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Existence of three solutions for a Navier boundary value problem involving the $p(x)$-biharmonic operator

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Abstract. The existence of at least three weak solutions is established for a class of quasilinear elliptic equations involving the $p(x)$-biharmonic operator with Navier boundary value conditions. The proof is mainly based on a three critical points theorem due to B. Ricceri [Nonlinear Anal. 70 (2009), 3084–3089].

1. Introduction. In this paper, we consider problems of the type

$$\begin{cases} \Delta^2_{p(x)}u + e(x)|u|^{p(x)-2}u = \lambda a(x)f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with boundary of class C^1; $\lambda, \mu \geq 0$ are real numbers; $p(\cdot) \in C^0(\overline{\Omega})$ with $\max\{2, N/2\} < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x)$; and $\Delta^2_{p(x)} := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the operator of fourth order called the $p(x)$-*biharmonic operator*, which is a natural generalization of the p-biharmonic operator (where $p > 1$ is a constant).

In [LS], the authors studied the following superlinear p-biharmonic elliptic problem with Navier boundary conditions:

$$\begin{cases} \Delta^2_p u = g(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

By means of Morse theory, they proved the existence of a nontrivial solution to (1.2) having a linking structure around the origin under the conditions: $\Omega \subseteq \mathbb{R}^N$ is bounded with smooth boundary; $N \geq 2p+1$; $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for some $C > 0, |g(x,t)| \leq C(1 + |t|^{q-1})$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$; $1 \leq q \leq p^* = Np/(N-2p)$. Moreover, in the case of both resonance near zero and nonresonance at ∞, the existence of two nontrivial solutions was obtained.

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In [LT], the authors considered the following problem:

$$\begin{cases} \Delta_p^2 u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

By the three critical points theorem due to Ricceri [Ri], they established the existence of three weak solutions to (1.3).

For more results on fourth-order elliptic equations with variable exponent, see [AA, AMM] and the references therein.

To obtain the existence of at least three solutions of problem (1.1), we apply the three critical points theorem by B. Ricceri [Ri]:

Theorem A. *Let X be a reflexive real Banach space; $I \subseteq \mathbb{R}$ an interval; $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose Gâteaux derivative admits a continuous inverse on X^*; and $\Psi : X \to \mathbb{R}$ a C^1 functional with compact Gâteaux derivative. Assume that*

(i) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$ *for all* $\lambda \in I$;
(ii) *there exists* $\rho \in \mathbb{R}$ *such that*

$$\sup_{\lambda\in I}\inf_{t\in X}(\Phi(t) + \lambda(\Psi(t) + \rho)) < \inf_{t\in X}\sup_{\lambda\in I}(\Phi(t) + \lambda(\Psi(t) + \rho)).$$

Then there exists a nonempty open set $\Lambda \subseteq I$ and a positive number σ with the following property: for each $\lambda \in \Lambda$ and every C^1 functional $J : X \to \mathbb{R}$ with compact Gâteaux derivative, there exists $\delta > 0$ such that for each $\mu \in [0,\delta]$, the equation

$$\Phi'(u) + \lambda\Psi'(u) + \mu J'(u) = 0 \tag{1.4}$$

has at least three solutions in X whose norms are less than σ.

On the basis of [B], we state an equivalent formulation of Theorem A:

Theorem B. *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose Gâteaux derivative admits a continuous inverse on X^*; and $\Psi : X \to \mathbb{R}$ a C^1 functional with compact Gâteaux derivative. Assume that*

(i) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$ *for all* $\lambda > 0$; *and there are* $r \in \mathbb{R}$ *and* $u_0, u_1 \in X$ *such that:*
(ii) $\Phi(u_0) < r < \Phi(u_1)$;
(iii) $\inf_{u\in\Phi^{-1}((-\infty,r])}\Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}$.

Then there exists a nonempty open set $\Lambda \subseteq [0,\infty)$ and a positive number σ with the following property: for each $\lambda \in \Lambda$ and every C^1 functional

$J : X \to \mathbb{R}$ with compact Gâteaux derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the equation

$$\varPhi'(u) + \lambda \varPsi'(u) + \mu J'(u) = 0 \tag{1.5}$$

has at least three solutions in X whose norms are less than σ.

