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On inhomogeneous self-similar measures and their L^q spectra

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Abstract. Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, N$ be contracting similarities, let $(p_1, \ldots, p_N, p)$ be a probability vector and let ν be a probability measure on $\mathbb{R}^d$ with compact support. It is well known that there exists a unique inhomogeneous self-similar probability measure μ on $\mathbb{R}^d$ such that $\mu = \sum_{i=1}^N p_i \mu \circ S_i^{-1} + p\nu$.

We give satisfactory estimates for the lower and upper bounds of the L^q spectra of inhomogeneous self-similar measures. The case in which there are a countable number of contracting similarities and probabilities is considered. In particular, we generalise some results obtained by Olsen and Snigireva [Nonlinearity 20 (2007), 151–175] and we give a partial answer to Question 2.7 in that paper.

1. Introduction. Inhomogeneous self-similar measures are a natural extension of the homogeneous measures that satisfy $\mu = \sum_{i=1}^N p_i \mu \circ S_i^{-1}$, which have been an object of study for the past 25 years (for instance, see [6] and the references therein). As homogeneous self-similar measures are solutions of the equation

$$\mu - \sum_{i=1}^{N} p_i \mu \circ S_i^{-1} = 0, \tag{1.1}$$

it is of interest to investigate the corresponding inhomogeneous equation

$$\mu - \sum_{i=1}^{N} p_i \mu \circ S_i^{-1} = p\nu, \tag{1.2}$$

where ν is a fixed probability measure with support in a compact set $C \subset \mathbb{R}^d$. At this point, it is worth referring to the inhomogeneous self-similar sets that satisfy $K = \bigcup_{i=1}^N S_i(K) \cup C$. The proof of the existence and uniqueness of such sets can be found in [1, Theorem 3.7.1]. These sets, which are actually

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fractals, are closely related to measures μ that satisfy (1.2). Specifically, it is proved in [15, Proposition 1.2] that $K = \operatorname{supp}\mu$ is as above.

Inhomogeneous self-similar measures and sets were introduced by Barnsley et al. [2, 4] as a tool for image compression; [4] gives a few examples of such measures. They also feature in [3].

One may ask whether any dynamical interpretation of inhomogeneous self-similar measures exists. The answer is affirmative: a measure μ which satisfies (1.2), can be viewed as an invariant measure for the operator $M\mu = \sum_{i=1}^{N} p_i \mu \circ S_i^{-1} + p\nu$, which acts on measures concentrated on the set K and is generated by the iterated function system $(S_1, \dots, S_N, p_1, \dots, p_N, p, \nu)$. The operator M is then the transition operator for a random sequence that takes values in K. Hence, an inhomogeneous self-similar measure can also be viewed as an invariant distribution of a Markov chain, and moreover, every measure μ that satisfies $M\mu = \mu$ is weakly asymptotically stable. These facts have been described in detail by Lasota [11]. In connection with the above-mentioned interpretation, a method of construction of such measures connected with probabilities will be presented.

Because μ is invariant for M, it is also natural to consider estimates for various dimensions of measures satisfying (1.2) or, if $p = 0$, (1.1). There is a huge body of literature on L^q spectra of homogeneous self-similar measures (for example, see [6] and the references therein). In the broad sense of investigating dimensions of invariant measures, the inhomogeneous case was considered, i.a., by Horbacz et al. (see [5], [9] and [10], for instance). The work of Olsen and Snigireva [15] is devoted to estimates of the L^q spectra and the Rényi dimensions of inhomogeneous self-similar measures under the assumption that the sets $(S_1K, \dots, S_NK, C)$ are pairwise disjoint. In this case, non-trivial lower and upper bounds for the above-mentioned dimensions were obtained.

When examining the L^q spectra of inhomogeneous self-similar measures, the assumption that the sets $(S_1K, \dots, S_NK, C)$ are pairwise disjoint is clearly unsatisfactory, as already stated by the authors of [15], who asked (see [15, Question 2.7]) whether the results obtained in [15, Section 2] are true when only the Finite Weak Inhomogeneous Open Set Condition (FWIOSC), a version of the standard Open Set Condition, is assumed. Namely, the FWIOSC is satisfied if there exists a non-empty and bounded open set U such that the following conditions are satisfied:

(FW1) $\quad S_i(U) \subseteq U, \quad i \in \{1, \dots, N\}, \quad C \subseteq \overline{U},$

(FW2) $\quad S_i(U) \cap S_j(U) = \emptyset, \quad i \neq j,\ i, j \in \{1, \dots, N\},$

(FW3) $\quad S_i(U) \cap C = \emptyset, \quad i \in \{1, \dots, N\}.$

In the present paper, we will consider the FWIOSC and provide a partial answer to this question. More precisely, we will prove [15, Theorem 2.1] assuming the FWIOSC, and we will also obtain much more accurate estimates. Moreover, our main theorem extends these results to the case of a countable number of contracting similarities.

For a more complete treatment of other problems relating to infinite iterated function systems, we refer the reader to Mauldin and Urbański's book [14].

