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Multiplicity results for a class of fractional boundary value problems

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Abstract. We prove the existence of at least three solutions to the following fractional boundary value problem:

$$\begin{cases} -\frac{d}{dt}\left(\frac{1}{2}\,{}_0D_t^{-\sigma}(u'(t))+\frac{1}{2}\,{}_tD_T^{-\sigma}(u'(t))\right)-\lambda\beta(t)f(u(t))-\mu\gamma(t)g(u(t))=0, & \text{a.e. } t\in[0,T],\\ u(0)=u(T)=0, \end{cases}$$

where ${}_0D_t^{-\sigma}$ and ${}_tD_T^{-\sigma}$ are the left and right Riemann–Liouville fractional integrals of order $0\le\sigma<1$ respectively. The approach is based on a recent three critical points theorem of Ricceri [B. Ricceri, A further refinement of a three critical points theorem, Nonlinear Anal. 74 (2011), 7446–7454].

1. Introduction. The aim of this paper is to establish the existence of at least three solutions to the fractional boundary value problem

$$(1.1)\quad \begin{cases} -\dfrac{d}{dt}\left(\dfrac{1}{2}\,{}_0D_t^{-\sigma}(u'(t))+\dfrac{1}{2}\,{}_tD_T^{-\sigma}(u'(t))\right)-\lambda\beta(t)f(u(t))\\ \qquad\qquad\qquad\qquad\qquad -\mu\gamma(t)g(u(t))=0, \quad \text{a.e. } t\in[0,T],\\ u(0)=u(T)=0, \end{cases}$$

where ${}_0D_t^{-\sigma}$ and ${}_tD_T^{-\sigma}$ are the left and right Riemann–Liouville fractional integrals of order $0\le\sigma<1$ respectively, $\lambda,\mu>0$ are parameters, $\beta,\gamma\in C([0,T];\mathbb{R})$, $\beta(t),\gamma(t)>0$ for all $t\in[0,T]$ and $f,g\in C([0,T];\mathbb{R})\setminus\{0\}$.

Fractional differential equations have been receiving great interest recently. This is due to both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. For details, see [E, KT1, KT2] and the references therein.

Solving differential equations of fractional order is rather involved. Some analytical methods have been presented, such as the popular Laplace trans-

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form method [P1, P2], Fourier transform method [MR], iteration method [SKM] and Green function method [SW, MLP]. Numerical schemes for solving fractional differential equations have been introduced, for example, in [DFF1, DFF2, OM1]. Recently, a great deal of effort has been expended to find robust and stable numerical as well as analytical methods for solving fractional differential equations of physical interest. The Adomian decomposition method [OM2], homotopy perturbation method [OM3], homotopy analysis method [CTXL], differential transform method [MO] and variational method [JZ] are relatively new approaches to provide an analytical approximate solution to linear and nonlinear fractional differential equations.

The existence of solutions of initial value problems for fractional order differential equations has been studied in [SKM, P1, LV] (see also references therein).

In [JZ], by using the Mountain Pass Theorem, Jiao and Zhou investigate the existence of solutions for the fractional boundary value problem

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right) + \nabla F(t,u(t)) = 0, & \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where $0 \le \beta < 1$ is a real number and ${}_0D_t^{-\beta}$ is the standard Riemann–Liouville derivative. Recently, many papers deal with the existence of multiple solutions of fractional boundary value problems: see [CT1, CT2] and the references therein.

In this paper, we investigate the existence of solutions for problem (1.1). We use variational methods.

The paper is organized as follows. In Section 2, we give preliminary facts and provide some basic properties which are needed later. Section 3 is devoted to our result on existence of three solutions.

2. Preliminaries and reminder about fractional calculus. In this section, we present some preliminaries and lemmas to be used in the proofs of the main results. For the convenience of the reader, we also present the necessary definitions. We refer the reader to [KST, P1, JZ] for basic fractional calculus.

Definition 2.1 ([KST, P1]). Let f be a function defined on $[a,b]$ and let $\gamma > 0$. The *left* and *right Riemann–Liouville fractional integrals* of order γ for f are defined by

$${}_aD_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s)\, ds, \qquad t \in [a,b],$$

$$ {}_tD_b^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} f(s)\, ds, \quad t \in [a,b], $$

provided the right-hand sides are pointwise defined on $[a,b]$; here $\Gamma > 0$ is the Gamma function.