To obtain the existence of at least three solutions of (1.1), we assume the following conditions:

(A) $e(\cdot) \in L^\infty(\varOmega)$ and $e^- > 0$; denote $\|e(\cdot)\|_1 := \int_\varOmega e(x)\, dx$;
(B) $g : \varOmega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $\sup_{|\zeta| \le s} |g(\cdot, \zeta)| \in L^1(\varOmega)$ for all $s > 0$;
(C) $a(\cdot) \in L^{r(\cdot)}(\varOmega)$, $f : \varOmega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $|f(x,t)| \le b(x) + c|t|^{q(x)-1}$ for $x \in \varOmega$ and $t \in \mathbb{R}$, where $c \ge 0$ is a constant, $b(\cdot) \in L^{q^0(\cdot) r^0(\cdot)}(\varOmega)$, $r(\cdot), q(\cdot) \in C(\varOmega)$, $r^- > 1$, $p^- > q^+ \ge q^- \ge 1$, and

$$q(x) < \frac{r(x) - 1}{r(x)} p^*(x), \qquad \forall x \in \varOmega.$$

Here

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)}, & p(x) < N, \\ \infty, & p(x) \ge N \end{cases}$$

and $r^0(\cdot)$ is the conjugate function of $r(\cdot)$, i.e., $1/r(x) + 1/r^0(x) = 1$.

The paper is organized as follows. In Section 2, we recall some facts that will be needed. In Section 3, we establish our main results.

2. Notation and preliminaries. For the reader's convenience, we recall some background facts concerning Lebesgue–Sobolev spaces with variable exponent and introduce some notation. For more details, we refer the reader to [FD, KR, Ru, S].

Set

$$C_+(\varOmega) = \{h \in C(\overline{\varOmega}) \mid h(x) > 1 \text{ for all } x \in \overline{\varOmega}\}.$$

For $p(\cdot) \in C_+(\varOmega)$, define

$$L^{p(\cdot)}(\varOmega) = \Big\{ u \;\Big|\; u \text{ is a measurable real-valued function on } \varOmega, \int_\varOmega |u(x)|^{p(x)}\, dx < \infty \Big\}.$$

We can introduce a norm on $L^{p(\cdot)}(\varOmega)$ by

$$|u|_{p(\cdot)} = \inf \Big\{ \lambda > 0 \;\Big|\; \int_\varOmega |u(x)/\lambda|^{p(x)}\, dx \le 1 \Big\}.$$

Thus $(L^{p(\cdot)}(\Omega), |u|_{p(\cdot)})$ becomes a Banach space, called a *variable exponent Lebesgue space.*

The space $W^{m,p(\cdot)}(\Omega)$ is defined by

$$W^{m,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \mid D^\alpha u \in L^{p(\cdot)}(\Omega) \text{ whenever } |\alpha| \le m\}.$$

Here α is a multi-index and $|\alpha|$ is its order; m is a positive integer. $W^{m,p(\cdot)}(\Omega)$ belongs to the class of so-called generalized Orlicz–Sobolev spaces. From [H], we know that $W^{m,p(\cdot)}(\Omega)$ can be equipped with the norm

$$\|u\|_{W^{m,p(\cdot)}(\Omega)} = \sum_{|\alpha|\le m} |D^\alpha u|_{p(\cdot)}.$$

By [FD], $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.

When $e(\cdot)$ satisfies (A), we define

$$L^{p(\cdot)}_{e(\cdot)}(\Omega) = \Big\{u \;\Big|\; u \text{ is a measurable real-valued function on } \Omega, \int_\Omega e(x)|u(x)|^{p(x)}\,dx < \infty\Big\},$$

with the norm

$$|u|_{(p(\cdot),e(\cdot))} = \inf\Big\{\lambda > 0 \;\Big|\; \int_\Omega e(x)|u(x)/\lambda|^{p(x)}\,dx \le 1\Big\}.$$

Then $L^{p(\cdot)}_{e(\cdot)}(\Omega)$ is a Banach space. Now we denote $X = W^{2,p(\cdot)}(\Omega) \cap W^{1,p(\cdot)}_0(\Omega)$, where $W^{1,p(\cdot)}_0(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For any $u \in X$, define

$$\|u\|_e = \inf\Big\{\lambda > 0 \;\Big|\; \int_\Omega (|\Delta u(x)/\lambda|^{p(x)} + e(x)|u(x)/\lambda|^{p(x)})\,dx \le 1\Big\}.$$

It is easy to see that X endowed with the above norm is also a separable, reflexive Banach space. We denote by X^* its dual.