2. Preliminaries. Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, 2, \ldots$, be contracting similarities and let r_i denote the contraction ratio of S_i. We assume that

$$\|\mathbf{r}\| := \sum_{i=1}^{\infty} r_i < \infty.$$

Let $B(x, r)$ denote the closed ball with centre $x \in \mathbb{R}^d$ and radius r; by $\operatorname{int} A$ we denote the interior of any set $A \subset \mathbb{R}^d$; $\overline{A}$ or $\operatorname{cl} A$ stands for the closure of A. Finally, $\mathcal{B}_X$ denotes the σ-algebra of Borel subsets of X.

Let $N \geq 2$ and let $C \subset \mathbb{R}^d$ be a fixed, non-empty, compact set. It is well known that there exists a unique homogeneous self-similar set $K_\emptyset$ (see [7, 8]) and a unique inhomogeneous self-similar set K such that

$$K_\emptyset = \bigcup_{i=1}^{N} S_i(K_\emptyset),$$

and

$$K = \bigcup_{i=1}^{N} S_i(K) \cup C \tag{2.1}$$

with respect to the contracting similarities $(S_1, \ldots, S_N)$. Both $K_\emptyset$ and K are non-empty and compact.

Our considerations are carried out under the assumption of the Infinite Weak Inhomogeneous Open Set Condition, briefly IWIOSC, which is as follows: there exists a non-empty and bounded open set U such that:

(IW1) $\quad S_i(U) \subseteq U, \quad i \in \mathbb{N}, \quad C \subseteq \overline{U},$

(IW2) $\quad S_i(U) \cap S_j(U) = \emptyset, \quad i \neq j,\ i, j \in \mathbb{N},$

(IW3) $\quad S_i(U) \cap C = \emptyset, \quad i \in \mathbb{N}.$

Note that the above condition differs from those in the introduction, due to countably many contracting similarities.

3. An inhomogeneous self-similar set. The main purpose of this section is to prove the existence, uniqueness and some properties of a non-empty, compact set satisfying the countable version of (2.1). We start with the following theorem.

THEOREM 3.1. *Let $(S_i)_{i=1}^{\infty}$ be contracting similarities and let r_i be the contraction ratios of S_i. Assume that the IWIOSC is satisfied and* $\sup_{i\in\mathbb{N}} r_i < 1$. *Then there exists a unique, non-empty, compact set $K_\infty \subseteq \overline{U}$ such that*

$$K_\infty = \mathrm{cl} \bigcup_{i=1}^{\infty} S_i(K_\infty) \cup C. \tag{3.1}$$

Proof. The proof is analogous to the proof of [16, Lemma 4.1] and therefore could be omitted. However, there are a few small differences, so we give a brief sketch. Consider the space $(\mathbf{K}(\mathbb{R}^d)_{|\overline{U}}, D)$, where $\mathbf{K}(\mathbb{R}^d)_{|\overline{U}}$ is the family of all compact subsets of $\mathbb{R}^d$ intersected with $\overline{U}$, and D denotes the Hausdorff metric. This space is complete as a closed subspace of a complete space. In addition, we consider the mapping $T : \mathbf{K}(\mathbb{R}^d)_{|\overline{U}} \to \mathbf{K}(\mathbb{R}^d)_{|\overline{U}}$ defined by $T(A) = \mathrm{cl} \bigcup_{i=1}^{\infty} S_i(A) \cup C$, which, by assumption, is a contraction with respect to D. Finally, the assertion follows from Banach's fixed point theorem. ∎

Define

$$\overline{\mathcal{F}} := \mathrm{cl} \bigcup_{N=2}^{\infty} K_N = \mathrm{cl} \bigcup_{N=2}^{\infty} \bigcup_{i=1}^{N} S_i(K_N) \cup C,$$

where, for each $N \geq 2$, the set K_N is the unique, non-empty, compact set satisfying (2.1). We will now show some properties of $\overline{\mathcal{F}}$.

LEMMA 3.1. *Assume that the IWIOSC is satisfied. Then the set $\overline{\mathcal{F}}$ is compact and $\overline{\mathcal{F}} \subseteq \overline{U}$.*

Proof. The proof is similar to the proof of Theorem 3.1 but, for the sake of clarity, we will provide a complete argument. Let $\mathbf{K}(\mathbb{R}^d)$ be the family of all compact subsets of $\mathbb{R}^d$, equipped with the Hausdorff metric D. For each $N \geq 2$, define $T_N : \mathbf{K}(\mathbb{R}^d) \to \mathbf{K}(\mathbb{R}^d)$ by $T_N(A) = \bigcup_{i=1}^{N} S_i(A) \cup C$. The space $(\mathbf{K}(\mathbb{R}^d), D)$ is complete and each T_N is a contraction mapping; thus, if A is compact, then, by Banach's fixed point theorem, $T_N^n(A) \to K_N$ as $n \to \infty$ with respect to the Hausdorff metric. In particular, for each $N \geq 2$, we have $T_N^n(\overline{U}) \to K_N$ as $n \to \infty$. From condition (IW1), we deduce that $\overline{U} \supseteq T_N(\overline{U}) \supseteq (T_N)^2(\overline{U}) \supseteq \cdots$, whence $K_N = \lim_n (T_N)^n(\overline{U}) \subseteq \overline{U}$ for each $N \geq 2$. Hence,

$$\bigcup_{i=1}^{N} S_i(K_N) \cup C = K_N \subseteq \overline{U}$$

and so

$$\overline{\mathcal{F}} = \mathrm{cl} \bigcup_{N=2}^{\infty} \bigcup_{i=1}^{N} S_i(K_N) \cup C = \mathrm{cl} \bigcup_{N=2}^{\infty} K_N \subseteq \overline{U}.$$

We have shown above that $\overline{\mathcal{F}} \subseteq \overline{U}$, which means that $\overline{\mathcal{F}}$ is bounded. The fact that $\overline{\mathcal{F}}$ is also closed completes the proof. ∎

The next theorem gives a connection between self-similar sets.