Remark. It is easy to see that for integer $\gamma = n$ the equations in Definition 2.1 take the form

$$ {}_aD_t^{-n} f(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)\, ds, \quad t \in [a,b], $$

$$ {}_tD_b^{-n} f(t) = \frac{1}{(n-1)!} \int_t^b (s-t)^{n-1} f(s)\, ds, \quad t \in [a,b]. $$

Definition 2.2 ([KST, P1]). Let f be a function defined on $[a,b]$ and let $\gamma > 0$. The *left* and *right Riemann–Liouville fractional derivatives* of order γ for f are defined by

$$ {}_aD_t^{\gamma} f(t) = \frac{d^n}{dt^n}\, {}_aD_t^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \Big(\int_a^t (t-s)^{n-\gamma-1} f(s)\, ds \Big), $$

$$ {}_tD_b^{\gamma} f(t) = (-1)^n \frac{d^n}{dt^n}\, {}_tD_b^{\gamma-n} f(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \Big(\int_t^b (s-t)^{n-\gamma-1} f(s)\, ds \Big), $$

where $t \in [a,b]$, $n-1 \le \gamma < n$ and $n \in \mathbb{N}$. In particular, if $0 \le \gamma < 1$, then

$$ \begin{aligned} {}_aD_t^{\gamma} f(t) &= \frac{d}{dt}\, {}_aD_t^{\gamma-1} f(t) \\ &= \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \Big(\int_a^t (t-s)^{-\gamma} f(s)\, ds \Big), \quad t \in [a,b], \end{aligned} \tag{2.1} $$

$$ \begin{aligned} {}_tD_b^{\gamma} f(t) &= -\frac{d}{dt}\, {}_tD_b^{\gamma-1} f(t) \\ &= -\frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \Big(\int_t^b (s-t)^{-\gamma} f(s)\, ds \Big), \quad t \in [a,b]. \end{aligned} \tag{2.2} $$

Remark. If $f \in C([a,b],\mathbb{R}^N)$, it is obvious that the Riemann–Liouville fractional integral of order $\gamma > 0$ exists on $[a,b]$. On the other hand, from [KST, Lemma 2.2, p. 73], we know that the Riemann–Liouville fractional derivative of order $\gamma \in [n-1,n)$ exists a.e. on $[a,b]$ if $f \in AC^n([a,b],\mathbb{R}^N)$, where $C^k([a,b],\mathbb{R}^N)$ $(k = 0,1,\ldots)$ denotes the set of k times continuously differentiable mappings on $[a,b]$, $AC([a,b],\mathbb{R}^N)$ is the space of absolutely continuous functions on $[a,b]$, and $AC^k([a,b],\mathbb{R}^N)$ $(k = 0,1,\ldots)$ is the space of functions f such that $f \in C^{k-1}([a,b],\mathbb{R}^N)$ and $f^{k-1} \in AC([a,b],\mathbb{R}^N)$.

In particular, $AC([a,b],\mathbb{R}^N) = AC^1([a,b],\mathbb{R}^N)$. The left and right Caputo fractional derivatives are defined via the above Riemann–Liouville fractional derivatives (see [KST, p. 91]). In particular, they are defined for absolutely continuous functions.

Definition 2.3 ([KST]). Let $\gamma \geq 0$ and $n \in \mathbb{N}$.

(i) If $\gamma \in (n-1,n)$ and $f \in AC^n([a,b],\mathbb{R}^N)$, then the *left* and *right Caputo fractional derivatives* of order γ for f, denoted by ${}_a^cD_t^\gamma f(t)$ and ${}_t^cD_b^\gamma f(t)$, respectively, exist almost everywhere on $[a,b]$ and are given by

$$ {}_a^cD_t^\gamma f(t) = {}_aD_t^{\gamma-n} f^{(n)}(t) = \frac{1}{\Gamma(n-\gamma)} \int_a^t (t-s)^{n-\gamma-1} f^{(n)}(s)\,ds, $$

$$ {}_t^cD_b^\gamma f(t) = (-1)^n {}_tD_b^{\gamma-n} f^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \int_t^b (s-t)^{n-\gamma-1} f^{(n)}(s)\,ds, $$

where $t \in [a,b]$. In particular, if $0 < \gamma < 1$, then

$$ \begin{aligned} {}_a^cD_t^\gamma f(t) &= {}_aD_t^{\gamma-1} f'(t) \\ &= \frac{1}{\Gamma(1-\gamma)} \int_a^t (t-s)^{-\gamma} f'(s)\,ds, \quad t \in [a,b], \end{aligned} \tag{2.3} $$

$$ \begin{aligned} {}_t^cD_b^\gamma f(t) &= -{}_tD_b^{\gamma-1} f'(t) \\ &= -\frac{1}{\Gamma(1-\gamma)} \int_t^b (s-t)^{-\gamma} f'(s)\,ds, \quad t \in [a,b]. \end{aligned} \tag{2.4} $$

(ii) If $\gamma = n-1$ and $f \in AC^n([a,b],\mathbb{R}^N)$, then ${}_a^cD_t^{n-1} f(t)$ and ${}_t^cD_b^{n-1} f(t)$ are given by

$$ \begin{aligned} {}_a^cD_t^{n-1} f(t) &= f^{(n-1)}(t), && t \in [a,b], \\ {}_t^cD_b^{n-1} f(t) &= (-1)^{(n-1)} f^{(n-1)}(t), && t \in [a,b]. \end{aligned} $$

In particular, ${}_a^cD_t^0 f(t) = {}_t^cD_b^0 f(t) = f(t)$, $t \in [a,b]$.

The first result yields the semigroup property of Riemann–Liouville fractional integral operators.