Remark. According to [ZF], $\|u\|_{W^{2,p(\cdot)}(\Omega)}$ is equivalent to $|\Delta u|_{p(\cdot)}$ in X. Consequently, the norms $\|u\|_{W^{2,p(\cdot)}(\Omega)}$, $|\Delta u|_{p(\cdot)}$ and $\|u\|_e$ are equivalent.

From now on, we will use $\|\cdot\|_e$ instead of $\|\cdot\|_{W^{2,p(\cdot)}(\Omega)}$ on X.

Proposition 2.1 (see [FD, Ru]). *The conjugate space of $L^{p(\cdot)}(\Omega)$ is $L^{p^0(\cdot)}(\Omega)$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p^0(\cdot)}(\Omega)$, we have*

$$\int_\Omega |uv|\,dx \le \Big(\frac{1}{p^-} + \frac{1}{(p^0)^-}\Big)|u|_{p(\cdot)}|v|_{p^0(\cdot)} \le 2|u|_{p(\cdot)}|v|_{p^0(\cdot)}.$$

Proposition 2.2 (see [FD, Ru]). *Denote $\rho(u) = \int_\Omega |u|^{p(x)}\,dx$ for $u \in L^{p(\cdot)}(\Omega)$. Then*

(i) $|u|_{p(\cdot)} < 1\ (= 1; > 1) \Leftrightarrow \rho(u) < 1\ (= 1; > 1)$;

(ii) $|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \le \rho(u) \le |u|_{p(\cdot)}^{p^+}$; $|u|_{p(x)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \le \rho(u) \le |u|_{p(\cdot)}^{p^-}$;

(iii) $|u|_{p(\cdot)} \to 0\ (\infty) \Leftrightarrow \rho(u) \to 0\ (\infty)$.

From Proposition 2.2, the following inequalities hold:

$$\|u\|_e^{p^-} \le \int_\Omega (|\nabla u(x)|^{p(x)} + e(x)|u(x)|^{p(x)})\,dx \le \|u\|_e^{p^+} \quad \text{if } \|u\|_e \ge 1, \tag{2.1}$$

$$\|u\|_e^{p^+} \le \int_\Omega (|\nabla u(x)|^{p(x)} + e(x)|u(x)|^{p(x)})\,dx \le \|u\|_e^{p^-} \quad \text{if } \|u\|_e \le 1. \tag{2.2}$$

Proposition 2.3 (see [F]). *Suppose that the boundary of Ω has the cone property and $a(\cdot) \in L^{r(\cdot)}(\Omega)$, $a(x) > 0$ for a.e. $x \in \Omega$, $r(\cdot) \in C(\overline{\Omega})$ and $r^- > 1$. If $p(\cdot), q(\cdot) \in C(\overline{\Omega})$ and*

$$1 \le q(x) < \frac{r(x)-1}{r(x)} p^*(x), \qquad \forall x \in \overline{\Omega},$$

then there is a compact embedding $X \hookrightarrow L_{a(\cdot)}^{q(\cdot)}(\Omega)$.

Proposition 2.4. *If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the embedding $X \hookrightarrow C^0(\overline{\Omega})$ is compact whenever $N/2 < p^-$.*

Proof. It is well known that $X \hookrightarrow W^{2,p^-}(\Omega) \cap W_0^{1,p^-}(\Omega)$ is a continuous embedding, and the embedding $W^{2,p^-}(\Omega) \cap W_0^{1,p^-}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is compact when $N/2 < p^-$ and Ω is bounded. So we obtain the compact embedding $X \hookrightarrow C^0(\overline{\Omega})$ whenever $N/2 < p^-$. ∎

From Proposition 2.4, there exists a positive constant d depending on $p(\cdot), N$ and Ω such that

$$\|u\|_\infty = \sup_{x \in \overline{\Omega}} |u(x)| \le d\|u\|_e, \qquad \forall u \in X. \tag{2.3}$$