THEOREM 3.2. *For each $N \geq 2$, let K_N be a homogeneous or inhomogeneous self-similar set. Assume that the IWIOSC is satisfied and that $\overline{U} = \operatorname{cl}\bigcup_{N=2}^{\infty} K_N$. Then*

$$K_\infty = \operatorname{cl}\bigcup_{N=2}^{\infty} K_N.$$

Proof. It suffices to check that

$$\operatorname{cl}\bigcup_{N=2}^{\infty} K_N = \operatorname{cl}\bigcup_{i=1}^{\infty} S_i\Big(\operatorname{cl}\bigcup_{N=2}^{\infty} K_N\Big) \cup C.$$

To do so, observe that

$$\begin{aligned}\operatorname{cl}\bigcup_{i=1}^{\infty} S_i\Big(\operatorname{cl}\bigcup_{N=2}^{\infty} K_N\Big) \cup C &= \operatorname{cl}\bigcup_{i=1}^{\infty}\bigcup_{N=2}^{\infty} S_i(K_N) \cup C \\ &= \operatorname{cl}\bigcup_{N=2}^{\infty}\Big(\bigcup_{i=1}^{N} S_i(K_N) \cup C \cup \bigcup_{i=N+1}^{\infty} S_i(K_N)\Big) \\ &= \operatorname{cl}\bigcup_{N=2}^{\infty} K_N \cup \operatorname{cl}\bigcup_{N=2}^{\infty}\bigcup_{i=N+1}^{\infty} S_i(K_N) = \operatorname{cl}\bigcup_{N=2}^{\infty} K_N,\end{aligned}$$

as, by assumption, $\operatorname{cl}\bigcup_{N=2}^{\infty}\bigcup_{i=N+1}^{\infty} S_i(K_N) \subseteq \overline{U} = \operatorname{cl}\bigcup_{N=2}^{\infty} K_N$. Because it was shown that $\operatorname{cl}\bigcup_{N=2}^{\infty} K_N$ also satisfies (3.1), this result proves the theorem. ∎

We finish this section by introducing the notation

$$K_{\infty|n} := \bigcup_{i=1}^{n} S_i(K_\infty) \cup C, \qquad n \in \mathbb{N},$$

which will be used later in Theorem 5.4.

4. An inhomogeneous self-similar measure. We begin by showing, for completeness, the existence and uniqueness of an inhomogeneous self-similar measure in the case of a countable number of contracting similarities and probabilities. Then, we will touch upon a question regarding this measure's interpretation that is connected with infinite iterated function systems.

Denote by $\mathcal{M}(\mathbb{R}^d)$ the family of all Borel signed measures on $\mathbb{R}^d$ and let $\mathcal{M}_1(\mathbb{R}^d)$ stand for the subspace of $\mathcal{M}(\mathbb{R}^d)$ that consists of all probability measures. It is well known that the following formula defines a norm, called the *total variation norm*, in $\mathcal{M}(\mathbb{R}^d)$:

$$\|\mu\|_{\mathrm{TV}} = \mu^+(\mathbb{R}^d) + \mu^-(\mathbb{R}^d),$$

where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ. Moreover, $\mathcal{M}(\mathbb{R}^d)$ is a Banach space for this norm. When $\mu_1, \mu_2 \in \mathcal{M}_1(\mathbb{R}^d)$, the distance between them in the total variation norm can be expressed as

$$\|\mu_1 - \mu_2\|_{\mathrm{TV}} = 2\sup\{|\mu_1(A) - \mu_2(A)| : A \in \mathcal{B}_{\mathbb{R}^d}\}.$$

Let $M : \mathcal{M}_1(\mathbb{R}^d) \to \mathcal{M}_1(\mathbb{R}^d)$ be a Markov operator. A measure μ is called *invariant* if $M\mu = \mu$. We say that M is *strongly asymptotically stable* if there exists an invariant measure μ such that for every measure η, the sequence $M^n\eta$ converges to μ in the total variation norm.

The following theorem strengthens the well known result on weak asymptotic stability. In particular, we obtain the existence and uniqueness of an inhomogeneous self-similar measure in the case of countably many contracting similarities.