Lemma 2.4 (see [KST]). *The left and right Riemann–Liouville fractional integral operators have the semigroup property:*

$$ \begin{aligned} &{}_aD_t^{-\gamma_1}({}_aD_t^{-\gamma_2} f(t)) = {}_aD_t^{-\gamma_1-\gamma_2} f(t), \\ &{}_tD_b^{-\gamma_1}({}_tD_b^{-\gamma_2} f(t)) = {}_tD_b^{-\gamma_1-\gamma_2} f(t), \qquad \forall \gamma_1,\gamma_2 > 0,\ \forall t \in [a,b], \end{aligned} $$

for every continuous function f; the equalities hold for almost every point in $[a,b]$ if $f \in L^1([a,b],\mathbb{R}^N)$.

Let us recall that for any fixed $t \in [0,T]$ and $1 \le r < \infty$,

$$\|u\|_{L^r([0,t])} = \Big(\int_0^t |u(\xi)|^r \, d\xi\Big)^{1/r}, \qquad \|u\|_{L^r} = \Big(\int_0^T |u(\xi)|^r \, d\xi\Big)^{1/r},$$
$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

Lemma 2.5 (see [JZ]). *Let $0 < \alpha \le 1$ and $1 \le r < \infty$. For any $f \in L^r([a,b],\mathbb{R}^N)$, we have*

$$\|{}_0D_\xi^{-\alpha} f\|_{L^r([0,t])} \le \frac{t^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^r([0,t])} \quad \textit{for } \xi \in [0,t],\ t \in [0,T].$$

Now, by Lemma 2.5, for any $h \in C_0^\infty([0,T],\mathbb{R}^N)$ and $1 < r < \infty$, we have $h \in L^r([0,T],\mathbb{R}^N)$ and ${}_0^cD_t^\alpha h \in L^r([0,T],\mathbb{R}^N)$. Thus, one can construct a subset $E_0^{\alpha,p}$, which depends on L^r-integrability of the Caputo fractional derivative of a function.

Definition 2.6. Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined to be the closure of $C_0^\infty([0,T],\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\alpha,p} = \Big(\int_0^T |u(t)|^p \, dt + \int_0^T |{}_0^cD_t^\alpha u(t)|^p \, dt\Big)^{1/p}. \tag{2.5}$$

Remark. (i) It is obvious that $E_0^{\alpha,p}$ is the space of functions $u \in L^p([0,T],\mathbb{R}^N)$ with ${}_0^cD_t^\alpha u \in L^p([0,T],\mathbb{R}^N)$ and $u(0) = u(T) = 0$.

(ii) For any $u \in E_0^{\alpha,p}$, noting that $u(0) = 0$, we have ${}_0^cD_t^\alpha u = {}_0D_t^\alpha u$ for $t \in [0,T]$ according to (2.3).

Lemma 2.7 ([JZ]). *Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

Lemma 2.8 ([JZ]). *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^cD_t^\alpha u\|_{L^p}. \tag{2.6}$$

Moreover, if $\alpha > 1/p$ and $1/p + 1/q = 1$, then

$$\|u\|_\infty \le \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0^cD_t^\alpha u\|_{L^p}. \tag{2.7}$$

According to (2.6), we can consider $E_0^{\alpha,p}$ with the norm

$$\|u\|_{\alpha,p} = \|{}_0^cD_t^\alpha u\|_{L^p} = \Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^p \, dt\Big)^{1/p}, \quad \forall u \in E_0^{\alpha,p}. \tag{2.8}$$

LEMMA 2.9 ([JZ]). *Let $0 < \alpha \le 1$ and $1 < p < \infty$. Assume that $\alpha > 1/p$ and $u_n \rightharpoonup u$ weakly in $E_0^{\alpha,p}$. Then $u_n \to u$ in $C([0,T],\mathbb{R}^N)$, i.e., $\|u_n - u\|_\infty \to 0$ as $n \to \infty$.*

Now, we prove that $E_0^{\alpha,p}$ is compactly embedded in $C([0,T],\mathbb{R}^N)$.

LEMMA 2.10. *Assume that $1 < p < \infty$ and $\alpha > 1/p$. Then $E_0^{\alpha,p}$ is compactly embedded in $C([0,T],\mathbb{R}^N)$.*

Proof. For $1 < p < \infty$ and $\alpha > 1/p$, from (2.7), we have $E_0^{\alpha,p} \subseteq C([0,T],\mathbb{R}^N)$, and the embedding is continuous.

Let $\{u_n\}$ be a sequence bounded in $E_0^{\alpha,p}$. Since $E_0^{\alpha,p}$ is a reflexive space, going to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ weakly in $E_0^{\alpha,p}$. Then by Lemma 2.9, $u_n \to u$ in $C([0,T],\mathbb{R}^N)$, i.e., $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Hence the embedding is compact. ■

Now, we will establish a variational structure which enables us to find solutions of problem (1.1). To that end we find the critical points of the corresponding functional defined on $E_0^{\alpha,2}$ with $1/2 < \alpha \le 1$. Then, by Lemma 2.4, for every $u \in AC([0,T],\mathbb{R})$, problem (1.1) transforms to

$$
(2.9) \qquad \begin{cases} -\dfrac{d}{dt}\left(\dfrac{1}{2}\,{}_0D_t^{-\sigma/2}({}_0D_t^{-\sigma/2}u'(t)) + \dfrac{1}{2}\,{}_tD_T^{-\sigma/2}({}_tD_T^{-\sigma/2}u'(t))\right) \\ \qquad\qquad - \lambda\beta(t)f(u(t)) - \mu\gamma(t)g(u(t)) = 0, \\ u(0) = u(T) = 0, \end{cases}
$$

for almost every $t \in [0,T]$, where $\sigma \in [0,1)$.