3. Existence of three solutions. We define $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega \left(\frac{1}{p(x)} |\Delta u(x)|^{p(x)} + \frac{e(x)}{p(x)} |u(x)|^{p(x)} \right) dx. \tag{3.1}$$

Then

$$(\Phi'(u), v) = \int_\Omega (|\Delta u|^{p(x)-2} \Delta u \Delta v + e(x)|u|^{p(x)-2} uv)\,dx, \qquad \forall u, v \in X.$$

Denote

$$F(x,u)=\int_0^u a(x)f(x,t)\,dt,\qquad G(x,u)=\int_0^u g(x,t)\,dt,$$
$$\Psi(u)=\int_\Omega F(x,u)\,dx,\qquad J(u)=\int_\Omega G(x,u)\,dx.$$

Then for all $u,v\in X$,

$$(\Psi'(u),v)=\int_\Omega a(x)f(x,u)v\,dx,\qquad (J'(u),v)=\int_\Omega g(x,u)v\,dx.$$

We say that $u\in X$ is a *weak solution* of problem (1.1) if

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v+e(x)|u|^{p(x)-2}uv\,dx$$
$$=\lambda\int_\Omega a(x)f(x,u)v\,dx+\mu\int_\Omega g(x,u)v\,dx,\qquad \forall v\in X,$$

i.e.,

$$(\Psi'(u),v)-\lambda(\Psi'(u),v)-\mu(J'(u),v)=0.$$

It follows that we can look for weak solutions of (1.1) by applying Theorem A or Theorem B.

We first give the following results.

LEMMA 3.1. *If Φ is as in* (3.1), *then $(\Phi')^{-1}:X^*\to X$ exists and is continuous.*

Proof. First, we show that Φ' is uniformly monotone. In fact, for any $\zeta,\eta\in\mathbb{R}^N$ (see [KV]),

$$(|\zeta|^{p-2}\zeta-|\eta|^{p-2}\eta)(\zeta-\eta)\ge\frac{1}{2^p}|\zeta-\eta|^p,\qquad p\ge 2.$$

Thus, we deduce that

$$(\Phi'(u)-\Phi'(v),u-v)$$
$$\ge\frac{1}{2^{p^+}}\int_\Omega(|\Delta u-\Delta v|^{p(x)}+e(x)|u-v|^{p(x)})\,dx,\qquad \forall u,v\in X,$$

i.e., Φ' is uniformly monotone.

From (2.1), we can see that for any $u\in X$ with $\|u\|_e\ge 1$,

$$\frac{(\Phi'(u),u)}{\|u\|_e}\ge\|u\|_e^{p^--1},$$

which means that Φ' is coercive on X.

By a standard argument, Φ' is hemicontinuous. Therefore, the conclusion follows immediately by applying [Z, Theorem 26.A]. ∎

LEMMA 3.2. *If* (A)–(C) *hold, then for any* $\lambda \in \mathbb{R}$, $\Phi(u) - \lambda\Psi(u)$ *is coercive on* X.

Proof. From $|f(x,t)| \le b(x) + c|t|^{q(x)-1}$ and the Young inequality, we have

$$\begin{aligned}|F(x,t)| &\le |a(x)|\left(b(x)|t| + \frac{c}{q(x)}|t|^{q(x)}\right)\\ &\le |a(x)|[(b(x))^{q^0(x)} + (1+c)|t|^{q(x)}].\end{aligned}$$

Then from condition (C) and Proposition 2.3 we know that $F(x,u)$ is integrable on Ω for any $u \in X$, and $\Psi(u)$ is well defined.