THEOREM 4.1. *Let ν be a Borel probability measure with compact support $C \subset \mathbb{R}^d$, let $(p_1, p_2, \ldots, p)$ be a probability vector with positive constant probability p and let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ be contracting similarities. Then the Markov operator $M : \mathcal{M}_1(\mathbb{R}^d) \to \mathcal{M}_1(\mathbb{R}^d)$ defined by*

$$M\mu(A) = \sum_{i=1}^{\infty} p_i \mu \circ S_i^{-1}(A) + p\nu(A)$$

is strongly asymptotically stable. In particular, there exists a unique probability measure μ that satisfies

$$\mu(A) = \sum_{i=1}^{\infty} p_i \mu \circ S_i^{-1}(A) + p\nu(A). \tag{4.1}$$

Proof. It is enough to observe that M is contractive in the total variation norm with a ratio $1 - p$. ∎

From now on, the unique probability measure satisfying (4.1) will be denoted by μ_∞. It should be clear that μ_∞ is an arbitrary inhomogeneous self-similar measure that consists of a countable number of contracting similarities and probabilities. The subscript "∞" is added to indicate this fact.

Because the support of every finite measure is a closed set (see [13, Remark 1.55]), the following theorem can be proved in much the same way as [15, Proposition 1.2].

THEOREM 4.2. *Let μ_∞ be the unique inhomogeneous self-similar measure given by* (4.1) *and let K_∞ be the unique, non-empty, compact set satisfying* (3.1). *Then* $\operatorname{supp} \mu_\infty = K_\infty$.

Proof. The proof is analogous to the proof of [15, Proposition 1.2]; therefore, it is omitted. ∎

As mentioned at the beginning of this section, it is important to recall a dynamical interpretation of inhomogeneous self-similar measures. This interpretation is connected with iterated function systems. For a fuller treatment, we refer the reader to [13] or, for even more details, to [5], [11].

Consider a probability space (Ω, Σ, P). Denote by $\mathcal{M}_1^{K_\infty}$ the family of all Borel probability measures μ on $\mathbb{R}^d$ such that $\mu(K_\infty) = 1$. Let $\mathcal{S}_i := S_{i|K_\infty} : K_\infty \to K_\infty$ be the restrictions of contracting similarities to the set K_∞ and let $(p_1, p_2, \dots, p)$ be a probability vector. Define the operator $M : \mathcal{M}_1^{K_\infty} \to \mathcal{M}_1^{K_\infty}$ by the formula

$$M\mu(A) = \sum_{i=1}^{\infty} p_i \mu \circ \mathcal{S}_i^{-1}(A) + p\nu(A). \tag{4.2}$$

Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of identically distributed random variables with values in $\mathbb{N}_0$. Moreover, let x_0, ξ be random elements that assume values in K_∞ and have distributions

$$\mu_0(A) = P(x_0 \in A), \quad \nu(A) = P(\xi \in A) \quad \text{for } A \in \mathcal{B}_{K_\infty}.$$

If x_0, ξ, k_n are independent, then M is the transition operator for a random sequence $x_n : \Omega \to K_\infty$ where

$$x_{n+1} = \begin{cases} \mathcal{S}_{k_n}(x_n) & \text{if } k_n \in \mathbb{N}, \\ \xi & \text{if } k_n = 0, \end{cases}$$

where

$$P(k_n = i) = p_i \quad \text{and} \quad P(k_n = 0) = p \quad \text{for } n \in \mathbb{N}_0,\ i \in \mathbb{N}.$$

Whereas p_i denotes the probability of the choice of $\mathcal{S}_i$, p denotes the probability that no $\mathcal{S}_i$ was chosen, which means that $x_{n+1} \in C$. For each set $A \subseteq \mathbb{R}^d$ such that $A \cap K_\infty \neq \emptyset$, we have

$$P(x_{n+1} \in A) = M\mu_n(A).$$

Because the random sequence takes values in K_∞ and $\mathcal{S}_i(K_\infty) = S_i(K_\infty)$, and moreover, because of conditions (IW2) and (IW3), it follows that

$$P(x_{n+1} \in \operatorname{int} S_i K_\infty) = M\mu_n(\operatorname{int} S_i K_\infty) = p_i \mu_n(\operatorname{int} K_\infty),$$
$$P(x_{n+1} \in \operatorname{int} C) = M\mu_n(\operatorname{int} C) = p\nu(\operatorname{int} C).$$

At this point, it becomes natural to assume that the probabilities p_i and p are proportional to the sizes of the sets $S_i K_\infty$ and C, respectively. It is also worth noting that K_∞ does not depend on the probabilities $(p_1, p_2, \dots, p)$. However, due to the above interpretation, these probabilities should depend on K_∞; thus, changing K_∞ should result in appropriate probabilities being found.

It would be of interest to ask how to construct the probabilities to be in accord with their interpretation. This leads to the following example.