Furthermore, in view of Definition 2.3, it is obvious that $u \in AC([0,T],\mathbb{R})$ is a solution of problem (2.9) if and only if u is a solution of

$$
(2.10) \qquad \begin{cases} -\dfrac{d}{dt}\left(\dfrac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - \dfrac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\right) \\ \qquad\qquad - \lambda\beta(t)f(u(t)) - \mu\gamma(t)g(u(t)) = 0, \\ u(0) = u(T) = 0, \end{cases}
$$

for almost every $t \in [0,T]$, where $\alpha = 1 - \sigma/2 \in (1/2, 1]$. Therefore, we seek a solution u of problem (2.10) which, of course, corresponds to the solution u of problem (1.1) provided that $u \in AC([0,T],\mathbb{R})$.

Let us denote

$$
(2.11) \qquad D^{\alpha}(u(t)) = \frac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - \frac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t)).
$$

We are now in a position to give a definition of a solution of (2.10).

DEFINITION 2.11. A function $u \in AC([0,T],\mathbb{R})$ is called a *solution* of problem (2.10) if

(i) $D^{\alpha}(u(t))$ is differentiable for almost every $t \in [0,T]$, and
(ii) u satisfies (2.10).

In what follows, we will treat problem (2.10) in the Hilbert space $E^\alpha = E_0^{\alpha,2}$ with the corresponding norm $\|u\|_\alpha = \|u\|_{\alpha,2}$ which we defined in (2.5).

The following estimate is useful for our further discussion.

LEMMA 2.12 ([JZ]). *If $1/2 < \alpha \le 1$, then for every $u \in E^\alpha$,*

$$|\cos(\pi\alpha)|\,\|u\|_\alpha^2 \le -\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\,dt \le \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2. \tag{2.12}$$

3. Main result. Let $J_{\lambda,\mu} : E^\alpha \to \mathbb{R}$ be defined by

$$\begin{aligned} J_{\lambda,\mu}(u) = -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\,dt - \lambda\int_0^T \beta(t)F(u(t))\,dt \\ - \mu\int_0^T \gamma(t)G(u(t))\,dt \quad \text{for all } u \in E^\alpha,\ 1/2 < \alpha \le 1, \end{aligned} \tag{3.1}$$

where $F(s) = \int_0^s f(t)\,dt$ and $G(s) = \int_0^s g(t)\,dt$. Clearly, $J_{\lambda,\mu}$ is continuously differentiable on E^α, and for every $u, v \in E^\alpha$ we have

$$\begin{aligned} \langle J'_{\lambda,\mu}(u), v\rangle = -\int_0^T \frac{1}{2}[({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t), {}_0^cD_t^\alpha v(t))]\,dt \\ - \lambda\int_0^T \beta(t)f(u(t))v(t)\,dt - \mu\int_0^T \gamma(t)g(u(t))v(t)\,dt. \end{aligned} \tag{3.2}$$

Now, by (2.11) we have the following lemma:

LEMMA 3.1. *Let $1/2 < \alpha \le 1$ and $J_{\lambda,\mu}$ be defined by (3.1). If $u \in E^\alpha$ is a solution of the Euler equation $J'_{\lambda,\mu} = 0$, then u is a solution of problem (2.10) which satisfies problem (1.1).*

Proof. The proof is similar to that of [JZ, Theorem 4.2] and is omitted. ■

The goal of this work is to establish some new criteria for system (1.1) to have at least three weak solutions in X, by means of a very recent abstract critical points result of B. Ricceri [R]. First, we recall [R, Theorem 1], with easy modifications, that we are going to use.

THEOREM 3.2. *Let X be a reflexive real Banach space; let $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; and let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact and*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\widetilde{x} \in X$, with $r < \Phi(\widetilde{x})$, such that (a1) $\dfrac{\sup_{x\in\Phi^{-1}(]-\infty,r])}\Psi(x)}{r} < \dfrac{\Psi(\widetilde{x})}{\Phi(\widetilde{x})}$;

(a2) *for each* λ *in*

$$\Lambda_r := \left]\frac{\Phi(\widetilde{x})}{\Psi(\widetilde{x})}, \frac{r}{\sup_{x\in\Phi^{-1}(]-\infty,r])}\Psi(x)}\right[$$

the functional $\Phi - \lambda\Psi$ *is coercive.*

Then, for each compact interval $[a,b] \subseteq \Lambda_r$, *there exists* $\rho > 0$ *with the following property: for every* $\lambda \in [a,b]$ *and every* C^1 *functional* $\Gamma : X \to \mathbb{R}$ *with compact derivative, there exists* $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$, *the equation*

$$\Phi'(x) - \lambda\Psi'(x) - \mu\Gamma'(x) = 0$$

has at least three solutions in X *whose norms are less than* ρ.

