}_{e\in E})$ *renders the dynamics* of the map f on K *with accuracy* $\{B_v\}_{v\in V}$ if the following conditions are satisfied:

(R1) $B_v \subset C_v$ for each $v \in V$, and for each $e \in E$ we have
$$\operatorname{dom} f_e = C_{i(e)} \cap f^{-1}(C_{t(e)}) \quad \text{and} \quad f_e|_{\operatorname{dom} f_e} = f|_{\operatorname{dom} f_e},$$

(R2) for each integer $n \geq 0$ and each path $\alpha = (e_0, \dots, e_n)$ in G there exists $x \in K$ such that α *renders the motion* of x until time n, i.e.,
$$f^k(x) \in \operatorname{dom} f_{e_k} \quad \text{for every } k \in \{0, \dots, n\}.$$

The following observations are immediate consequences of the above definition:

(R3) for each $n \geq 0$ and each $x \in K$ there exists a path $(e_0, \dots, e_n)$ in G that renders the motion of x until time n,

(R4) for each $n \geq 0$ the family
$$\mathcal{B}_n = \Big\{ \bigcap_{k=-n}^{n} f^{-k}(B_{i(e_k)}) : (e_{-n}, \dots, e_n) \text{ is a path in } G \Big\}$$
forms a covering of K by sets open in K,

(R5) the families $\{C_v\}_{v\in V}$, $\{\operatorname{dom} f_e\}_{e\in E}$, and

$$\mathcal{C}_n = \Big\{ \bigcap_{k=-n}^{n} f^{-k}(C_{i(e_k)}) : (e_{-n},\ldots,e_n) \text{ is a path in } G \Big\}$$

for each $n \geq 0$, form coverings of K by nonempty compact sets (note that in this case the assumption that the graphs of the maps f_e are closed means that their domains are compact).

Now we can state our main result.

Main Theorem 3.1. *Assume that $f\colon X \to X$ is a continuous map and K is a compact invariant subset of the metric space X such that $f|_K\colon K \to K$ is a homeomorphism. Then the following conditions are equivalent:*

1. *$f|_K$ is topologically hyperbolic,*
2. *there exist a finite covering $\{B_v\}_{v\in V}$ of K by nonempty sets open in K, together with the corresponding graph $G = (V,E)$, and a cone-hyperbolic G-directed system $(G, \{C_v\}_{v\in V}, \{f_e\}_{e\in E})$ that renders the dynamics of f on K with accuracy $\{B_v\}_{v\in V}$,*
3. *there exist a finite covering $\{B_v\}_{v\in V}$ of K by nonempty sets open in K, together with the corresponding graph $G = (V,E)$, and a cone-hyperbolic G-directed system $(G, \{C_v\}_{v\in V}, \{f_e\}_{e\in E})$ that renders the dynamics of f on K with accuracy $\{B_v\}_{v\in V}$, such that the diameters of all C_v are uniformly bounded by an arbitrarily small constant.*

Proof. It is enough to show the implications (1)⇒(3), (3)⇒(2) (obvious) and (2)⇒(1).

(1)⇒(3). Assume that the homeomorphism $f|_K$ has the shadowing property and is expansive. By Theorem 2.1, $(K, f|_K)$ is a Smale space satisfying condition (LC) with respect to some compatible metric d. Let $\eta, \nu, L > 0$ and $\lambda \in (0,1)$ be constants given by conditions (LPS), (HS) and (LC). Let $\varepsilon > 0$ be an arbitrary constant. Without loss of generality we can assume that

$$\varepsilon < \frac{\min\{\eta/2, \nu/2\}}{\max\{L, 1\}}.$$

Let $\delta < \varepsilon$ be a constant corresponding to $\varepsilon/2$ by the shadowing property. Let V be a finite set of points in K such that $\{B_{\delta/2}(v)\}_{v\in V}$ is a covering of K. (Here and subsequently, $B_r(x)$ denotes the open ball in X with radius $r > 0$, centered at $x \in X$.) For each $v \in V$ put $C_v = \overline{B}_\delta(v) \cap K$, $B_v = B_{\delta/2}(v) \cap K$ and define

$$E = \{e = (i(e), t(e)) \in V \times V : B_{i(e)} \cap f^{-1}(B_{t(e)}) \neq \emptyset\},$$
$$f_e = f|_{C_{i(e)} \cap f^{-1}(C_{t(e)})} \colon C_{i(e)} \rightharpoonup C_{t(e)} \quad \text{for all } e \in E.$$

(This means in particular that $\operatorname{dom} f_e = C_{i(e)} \cap f^{-1}(C_{t(e)})$ is compact.) So we have obtained the covering $\{B_v\}_{v\in V}$, together with the corresponding graph $G = (V, E)$, and the triple $(G, \{C_v\}_{v\in V}, \{f_e\}_{e\in E})$ for which condition (R1) is obviously satisfied.