EXAMPLE 4.1. Define the constants

$$c_i = \begin{cases} r_i & \text{when } \|\mathbf{r}\| < 1, \\ \dfrac{r_i}{\|\mathbf{r}\| + 1} & \text{when } \|\mathbf{r}\| \geq 1. \end{cases}$$

Let $g : \mathbb{R}^d \to \mathbb{R}$ be an arbitrary continuous function such that $g > 0$. Let $f := g_{|K_\infty}$ and define

$$p_i(x) = \frac{c_i f(x)}{\sup_{x \in K_\infty} f(x)}, \qquad p(x) = 1 - \sum_{i=1}^{\infty} p_i(x).$$

Note that

$$\sum_{i=1}^{\infty} p_i(x) = \sum_{i=1}^{\infty} \frac{c_i f(x)}{\sup_{x \in K_\infty} f(x)} \leq \sum_{i=1}^{\infty} c_i \in (0,1), \quad x \in K_\infty,$$

and

$$\sum_{i=1}^{\infty} p_i(x) + p(x) = 1, \quad x \in K_\infty.$$

Let

$$\eta(A) := \frac{l_d(A \cap K_\infty)}{l_d(K_\infty)},$$

where l_d denotes the Lebesgue measure on $\mathbb{R}^d$ and put

$$\rho_i = \int_{K_\infty} p_i(x)\, d\eta(x), \qquad \rho = \int_{K_\infty} p(x)\, d\eta(x).$$

Then ρ_i, ρ are proportional to the sizes of the sets $S_i K_\infty$, C. Moreover,

$$\sum_{i=1}^{\infty} \rho_i + \rho = 1.$$

REMARK 4.1. In practice, we can replace K_∞ by $\overline{U}$ in the definitions of ρ_i, ρ in Example 4.1. Furthermore, ρ_i, ρ can be calculated as

$$\rho_i = \frac{\int_{\overline{U}} p_i(x)\, dl_d(x)}{l_d(\overline{U})}, \qquad \rho = \frac{\int_{\overline{U}} p(x)\, dl_d(x)}{l_d(\overline{U})}$$

and in formula (4.2), we can also use $\mathcal{S}_i := S_{i|\overline{U}} : \overline{U} \to \overline{U}$.

Thus, by substituting ρ_i and ρ from Example 4.1 in (4.2), i.e., by setting $p_i = \rho_i$, $p = \rho$, we obtain the following form of the operator M:

$$M\mu(A) = \sum_{i=1}^{\infty} \rho_i \mu \circ \mathcal{S}_i^{-1}(A) + \rho\nu(A),$$

where the probabilities, and consequently M, are constructed in order for them to be in accord with their interpretation.

Furthermore, the existence and uniqueness of the following measure can be shown in a way that is analogous to the proof of Theorem 4.1:

$$\mu(A) = \sum_{i=1}^{\infty} \rho_i \mu \circ \mathcal{S}_i^{-1}(A) + \rho\nu(A).$$

The measure μ is invariant for the operator M and μ can be interpreted as an invariant distribution of the above-mentioned Markov chain that takes values in the set K_∞. Moreover, μ is strongly asymptotically stable.

We will now extend a function related to L^q spectra so that it is well defined in the case of a countable number of contracting similarities. This extension is necessary to provide our estimation results in the next section. For this purpose, we will additionally assume that for some constant $\alpha > 0$, the probabilities satisfy the following inequalities:

$$\alpha r_i \leq p_i \leq r_i, \quad i \in \mathbb{N}. \tag{4.3}$$

The necessity of this assumption will be discussed later in this section. For $q \in \mathbb{R}$, define the functions $\psi_q(t) := \sum_{i=1}^{\infty} p_i^q r_i^t$, $\phi_q(t) := \sum_{i=1}^{\infty} r_i^{q+t}$. By assumption (4.3),

$$\alpha^q \phi_q(t) \leq \psi_q(t) \leq \phi_q(t) \quad \text{for } q \geq 0,$$
$$\alpha^q \phi_q(t) \geq \psi_q(t) \geq \phi_q(t) \quad \text{for } q < 0.$$

Since $\phi_q(t)$ is continuous with $\lim_{t\to -q^+} \phi_q(t) = \infty$, $\lim_{t\to\infty} \phi_q(t) = 0$, and since $\|\mathbf{r}\| < \infty$, we deduce from the above inequalities that $\psi_q(t) \neq \infty$. Observe that $\psi_q(t)$ is strictly decreasing. Moreover, $\lim_{t\to -q^+} \psi_q(t) = \infty$ and $\lim_{t\to\infty} \psi_q(t) = 0$.

As a consequence, there exists a finite $\beta_\infty(q)$ such that

$$\psi_q(\beta_\infty(q)) = \sum_{i=1}^{\infty} p_i^q r_i^{\beta_\infty(q)} = 1.$$

We are now in a position to emphasize the necessity of assumption (4.3). The following example shows that, without it, the function $\psi_q(t)$ may not be well defined for $q < 0$.

Example 4.2. Let $q < 0$, and let

$$p_i = \frac{1}{3^i} \quad \text{and} \quad r_i = \frac{5}{\pi^2}\frac{1}{i^2}.$$

Then $p_i \le r_i$ for infinitely many i and, for each $t \ge 0$,

$$\lim_{i\to\infty} p_i^q r_i^t = \left(\frac{5}{\pi^2}\right)^t \lim_{i\to\infty} 3^{-iq}\,\frac{1}{i^{2t}} = \infty.$$

Thus, for each $t \ge 0$, the series $\sum_{i=1}^{\infty} p_i^q r_i^t$ is divergent, hence $\psi_q(t) = \infty$. The same holds for all $t < 0$, so $\psi_q(t) \equiv \infty$.