We recall that the derivative of Φ *admits a continuous inverse* on X when there exists a continuous operator $T : X^* \to X$ such that $T(\Phi'(x)) = x$ for all $x \in X$.

Lemma 3.3. *Let* $T : E^\alpha \to (E^\alpha)^*$ *be the operator defined by*

$$\langle T(u), v\rangle = -\int_0^T \frac{1}{2}[({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t), {}_0^cD_t^\alpha v(t))]\,dt$$

for all $u, v \in E^\alpha$, *where* $(E^\alpha)^*$ *denotes the dual of* E^α. *Then* T *admits a continuous inverse on* $(E^\alpha)^*$.

Proof. By (2.12), for every $u, v \in E^\alpha$ we have

$$\begin{aligned}\langle T(u_1) - T(u_2), u_1 - u_2\rangle &= -\int_0^T \frac{1}{2}\big[\big({}_0^cD_t^\alpha(u_1(t) - u_2(t)), {}_t^cD_T^\alpha(u_1(t) - u_2(t))\big) \\ &\quad + \big({}_t^cD_T^\alpha(u_1(t) - u_2(t)), {}_0^cD_t^\alpha(u_1(t) - u_2(t))\big)\big]\,dt \\ &= -\int_0^T \big({}_0^cD_t^\alpha(u_1(t) - u_2(t)), {}_t^cD_T^\alpha(u_1(t) - u_2(t))\big)\,dt \\ &\ge |\cos(\pi\alpha)|\,\|u_1 - u_2\|_\alpha^2 > 0.\end{aligned}$$

So T is a strictly monotone operator.

Moreover, for $u_n \to u$ in E^α, we have $T(u_n) \rightharpoonup T(u)$ in $(E^\alpha)^*$. Since E^α is reflexive, we get $T(u_n) \rightharpoonup T(u)$ in $(E^\alpha)^*$. Hence T is demicontinuous. On the other hand, T is coercive since

$$\langle T(u), u\rangle \ge |\cos(\pi\alpha)|\,\|u\|_\alpha^2.$$

Now, we show that

$$\text{if}\quad u_n \rightharpoonup u \text{ and } T(u_n) \to T(u) \quad\text{then}\quad u_n \to u. \tag{3.3}$$

Let us take a sequence $\{u_n\} \subseteq E^\alpha$ such that $u_n \rightharpoonup u$ in E^α and $T(u_n) \to T(u)$ in $(E^\alpha)^*$. Then

$$\langle T(u_n) - T(u), u_n - u\rangle \le \|T(u_n) - T(u)\|_\alpha \|u_n - u\|_\alpha \to 0. \tag{3.4}$$

Note that

$$\begin{aligned}\langle T(u_n) - T(u), u_n - u\rangle &= -\int_0^T \big({}_0^c D_t^\alpha (u_n(t) - u(t)), {}_t^c D_T^\alpha (u_n(t) - u(t))\big)\, dt \\ &\ge |\cos(\pi\alpha)|\, \|u_n - u\|_\alpha^2.\end{aligned}$$

So, by (3.4), we have $\|u_n - u\|_\alpha^2 \to 0$ as $n \to \infty$, and hence $u_n \to u$ in E^α.

Note that the strict monotonicity of T implies its injectivity. Moreover, T is coercive and demicontinuous, so it is semicontinuous. Consequently, thanks to the Minty–Browder theorem [Z1], the operator T is a surjection and admits an inverse mapping.

It then suffices to show the continuity of T^{-1}. Let $f_n \to f$ in $(E^\alpha)^*$. Let $\{u_n\}$ in E^α be such that

$$T^{-1}(f_n) = u_n \quad \text{and} \quad T^{-1}(f) = u.$$

By the coercivity of T, $\{u_n\}$ is bounded in the reflexive space E^α. For a suitable subsequence, we have $u_n \rightharpoonup \widetilde{u}$ in E^α, which implies

$$\lim_{n\to\infty} \langle T(u_n) - T(u), u_n - \widetilde{u}\rangle = \lim_{n\to\infty} \langle f_n - f, u_n - \widetilde{u}\rangle = 0.$$

It follows from (3.3) and the continuity of T that $u_n \to \widetilde{u}$ in E^α and $T(u_n) \to T(\widetilde{u})$ in $(E^\alpha)^*$.

Moreover, since T is an injection, we conclude that $u = \widetilde{u}$. ■

We assume that the nonlinear term $f \in C(\mathbb{R}, \mathbb{R})$ has the following properties:

(H1) There exist constants $a_1, a_2, a_3 > 0$ such that

$$|F(s)| \le a_1 |s|^2 + a_2 |s|^{2-q} + a_3, \quad s \in \mathbb{R},$$

for some $q \in (0, 2)$ and

$$a_1 \in \left[0, \frac{|\cos(\pi\alpha)|}{2} \|\beta\|_\infty^{-1} \left(\frac{\sqrt{2}\, T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\right)^{-2}\right).$$

(G1) $g \in C(\mathbb{R}^+, \mathbb{R})$ and there exist constants $M > 0$ and $\gamma \in (0, 2)$ such that

$$g(0) = 0 \quad \text{and} \quad |g(s)| \le M + s^{2-\gamma}.$$

Our main result reads as follows.