By Proposition 2.3, we have

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &= \int_\Omega \left(\frac{1}{p(x)}|\nabla u(x)|^{p(x)} + \frac{e(x)}{p(x)}|u(x)|^{p(x)}\right) dx - \lambda \int_\Omega F(x,u)\,dx\\ &\ge \frac{\|u\|_e^{p^-}}{p^+} - |\lambda| \int_\Omega |a(x)|[|b(x)|^{q^0(x)} + (1+c)|u|^{q(x)}]\,dx\\ &\ge \frac{\|u\|_e^{p^-}}{p^+} - |\lambda|C_1 - |\lambda|(1+c)|u|^{q^+}_{(q(\cdot),|a(\cdot)|)}\\ &\ge \frac{\|u\|_e^{p^-}}{p^+} - |\lambda|C_1 - |\lambda|C_2\|u\|_e^{q^+},\end{aligned}$$

where C_1, C_2 are positive constants. Since $q^+ < p^-$, we see that $\Phi(u) - \lambda\Psi(u)$ is coercive. ■

Furthermore, we suppose:

(D) there are $0 < r < 1/p^+$ and $1 < |\xi_1| \in \mathbb{R}$ with $\operatorname{meas}(\Omega)|\xi_1|^{p^-}\|e\|_1 > p^+ r$ such that

$$\operatorname{meas}(\Omega) \sup_{|\xi|\le d} F(x,\xi) < \frac{p^- r}{|\xi_1|^{p^+}\|e\|_1} F(x,\xi_1), \quad \text{a.e. } x \in \Omega. \tag{3.2}$$

Then we have the following main theorem.

THEOREM 3.3. *Assume* (A)–(D) *hold. Then there exist a nonempty open set* $\Lambda \subseteq \mathbb{R}$ *and a positive number* σ *with the following property: for each* $\lambda \in \Lambda$, *there exists* $\delta > 0$ *such that for each* $\mu \in [0,\delta]$, *problem* (1.1) *has at least three weak solutions whose norms are less than* σ.

Proof. From Lemma 3.1 we can see that $(\Phi')^{-1}$ is well defined, so we can use Theorem A to obtain the result. Now we show that the other hypotheses of Theorem A are satisfied.

From Lemma 3.2, we can see that (i) is satisfied.

From $\Phi(u) \le r$ we deduce that $\|u\|_e \le 1$, so

$$\|u\|_\infty \le d(rp^+)^{1/p^+} \le d. \tag{3.3}$$

Condition (D) implies that

$$\Phi(\xi_1) \ge \frac{|\xi_1|^{p^-}}{p^+}\|e\|_1 \operatorname{meas}(\Omega) > r.$$

Let $u_1(x) \equiv \xi_1$ on Ω. Then we have

$$\begin{aligned}\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) &\le \int_\Omega \sup_{|u|\le d} F(x,u)\,dx \\ &< \frac{p^- r}{\operatorname{meas}(\Omega)|u_1|^{p^+}\|e\|_1}\int_\Omega F(x,u_1)\,dx \le r\frac{\Psi(u_1)}{\Phi(u_1)}.\end{aligned} \tag{3.4}$$

Fix any $h > 1$; it is easy to see that

$$\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) + \frac{r\frac{\Psi(u_1)}{\Phi(u_1)} - \sup_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)}{h} < r\frac{\Psi(u_1)}{\Phi(u_1)}.$$

By [B1, Proposition 1.3], for ρ satisfying

$$\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) + \frac{r\frac{\Psi(u_1)}{\Phi(u_1)} - \sup_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)}{h} < \rho < r\frac{\Psi(u_1)}{\Phi(u_1)},$$

we have

$$\sup_{\lambda\in R}\inf_{u\in X}(\Phi(u)+\lambda(\rho-\Psi(u))) < \inf_{u\in X}\sup_{\lambda\in[0,\alpha_1]}(\Phi(u)+\lambda(\rho-\Psi(u))),$$

where

$$\alpha_1 = \frac{hr}{r\frac{\Psi(u_1)}{\Phi(u_1)} - \sup_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)} > 0.$$

Thus (ii) of Theorem A holds with $I = [0,\alpha_1]$. Hence all the hypotheses of Theorem A are satisfied. Consequently, there exist an open interval $\Lambda \subseteq I$ and a positive constant σ such that for any $\lambda \in \Lambda$, there exists $\delta > 0$ such that for each $\mu \in [0,\delta]$, problem (1.1) has at least three weak solutions whose norms are less than σ. ∎

Next, we consider the case when $f(x,t)$ in (1.1) is independent of x, i.e., we have the problem

$$\begin{cases} -\Delta^2_{p(x)}u + e(x)|u|^{p(x)-2}u = \lambda a(x)f(u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega. \end{cases} \tag{3.5}$$