From the above considerations, it follows that assumption (4.3) asserts the existence of finite $\beta_\infty(q)$ for all $q \in \mathbb{R}$. Moreover,

$$\lim_{n\to\infty} \beta_n(q) = \beta_\infty(q),$$

where, for all $n \in \mathbb{N}$, the numbers $\beta_n(q)$ are such that

$$\sum_{i=1}^{n} p_i^q r_i^{\beta_n(q)} = 1.$$

For a deeper discussion of the function β_n we refer the reader to [15].

5. The L^q spectra of μ_∞. This section is devoted to providing estimates for the L^q spectra of the measure μ_∞, which is the main result of the paper.

Let us recall the following notation, introduced in [15]: for $l, m \in \mathcal{M}_1(\mathbb{R}^d)$, $q \in \mathbb{R}$ and $A \subseteq \operatorname{supp} m$, write

$$I_m(q,r) = \int_{\operatorname{supp} m} m(B(x,r))^{q-1}\,dm(x),$$

$$I_{m|A}(q,r) = \int_{\operatorname{supp} m\cap A} m(B(x,r))^{q-1}\,dm(x),$$

$$I_{l,m,A}(q,r) = \int_A l(B(x,r))^{q-1}\,dm(x).$$

From this point forwards, we fix an inhomogeneous self-similar measure μ_∞ satisfying (4.1).

LEMMA 5.1. *Assume that the IWIOSC is satisfied. Then, for all $q \in \mathbb{R}$ and $r > 0$, we have*

$$I_{\mu_\infty}(q,r) = \sum_{i=1}^{\infty} p_i I_{\mu_\infty,\mu_\infty\circ S_i^{-1},S_iK_\infty}(q,r) + pI_{\mu_\infty,\nu,C}(q,r).$$

Proof. Fix any $q \in \mathbb{R}$ and $r > 0$. From (4.1), it follows that

$$I_{\mu_\infty}(q,r) = \sum_{i=1}^{\infty} p_i I_{\mu_\infty,\mu_\infty\circ S_i^{-1},K_\infty}(q,r) + pI_{\mu_\infty,\nu,K_\infty}(q,r).$$

Let $l, m \in \mathcal{M}_1(\mathbb{R}^d)$ be such that $\operatorname{supp} l = K_\infty$ and $\operatorname{supp} m \subseteq K_\infty$. Then

$$\int_{K_\infty} l(B(x,r))^{q-1}\, dm(x) = \int_{K_\infty \cap \operatorname{supp} m} l(B(x,r))^{q-1}\, dm(x).$$

By using the above formula for the couples

$$(l = \mu_\infty,\ m = \mu_\infty \circ S_i^{-1}), \qquad (l = \mu_\infty,\ m = \nu)$$

together with the relations

$$\operatorname{supp} \mu_\infty \circ S_i^{-1} = S_i K_\infty \subseteq K_\infty, \qquad \operatorname{supp} \nu = C \subseteq K_\infty,$$

we immediately conclude that

$$\begin{aligned} I_{\mu_\infty}(q,r) &= \sum_{i=1}^{\infty} p_i I_{\mu_\infty, \mu_\infty \circ S_i^{-1}, K_\infty}(q,r) + p I_{\mu_\infty, \nu, K_\infty}(q,r) \\ &= \sum_{i=1}^{\infty} p_i I_{\mu_\infty, \mu_\infty \circ S_i^{-1}, S_i K_\infty}(q,r) + p I_{\mu_\infty, \nu, C}(q,r). \ \blacksquare \end{aligned}$$

The following notation will be needed in our further considerations. Introducing it also allows us to record some crucial observations in Theorem 5.1.

For $l, m \in \mathcal{M}_1(\mathbb{R}^d)$, $A \subseteq \operatorname{supp} m$ and $x \in A$, write

$$\begin{aligned} J_{i,m,A}(x,r) &= \sum_{j \neq i} p_j m\big(S_j^{-1}(B(x,r) \cap S_j(\operatorname{supp} m))\big) + p\nu(B(x,r) \cap C), \\ J_{C,m,A}(x,r) &= \sum_{i=1}^{\infty} p_i m\big(S_i^{-1}(B(x,r) \cap S_i(\operatorname{supp} m))\big), \\ F_{i,l,m,A}(q,r) &= \int_A \big(l(B(x,r/r_i)) + J_{i,l,A}(S_i x, r)/p_i\big)^{q-1}\, dm(x). \end{aligned}$$

We will simply write $J_i(x,r)$ if $m = \mu_\infty$ and $A = K_\infty$; $J_C(x,r)$ when $m = \mu_\infty$ and $A = C$; and $F_i(q,r)$ if $l = m = \mu_\infty$ and $A = K_\infty$.