THEOREM 3.4. *Let $\alpha \in (1/2, 1]$ and $f \in C(\mathbb{R}, \mathbb{R})$ be a function such that* (H1) *and* (G1) *hold. Assume that there exist positive constants $\alpha_1, \beta_1, \delta, \gamma$ with $\alpha_1 + \beta_1 < 1$ and $\delta > \gamma L_{\alpha_1, \beta_1}$ and a function $\omega_\delta \in E^\alpha$ such that*

$$-\int_0^T ({}_0^cD_t^\alpha \omega_\delta(t), {}_t^cD_T^\alpha \omega_\delta(t))\,dt + \int_0^T |\omega_\delta(t)|^2\,dt > 2\gamma, \tag{3.5}$$

$$\frac{\int_0^T \beta(t)F(\omega_\delta(t))\,dt}{\delta^2} > K_{\alpha_1,\beta_1}\|\beta\|_\infty \bigg[a_1\bigg(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\bigg)^2 \frac{2}{|\cos(\pi\alpha)|} + a_2T^{q/2}\bigg(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\bigg)^{2-q}\bigg(\frac{2}{|\cos(\pi\alpha)|}\bigg)^{1-q/2}\frac{1}{\gamma^q} + a_3T\bigg], \tag{3.6}$$

where

$$K_{\alpha_1,\beta_1} = \frac{1}{L^2_{\alpha_1,\beta_1}} = \frac{1}{2|\cos(\pi\alpha)|}\bigg[\frac{1}{\Gamma^2(2-\alpha)(3-2\alpha)} \times \big(\beta_1^2\alpha_1^{3-2\alpha} + T^{3-2\alpha} - (T-\beta_1)^{3-2\alpha}\big)\bigg] > 0.$$

Then, for each compact interval

$$\begin{aligned}[a,b] &\subset \Lambda_{\alpha_1,\beta_1,\gamma,\delta} \\ &:= \bigg[\frac{\int_0^T \beta(t)F(\omega_\delta(t))\,dt}{K_{\alpha_1,\beta_1}\delta^2}, \|\beta\|_\infty^{-1}\bigg[a_1\bigg(\frac{\sqrt{2}\,T^{(\alpha+1)/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\bigg)^2\frac{2}{|\cos(\pi\alpha)|} \\ &\quad + a_2T^{q/2}\bigg(\frac{\sqrt{2}\,T^{(\alpha+1)/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\bigg)^{2-q}\bigg(\frac{2}{|\cos(\pi\alpha)|}\bigg)^{1-q/2}\frac{1}{\gamma^q} + a_3T\bigg]^{-1}\bigg] \\ &\subseteq \bigg]\frac{\Phi(\omega_\delta)}{\Psi(\omega_\delta)}, \frac{\gamma^2}{\sup_{u\in\Phi^{-1}(]-\infty,\gamma^2])}\Psi(u)}\bigg[,\end{aligned}$$

there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$ and any g which satisfies (G1)*, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the problem* (1.1) *has at least three distinct solutions in E^α whose norms are less than ρ.*

Proof. In order to apply Theorem 3.2 to our problem, let $X := E^\alpha$ and consider the functionals $\Phi, \Psi : E^\alpha \to \mathbb{R}$ defined by

$$\begin{aligned}\Phi(u) &= -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\,dt, \\ \Psi(u) &= \int_0^T \beta(t)F(u(t))\,dt, \qquad \forall u \in E^\alpha.\end{aligned} \tag{3.7}$$

It is clear that both Φ and Ψ are well-defined and continuously Gâteaux differentiable. This follows from Lemma 3.3 and the standard fact that Φ is a coercive, sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $(E^\alpha)^*$.

We claim that $\Psi' : E^\alpha \to (E^\alpha)^*$ is a compact operator. To see this, it is enough to show that Ψ' is strongly continuous on E^α. For this, for fixed $u \in E^\alpha$ let $u_n \rightharpoonup u$ weakly in E^α as $n \to \infty$. According to (2.7) and Lemma 2.9, we have $u_n \to u$ in $C([0,T],\mathbb{R})$, which yields

$$\int_0^T \beta(t)f(u_n(t))dt \to \int_0^T \beta(t)f(u(t))\,dt \qquad \text{strongly as } n \to \infty.$$

Thus, Ψ' is strongly continuous on E^α, which implies that Ψ' is a compact operator by [Z2, Proposition 26.2]. Hence the claim is true.