We suppose $a(x)$ and $f(t)$ satisfy

(E) $a(\cdot) \in C^0(\overline{\Omega})$ and is nonnegative, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\liminf_{\rho \to 0} \frac{\sup_{|\xi| \le \rho} F(\xi)}{\rho^{p^+}} = 0, \tag{3.6}$$

$$\max\left\{0, \limsup_{|\xi| \to \infty} \frac{F(\xi)}{|\xi|^{p^-}}\right\} < \sup_{\xi \in \mathbb{R}} \frac{F(\xi)}{|\xi|^{p^-}}, \tag{3.7}$$

where $F(u) = \int_0^u f(t)\,dt$. Then we have the following result.

THEOREM 3.4. *Assume* (A), (B) *and* (E) *hold and e^- is small enough. Then there exist a nonempty open set $\Lambda \subseteq \mathbb{R}$ and a positive real number σ with the following property: for each $\lambda \in \Lambda$, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, problem* (3.5) *has at least three weak solutions whose norms are less than σ.*

Proof. From (3.7) we can find a $\xi_2 \in \mathbb{R}$ such that

$$\limsup_{|\xi| \to \infty} \frac{F(\xi)}{|\xi|^{p^-}} < \frac{F(\xi_2)}{|\xi_2|^{p^-}}.$$

Fix $b > 0$ such that

$$\limsup_{|\xi| \to \infty} \frac{F(\xi)}{|\xi|^{p^-}} < \frac{b}{p^+} < \frac{F(\xi_2)}{|\xi_2|^{p^-}}.$$

Then for $0 \le \lambda < \frac{1}{ba^+e^-}$ and a suitable $\beta > 0$, we have

$$\begin{aligned}
\Phi(u) - \lambda \Psi(u) &= \int_\Omega \left(\frac{1}{p(x)} |\nabla u(x)|^{p(x)} + \frac{e(x)}{p(x)} |u(x)|^{p(x)} \right) dx - \lambda \int_\Omega a(x) F(u)\,dx \\
&\ge \frac{\|u\|_e^{p^-}}{p^+} - \lambda a^+ e^- \frac{b}{p^+} \int_\Omega e(x) |u|^{p^-}\,dx - \beta \\
&\ge (1 - \lambda b a^+ e^-) \frac{\|u\|_e^{p^-}}{p^+} - \beta.
\end{aligned}$$

We can see that $\Phi(u) - \lambda \Psi(u)$ is coercive on X when $0 \le \lambda < 1/(ba^+e^-)$.

Now we denote $u_2(x) \equiv \xi_2$ on Ω. From (3.6), we can choose

$$0 < r < \min\left\{ \frac{1}{p^+}, \frac{|\xi_2|^{p^+}}{p^+} \operatorname{meas}(\Omega) \|e\|_1, \frac{|\xi_2|^{p^-}}{p^+} \operatorname{meas}(\Omega) \|e\|_1 \right\}$$

such that

$$\int_\Omega a(x) \sup_{|u| \le d(p^+ r)^{1/p^+}} F(u) \le r \frac{\Psi(u_2)}{\Phi(u_2)} \tag{3.8}$$

and

$$\Phi(u_2) = \int_\Omega e(x) \frac{|\xi_2|^{p(x)}}{p(x)}\, dx > r.$$

When $\Phi(u) \le r$, (3.3) still holds, so by (3.8),

$$\sup_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) \le r \frac{\Psi(u_2)}{\Phi(u_2)}.$$

By [B1, Proposition 1.3], for any $h > 1$ and ρ satisfying

$$\sup_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) + \frac{r\frac{\Psi(u_2)}{\Phi(u_2)} - \sup_{u \in \Phi^{-1}((-\infty, r])} \Psi(u)}{h} < \rho < r \frac{\Psi(u_2)}{\Phi(u_2)},$$

we have

$$\sup_{\lambda \in R} \inf_{u \in X} (\Phi(u) + \lambda(\rho - \Psi(u))) < \inf_{u \in X} \sup_{\lambda \in [0, \alpha_2]} (\Phi(u) + \lambda(\rho - \Psi(u))),$$

where

$$\alpha_2 = \frac{hr}{r\frac{\Psi(u_2)}{\Phi(u_2)} - \sup_{u \in \Phi^{-1}((-\infty, r])} \Psi(u)} > 0.$$