It is easily observed from (4.1) that, under the assumption of the IWIOSC, for all $q \in \mathbb{R}$, $r > 0$ and $x \in S_i K_\infty$, we have

$$\begin{aligned} \mu_\infty(B(x,r))^{q-1} &= \big(p_i \mu_\infty(S_i^{-1}(B(x,r) \cap S_i K_\infty)) + J_i(x,r)\big)^{q-1} \\ &= \big(p_i \mu_\infty(B(S_i^{-1}x, r/r_i)) + J_i(x,r)\big)^{q-1}, \end{aligned}$$

where

$$J_i(x,r) = \sum_{j \neq i} p_j \mu_\infty(S_j^{-1}(B(x,r) \cap S_j K_\infty)) + p\nu(B(x,r) \cap C).$$

Analogously,

$$\mu_\infty(B(x,r))^{q-1} = \big(p\nu(B(x,r) \cap C) + J_C(x,r)\big)^{q-1}$$

for $x \in C$, where in this case

$$J_C(x,r) = \sum_{i=1}^{\infty} p_i \mu_\infty(S_i^{-1}(B(x,r) \cap S_i K_\infty)).$$

Let us also introduce the following notation:

$$F_{C,\nu,m,A}(q,r) = \int_A \big(\nu(B(x,r)) + J_{C,\mu_\infty,A}(x,r)/p\big)^{q-1}\, dm(x).$$

When $m = \nu$ and $A = C$, we simply write

$$F_C(q,r) = \int_C \big(\nu(B(x,r)) + J_C(x,r)/p\big)^{q-1}\, d\nu(x).$$

We are now in a position to formulate the following theorem, which refers to Lemma 5.1 but goes a step further.

THEOREM 5.1. *Assume that the IWIOSC is satisfied. Then, for all $q \in \mathbb{R}$ and $r > 0$, we have*

$$I_{\mu_\infty}(q,r) = \sum_{i=1}^{\infty} p_i^q F_i(q,r) + p^q F_C(q,r).$$

Proof. Fix $q \in \mathbb{R}$ and let $r > 0$. Recall that

$$\mu_\infty(B(x,r))^{q-1} = \begin{cases} \big(p_i \mu_\infty(B(S_i^{-1}x, r/r_i)) + J_i(x,r)\big)^{q-1} & \text{for } x \in S_i K_\infty, \\ \big(p\nu(B(x,r)) + J_C(x,r)\big)^{q-1} & \text{for } x \in C. \end{cases}$$

It follows that

$$\begin{aligned} I&_{\mu_\infty,\mu_\infty \circ S_i^{-1}, S_i K_\infty}(q,r) \\ &= \int_{S_i K_\infty} \big(p_i \mu_\infty(B(S_i^{-1}x, r/r_i)) + J_i(x,r)\big)^{q-1}\, d(\mu_\infty \circ S_i^{-1})(x) \\ &= \int_{K_\infty} \big(p_i \mu_\infty(B(x, r/r_i)) + J_i(S_i x, r)\big)^{q-1}\, d\mu_\infty(x) \\ &= p_i^{q-1} \int_{K_\infty} \big(\mu_\infty(B(x, r/r_i)) + J_i(S_i x, r)/p_i\big)^{q-1}\, d\mu_\infty(x) = p_i^{q-1} F_i(q,r) \end{aligned}$$

and similarly

$$I_{\mu_\infty,\nu,C}(q,r) = p^{q-1} F_C(q,r).$$

Finally, from Lemma 5.1 and from what has already been proved, we have

$$\begin{aligned} I_{\mu_\infty}(q,r) &= \sum_{i=1}^{\infty} p_i I_{\mu_\infty,\mu_\infty \circ S_i^{-1}, S_i K_\infty}(q,r) + p I_{\mu_\infty,\nu,C}(q,r) \\ &= \sum_{i=1}^{\infty} p_i^q F_i(q,r) + p^q F_C(q,r). \ \blacksquare \end{aligned}$$

To define the L^q spectra for $l, m \in \mathcal{M}_1(\mathbb{R}^d)$, $A \subseteq \operatorname{supp} m$ and $q \in \mathbb{R}$ we set

$$\begin{aligned}
\overline{\tau}_{m|A}(q) &:= \limsup_{r\to 0} \frac{\log \int_A m(B(x,r))^{q-1}\,dm(x)}{-\log r},\\
\underline{\tau}_{m|A}(q) &:= \liminf_{r\to 0} \frac{\log \int_A m(B(x,r))^{q-1}\,dm(x)}{-\log r},\\
\overline{\tau}_{l,m,A}(q) &:= \limsup_{r\to 0} \frac{\log \int_A l(B(x,r))^{q-1}\,dm(x)}{-\log r},\\
\underline{\tau}_{l,m,A}(q) &:= \liminf_{r\to 0} \frac{\log \int_A l(B(x,r))^{q-1}\,dm(x)}{-\log r}.
\end{aligned}$$

In particular, for $l = m = \mu_\infty$ and $A = K_\infty$, we obtain the *upper* and *lower* L^q *spectrum* of the measure μ_∞:

$$\begin{aligned}
\overline{\tau}_{\mu_\infty}(q) &:= \limsup_{r\to 0} \frac{\log \int_{K_\infty} \mu_\infty(B(x,r))^{q-1}\,d\mu_\infty(x)}{-\log r},\\
\underline{\tau}_{\mu_\infty}(q) &:= \liminf_{r\to 0} \frac{\log \int_{K_\infty} \mu_\infty(B(x,r))^{q-1}\,d\mu_\infty(x)}{-\log r}.
\end{aligned}$$

Theorem 5.2 below plays a crucial role in establishing the inequality "$\geq$" of our main result. The following lemma will simplify the theorem's proof.