Now, $\Phi(0) = \Psi(0) = 0$. Let $\rho \in \,]0,+\infty[$ and consider the function

$$\chi(\rho) := \frac{\sup_{\omega\in\Phi^{-1}(]-\infty,\rho])}\Psi(\omega)}{\rho}.$$

Taking into account (H1) and by the Hölder inequality, it follows that

$$\begin{aligned}
\Psi(u) &= \int_0^T \beta(t)F(u(t))\,dt \le \int_0^T \beta(t)[a_1(u(t))^2 + a_2(u(t))^{2-q} + a_3]\,dt \\
&\le \|\beta\|_\infty\Big[a_1\int_0^T |u(t)|^2\,dt + a_2\Big(\int_0^T 1^{2/q}\,dt\Big)^{q/2}\Big(\int_0^T |u(t)|^2\,dt\Big)^{1-p/2} + a_3T\Big] \\
&\le \|\beta\|_\infty[a_1\|u\|_\infty^2 + a_2T^{q/2}\|u\|_\infty^{2-q} + a_3T] \\
&\le a_1\|\beta\|_\infty\bigg(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\bigg)^2\|u\|_\alpha^2 \\
&\quad + a_2T^{q/2}\|\beta\|_\infty\bigg(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\bigg)^{2-q}\|u\|_\alpha^{2-q} + \|\beta\|_\infty a_3T.
\end{aligned}$$

Then, for every $u \in E^\alpha$ such that $u \in \Phi^{-1}(]-\infty,\rho]$, owing to (2.12), we get

$$\begin{aligned}
\Psi(u) \le \|\beta\|_\infty\bigg[&a_1\bigg(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\bigg)^2\frac{2\rho}{|\cos(\pi\alpha)|} \\
&+ a_2T^{q/2}\bigg(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\bigg)^{2-q}\bigg(\frac{2}{|\cos(\pi\alpha)|}\bigg)^{1-q/2}\rho^{1-q/2} + a_3T\bigg].
\end{aligned}$$

Hence, by using the definition of Φ,

$$\begin{aligned}
\sup_{u\in\Phi^{-1}(]-\infty,\rho])}\Psi(u) \le \|\beta\|_\infty\bigg[&a_1\bigg(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\bigg)^2\frac{2\rho}{|\cos(\pi\alpha)|} \\
&+ a_2T^{q/2}\bigg(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\bigg)^{2-q}\bigg(\frac{2}{|\cos(\pi\alpha)|}\bigg)^{1-q/2}\rho^{1-q/2} + a_3T\bigg].
\end{aligned}$$

This yields

$$\begin{aligned}(3.8)\qquad \chi(\rho) \le \|\beta\|_\infty\bigg[&a_1\left(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\right)^2\frac{2}{|\cos(\pi\alpha)|}\\ &+ a_2T^{q/2}\left(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\right)^{2-q}\left(\frac{2}{|\cos(\pi\alpha)|}\right)^{1-q/2}\frac{1}{\rho^{q/2}} + a_3T\bigg]\end{aligned}$$

for every $\rho > 0$. Let

$$\omega_\delta(t) = \begin{cases} \delta\beta_1 t & \text{if } 0 \le t < \alpha_1,\\ \delta\alpha_1\beta_1 & \text{if } \delta_1 \le t \le T-\beta_1,\\ \delta\alpha_1(T-t) & \text{if } T-\beta_1 < t \le T.\end{cases}$$

It is easy to see that $\omega_\delta \in E^\alpha$ and

$$\begin{aligned}(3.9)\qquad 0 < \Phi(\omega_\delta) \le \frac{1}{2|\cos(\pi\alpha)|}\bigg[&\frac{1}{\Gamma^2(2-\alpha)(3-2\alpha)}\big(\beta_1^2\alpha_1^{3-2\alpha} + T^{3-2\alpha}\\ &- (T-\beta_1)^{3-2\alpha}\big)\bigg]\delta^2.\end{aligned}$$

A direct computation taking into account that $\delta > \gamma L_{\alpha_1,\beta_1}$ yields $\gamma^2 < \Phi(\omega_\delta)$.

Moreover,

$$(3.10)\qquad \Psi(\omega_\delta) = \int_0^T \beta(t)F(\omega_\delta(t))\,dt.$$

Hence, from (3.9) and (3.10), one has

$$(3.11)\qquad \frac{\Psi(\omega_\delta)}{\Phi(\omega_\delta)} \ge \frac{\int_0^T \beta(t)F(\omega_\delta(t))\,dt}{K_{\alpha_1,\beta_1}\delta^2}.$$

In view of (3.6) and taking into account (3.8) and (3.11), we get

$$\begin{aligned}\chi(\gamma^2) &= \frac{\sup_{\omega\in\Phi^{-1}(]-\infty,\rho])}\Psi(\omega)}{\gamma^2}\\ &\le \|\beta\|_\infty\bigg[a_1\left(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\right)^2\frac{2}{|\cos(\pi\alpha)|}\\ &\quad + a_2T^{q/2}\left(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\right)^{2-q}\left(\frac{2}{|\cos(\pi\alpha)|}\right)^{1-q/2}\frac{1}{\gamma^q} + a_3T\bigg]\\ &< \frac{\int_0^T \beta(t)F(\omega_\delta(t))\,dt}{K_{\alpha_1,\beta_1}\delta^2} \le \frac{\Psi(\omega_\delta)}{\Phi(\omega_\delta)}.\end{aligned}$$