To show condition (R2) assume that $\alpha = (e_0, \dots, e_n)$ is a path in G. Then there exist $x_0, \dots, x_n \in K$ such that

$$x_k \in B_{\delta/2}(i(e_k)) \quad \text{and} \quad f(x_k) \in B_{\delta/2}(t(e_k)) \quad \text{for } k = 0, \dots, n,$$

which means that $\xi = (x_0, \dots, x_n, f(x_n))$ is a δ-pseudo-orbit of $f|_K$. Let $y \in K$ be a point that $\varepsilon/2$-shadows ξ, i.e.,

$$d(x_k, f^k(y)) \le \varepsilon/2 \quad \text{for } k = 0, \dots, n, \quad d(f(x_n), f^{n+1}(y)) \le \varepsilon/2.$$

Hence $y \in C_{i(e_0)}$, and

$$f^k(y) \in C_{i(e_k)} = C_{t(e_{k-1})} \quad \text{for } k = 1, \dots, n, \quad f^{n+1}(y) \in C_{t(e_n)}$$

and so

$$f^k(y) \in \operatorname{dom} f_{e_k} \quad \text{for } k = 0, \dots, n.$$

To finish the proof of this part we need to define a cone-field on each C_v, with respect to which each f_e is cone-hyperbolic.

For each $v \in V$ and $x, y \in C_v$ we define

$$c_s(x, y) = d(x, [x, y]), \qquad c_u(x, y) = d(y, [x, y]),$$

where $[\,\cdot\,, \cdot\,]$ comes from condition (LPS). Note that

$$d(x, y) \le d(x, [x, y]) + d(y, [x, y]) = c_s(x, y) + c_u(x, y) \le 2c(x, y),$$
$$c(x, y) = \max\{d(x, [x, y]), d(y, [x, y])\} \le Ld(x, y) \le \min\{\eta, \nu\}$$

for all $x, y \in C_v$.

Take any map $f_e \colon C_{i(e)} \rightharpoonup C_{t(e)}$. Then, by condition (HS),

$$\begin{aligned} c(f_e(x), f_e(y)) &= c_s(f(x), f(y)) = d(f(x), [f(x), f(y)]) \le \lambda d(x, [x, y]) \\ &= \lambda c_s(x, y) \le \lambda c(x, y) \end{aligned}$$

for $x, y \in \operatorname{dom} f_e$ with $(f_e(x), f_e(y)) \in C^s_{t(e)}$, while

$$\begin{aligned} c(f_e(x), f_e(y)) &\ge c_u(f(x), f(y)) = d(f(y), [f(x), f(y)]) \ge \lambda^{-1} d(y, [x, y]) \\ &= \lambda^{-1} c_u(x, y) = \lambda^{-1} c(x, y) \end{aligned}$$

for $x, y \in \operatorname{dom} f_e$ with $(x, y) \in C^u_{i(e)}$.

Hence

$$|f_e|_s \le \lambda \quad \text{and} \quad \langle f_e \rangle_u \ge \lambda^{-1},$$

which means that f_e is cone-hyperbolic.

(2)⇒(1). Assume that $\{B_v\}_{v\in V}$ is a covering of K by nonempty sets open in K and $G = (V, E)$ is the corresponding graph, and $(G, \{C_v\}_{v\in V}, \{f_e\}_{e\in E})$ is a G-directed system that renders the dynamics of f on K with accuracy

$\{B_v\}_{v\in V}$, which is cone-hyperbolic with respect to some compatible metric d on $\bigcup_{v\in V} C_v$. Let $\beta > 0$ be a Lebesgue number of the covering $\{B_v\}_{v\in V}$. Since our subsequent considerations are, in fact, restricted to the set K, we use the notation f for the homeomorphism $f|_K$.