When e^- is small enough and $I = [0, \alpha_2]$, we see that all the hypotheses of Theorem A are satisfied. Hence the conclusion follows. ∎

If moreover

(F) $F(x,t) > 0$ for any $x \in \Omega$ and $|t|$ large enough, and there exists $M > 0$ such that

$$F(x,t) \le 0, \qquad x \in \Omega,\ |t| \le M, \quad \text{where} \quad F(x,t) = \int_0^t a(x) f(x,s)\, ds,$$

then we can obtain the following result.

THEOREM 3.5. *Assume* (A)–(C) *and* (F) *hold. Then there exist a nonempty open set $\Lambda \subseteq \mathbb{R}$ and a positive number σ with the following property: for each $\lambda \in \Lambda$, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, problem* (1.1) *has at least three weak solutions whose norms are less than σ.*

Proof. We define $\Psi(u) = -\int_\Omega F(x,u)\, dx$; we will prove that all hypotheses of Theorem B are satisfied.

From (C) and the argument in Theorem 3.4, we can see that (i) of Theorem B holds.

By (F), there exists $|\xi_3| > 1$ such that $F(x, \xi_3) > 0$ for any $x \in \Omega$ and $|\xi_3|^{p^-} \|e\|_1 \ge 1$. Set $a = \min\{d, M\}$. Then

$$\int_\Omega \sup_{|t| \in [0,a]} F(x,t)\, dx \le 0 < \int_\Omega F(x, \xi_3)\, dx. \tag{3.9}$$

We denote $u_0 = 0, u_3 = \xi_3$ and $r = \frac{1}{p^+}(a/d)^{p^+}$. Then it is easy to see that

$$\Phi(u_3) > r > \Phi(u_0).$$

So, (ii) of Theorem B is satisfied.

When $\Phi(u) \leq r$, similarly to the above arguments, we obtain

$$\|u\|_\infty \leq a. \tag{3.10}$$

At last, we see that

$$\frac{(\Phi(u_3) - r)\Psi(u_0) + (r - \Phi(u_0))\Psi(u_3)}{\Phi(u_3) - \Phi(u_0)} \tag{3.11}$$
$$= r\frac{\Psi(u_3)}{\Phi(u_3)} \leq -r\frac{\int_\Omega F(x, \xi_3)\, dx}{\frac{|\xi_3|^{p^+}}{p^-}\|e\|_1} < 0.$$

From the definition of Ψ and (3.10), we have

$$\begin{aligned} -\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) &= \sup_{u\in\Phi^{-1}((-\infty,r])} -\Psi(u) \\ &\leq \int_\Omega \sup_{|u|\in[0,a]} F(x, u)\, dx \leq 0. \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we can see that (iii) of Theorem B holds.

Thus all the hypotheses of Theorem B are satisfied. Hence the conclusion follows. ∎

Finally, as an application of Theorem 3.5, we consider the problem

$$\begin{cases} \Delta^2_{p(x)}u + e(x)|u|^{p(x)-2}u = \lambda(|u|^{q(x)-2}u - u), & x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega, \end{cases} \tag{3.13}$$

where $q(\cdot) \in C_+(\Omega)$ and $2 < q(x) < p^-$ for any $x \in \Omega$, and $e(\cdot)$ satisfies (A). Set

$$F(x, u) = \frac{1}{q(x)}|u|^{q(x)} - \frac{1}{2}|u|^2.$$

Then it is easy to see that $F(x, u) < 0$ when $|u| \leq 1$, and $\lim_{|u|\to\infty} F(x, u) \to +\infty$. By Theorem 3.5, there exist an open interval $\Lambda \subseteq [0, \infty)$ and a positive constant ρ such that for any $\lambda \in \Lambda$, problem (3.13) has at least three weak solutions whose norms are less than ρ.

Remark. We remark that problem (3.13) can be regarded as an eigenvalue problem. In that context, we have established the existence of a continuous family of eigenvalues for problem (3.13) which are not simple.

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