Therefore, the assumption (a1) of Theorem 3.2 is satisfied with $\widetilde{x} := \omega_\delta$ and $r := \gamma^2$. Moreover, owing to (H1), by (2.12) and the Hölder inequality,

$$\begin{aligned}
\Phi(u)-\lambda\Psi(u) &= -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\,dt - \lambda\int_0^T \beta(t)F(u(t))\,dt \\
&\geq -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\,dt - \int_0^T \beta(t)[a_1(u(t))^2 + a_2(u(t))^{2-q} + a_3]\,dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,dt \\
&\quad - \|\beta\|_\infty\Big[a_1\int_0^T |u(t)|^2\,dt + a_2\Big(\int_0^T 1^{2/q}\,dt\Big)^{q/2}\Big(\int_0^T |u(t)|^2\,dt\Big)^{1-p/2} + a_3T\Big] \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \|\beta\|_\infty[a_1\|u\|_\infty^2 + a_2T^{q/2}\|u\|_\infty^{2-q} + a_3T] \\
&\geq \bigg[\frac{|\cos(\pi\alpha)|}{2} - a_1\|\beta\|_\infty\bigg(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\bigg)^2\bigg]\|u\|_\alpha^2 \\
&\quad - a_2T^{q/2}\|\beta\|_\infty\bigg(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\bigg)^{2-q}\|u\|_\alpha^{2-q} - \|\beta\|_\infty a_3T.
\end{aligned}$$

Since $q \in (0,2)$, the functional $\Phi - \lambda\Psi$ is coercive for every positive parameter, in particular, for every

$$\lambda \in \Lambda_{\alpha_1,\beta_1,\gamma,\delta} \subseteq \bigg]\frac{\Phi(\omega_\delta)}{\Psi(\omega_\delta)}, \frac{\gamma^2}{\sup_{u\in\Phi^{-1}(]-\infty,\gamma^2])}\Psi(u)}\bigg[.$$

In particular, on account of Theorem 3.2, for every interval $[a,b] \subset \Lambda_{\alpha_1,\beta_1,\gamma,\delta}$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$ and every $g \in C(\mathbb{R}^+,\mathbb{R})$ which satisfies (G1), there exists $\delta > 0$ such that for every $\mu \in [0,\delta]$, the equation $\Phi'(x) - \lambda\Psi'(x) - \mu\Gamma'(x) = 0$ admits at least three solutions in E^α whose norms are less than ρ, where $\Gamma : E^\alpha \to \mathbb{R}$ is defined by

$$\Gamma(u) = \int_0^T \gamma(t)G(u(t))\,dt. \tag{3.12}$$

Here, we have exploited again the fact that E^α is compactly embedded in $C([0,T],\mathbb{R}^N)$ for every $1/2 < \alpha \leq 1$, thus Γ is of class C^1 with compact derivative. Since the solutions of system (1.1) are exactly the critical points of the functional $J_{\lambda,\mu} = \Phi'(x) - \lambda\Psi'(x) - \mu\Gamma'(x) = 0$, the proof is complete. ∎

Remark. The hypothesis $a_1 \in \big[0, \frac{|\cos(\pi\alpha)|}{2}\|\beta\|_\infty^{-1}\big(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\big)^{-2}\big)$ in (H1) can be substituted by the following growth condition.

(H2) $\limsup_{|s|\to\infty} F(s)/s^2 = 0$.

Indeed, owing to (H1), problem (1.1) is well defined. Therefore, the functional $\Phi - \lambda\Psi$ is coercive for every $\lambda \in (0,\infty)$. Indeed, for every $\epsilon > 0$ we have $|F(s)| \le \epsilon|s|^2 + c(\epsilon)$ for every $s \in \mathbb{R}$. Consequently, for every $u \in E^\alpha$,

$$\begin{aligned}\Phi(u) &- \lambda\Psi(u)\\ &\ge \left[\frac{|\cos(\pi\alpha)|}{2} - \epsilon\|\beta\|_\infty\left(\frac{\sqrt{2}\,T^{(2\alpha-1)/2}}{\Gamma(\alpha)(\alpha+1)^{1/2}}\right)^2\right]\|u\|_\alpha^2 - c(\epsilon)T\|\beta\|_\infty.\end{aligned}$$

Hence, $\Phi - \lambda\Psi$ is coercive for every real positive parameter λ.

Remark. Let $f \in C(\mathbb{R}^+,\mathbb{R}^+)$ be such that

$$|f(s)| \le c_1|s|^{r-1}, \qquad \forall s \in \mathbb{R},$$

for some $c_1 > 0$ and $r \in (2,\infty)$. Clearly, the above growth condition is a particular case of hypothesis (H1) and implies $f(0) = 0$. In this setting, under the additional hypothesis (H2), Theorem 3.4 ensures the existence of at least three solutions for every

$$\lambda > \lambda^* := \frac{1}{K_{\alpha_1,\beta_1}} \inf_{\substack{\delta>0\\ \omega_\delta\in E^\alpha}} \frac{\int_0^T \beta(t)F(\omega_\delta(t))\,dt}{\delta^2}.$$

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