First we show that β is an expansive constant for f. To do this, take $x, y \in K$ satisfying

$$d(f^k(x), f^k(y)) < \beta \quad \text{for all } k \in \mathbb{Z}.$$

Then, by (R1), there exists an infinite path $\alpha = (\ldots, e_{-1}, e_0, e_1, \ldots)$ (i.e., for each $n > 0$ the sequence $(e_{-n}, \ldots, e_0, \ldots, e_n)$ is a path in G) such that

$$f^k(x), f^k(y) \in B_{i(e_k)} \cap f^{-1}(B_{t(e_k)}) \subset \operatorname{dom} f_{e_k} \quad \text{for all } k \in \mathbb{Z}.$$

Using Lemma 2.2 we obtain

$$\begin{aligned} d(x,y) &\le Dc(x,y) \\ &\le D \max\{\langle f_{e_{n-1}}\rangle_u^{-1} \cdots \langle f_{e_0}\rangle_u^{-1} c(f^n(x), f^n(y)), \\ &\qquad |f_{e_{-1}}|_s \cdots |f_{e_{-n}}|_s c(f^{-n}(x), f^{-n}(y))\} \\ &\le D^2 \max_{v\in V} \operatorname{diam} C_v \cdot \max\{\langle f_{e_{n-1}}\rangle_u^{-1} \cdots \langle f_{e_0}\rangle_u^{-1}, |f_{e_{-1}}|_s \cdots |f_{e_{-n}}|_s\} \xrightarrow{n\to\infty} 0 \end{aligned}$$

and so $x = y$, which shows that $f|_K$ is expansive.

To show that f has the shadowing property take any $\gamma > 0$ and for every $n \ge 0$ consider the coverings $\mathcal{B}_n$ and $\mathcal{C}_n$ defined by (R4) and (R5). Note that, by Lemma 2.2, we can find $N > 0$ for which the diameters of all the sets from $\mathcal{C}_N$ are smaller than γ. Let $\delta > 0$ be such that for any $x, y \in K$ the following implication holds:

$$d(x,y) < \delta \ \Rightarrow \max_{k=-N,\ldots,N} d(f^k(x), f^k(y)) < \beta/(2N).$$

Let $\xi = (x_0, \ldots, x_{n-1})$ be a finite δ-pseudo-orbit in K. Then, putting $x_n = f(x_{n-1})$, we get

$$d(f^{i+1}(x_j), f^i(x_{j+1})) < \beta/(2N) \quad \text{for } i = -N, \ldots, N,\ j = 0, \ldots, n-1,$$

hence for each $k \in \{-N, \ldots, N+n\}$ there exists $v_k \in V$ such that

$$f^i(x_j) \in B_{v_k} \quad \text{for } i = -N, \ldots, N,\ j = 0, \ldots, n,\ i + j = k.$$

(Note that whenever

$$\max\{k - N, 0\} \le j_1 < j_2 \le \min\{k + N, n\},$$

then

$$\begin{aligned} d(f^{k-j_1}(x_{j_1}), f^{k-j_2}(x_{j_2})) &\le \sum_{j=j_1}^{j_2-1} d(f^{k-j}(x_j), f^{k-j-1}(x_{j+1})) \\ &\le 2N\beta/(2N) = \beta.) \end{aligned}$$

It follows that for each $k \in \{-N, \ldots, N+n-1\}$ there exist $i \in \{-N, \ldots, N\}$ and $j \in \{0, \ldots, n\}$ such that $i + j = k$ and

$$f^i(x_j) \in B_{v_k} \cap f^{-1}(B_{v_{k+1}}).$$

Hence there is a path $\alpha = (e_{-N}, \ldots, e_{N+n-1})$ in G passing through all the vertices v_k, i.e.,

$$i(e_k) = v_k, \quad t(e_k) = v_{k+1} \quad \text{for } k = -N, \ldots, N + n - 1,$$

and consequently, by (R2), we can find $y \in K$ such that α renders the motion of $f^{-N}(y)$ until time $2N + n - 1$, i.e.,

$$f^{k+i}(y) \in \operatorname{dom} f_{e_{k+i}} \subset C_{v_{k+i}} \quad \text{for } k = 0, \ldots, n - 1,\ i = -N, \ldots, N.$$

Thus for $k = 0, \ldots, n - 1$ we obtain

$$x_k, f^k(y) \in \bigcap_{i=-N}^{N} f^{-i}(C_{v_{k+i}}) \in \mathcal{C}_N,$$

and since the diameters of all the sets in $\mathcal{C}_N$ are less than γ, we conclude that the δ-pseudo-orbit ξ is γ-traced by y, which means that f has the shadowing property. This finishes the proof. ■

Remark 3.2. It is known (see, e.g., [1]) that if a continuous map restricted to a compact invariant set K is a topologically hyperbolic homeomorphism, then it admits a Markov partition of K. Let us note that our construction of a finite covering and a corresponding cone-hyperbolic graph-directed system for the set K seems to be, in a sense, a similar but more flexible concept (we do not work with partitions but with coverings of K).

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