LEMMA 5.2. *Let $G : (0,\infty) \to \mathbb{R}$ be a real-valued function, let μ be a Borel probability measure and let $A \subseteq \operatorname{supp}\mu$. Assume that*

$$I_{\mu|A}(q,r) \geq \sum_{i=1}^{n} p_i^q I_{\mu|A}(q, r/r_i) \quad \textit{and} \quad \sum_{i=1}^{n} p_i^q G(r/r_i) \geq G(r)$$

for all $r > 0$. If

$$I_{\mu|A}(q,r) \geq G(r) \quad \textit{for all } r \in [r_{\min}, 1],$$

then

$$I_{\mu|A}(q,r) \geq G(r) \quad \textit{for all } 0 < r \leq 1.$$

Proof. For the method of proving such lemmas, we refer the reader to [11] and [12]. The proof in a more general case can also be found in [15, Lemma 3.1], so we omit it here. ■

THEOREM 5.2. *Let $q \in \mathbb{R}$ and $n \in \mathbb{N}$, and let μ be a probability measure. Let $A \subseteq \operatorname{supp}\mu$ and assume that*

$$I_{\mu|A}(q,r) \geq \sum_{i=1}^{n} p_i^q I_{\mu|A}(q, r/r_i) \quad \textit{for all } r > 0.$$

In addition, let t be such that $\beta_n(q) > t$. Then:

(1) *There exists a constant* $c_0 > 0$ *such that the function* $G : (0,\infty) \to \mathbb{R}$ *defined by the formula* $G(r) = c_0 r^{-t}$ *satisfies*

$$\sum_{i=1}^{n} p_i^q G(r/r_i) \geq G(r) \quad \textit{for all } r > 0$$

and $I_{\mu|A}(q,r) \geq G(r)$ *for all* $r \in [r_{\min}, 1]$.

(2) $\underline{\tau}_{\mu|A}(q) \geq \beta_n(q)$.

Proof. The proof is similar to the proofs in [15, Section 4]. However, the proof of this particular result is omitted there, so we will provide it for the sake of completeness.

(1) Because $\beta_n(q) > t$, we conclude that

$$\sum_{i=1}^{n} p_i^q r_i^t > 1.$$

Let $c_0 > 0$ be such that

$$\frac{\min\big(I_{\mu|A}(q, r_{\min}), I_{\mu|A}(q,1)\big)}{\max(1, (r_{\min})^{-t})} \geq c_0,$$

where $r_{\min} = \min\{r_1, \ldots, r_n\}$. By the above inequalities, it follows that $\sum_{i=1}^{n} p_i^q G(r/r_i) \geq c_0 r^{-t} = G(r)$ for all $r > 0$, and

$$\begin{aligned} I_{\mu|A}(q,r) &\geq \min\big(I_{\mu|A}(q, r_{\min}), I_{\mu|A}(q,1)\big) \\ &\geq \frac{\min\big(I_{\mu|A}(q, r_{\min}), I_{\mu|A}(q,1)\big)}{\max(1, (r_{\min})^{-t})} \cdot r^{-t} \geq c_0 r^{-t} = G(r) \end{aligned}$$

for all $r \in [r_{\min}, 1]$.

(2) From Lemma 5.2, it follows that

$$I_{\mu|A}(q,r) \geq G(r) = c_0 r^{-t} \quad \text{for all } 0 < r \leq 1,$$

whence $\underline{\tau}_{\mu|A}(q) \geq t$. Because t in the inequality $\beta_n(q) > t$ was arbitrary, we conclude that $\underline{\tau}_{\mu|A}(q) \geq \beta_n(q)$. ■

The key to showing the opposite inequality is the next theorem, which also enables us to formulate our main result. The reader is invited to compare it with [15, Proposition 4.2].

THEOREM 5.3. *Let* μ *and* ν *be probability measures, and let* $(\mu_m)_{m\in\mathbb{N}}$ *and* $(\nu_m)_{m\in\mathbb{N}}$ *be sequences of probability measures. Let* K_m *and* C_m *denote the supports of* μ_m, ν_m, *respectively, and let* $q \in \mathbb{R}$, $n \in \mathbb{N}$. *Assume that, for each* $m \in \mathbb{N}$,

$$I_{\mu,\mu_m,K_m}(q,r) \leq \sum_{i=1}^{n} p_i^q I_{\mu,\mu_m,K_m}(q, r/r_i) + p^q I_{\nu,\nu_m,C_m}(q,r).$$

Then

$$\overline{\tau}_{\mu,\mu_m,K_m}(q) \le \max(\beta_n(q), \overline{\tau}_{\nu,\nu_m,C_m}(q)), \qquad m \in \mathbb{N}.$$

Proof. The proof is the same as the proof of [15, Proposition 4.2], applied for each $m \in \mathbb{N}$. ■

We are now in a position to state the main theorem of this paper, which presents satisfactory estimates for the lower and upper bounds of the L^q spectra of the measure μ_∞. It provides a much more accurate result under the assumption of the IWIOSC for all $q \in \mathbb{R}$ than does [15, Theorem 2.1] and extends it to the case of countably many contracting similarities.

THEOREM 5.4. *Assume that the IWIOSC is satisfied. Then, for all $q \in \mathbb{R}$, we have*

$$\overline{\tau}_{\mu_\infty}(q) = \max(\beta_\infty(q), \overline{\tau}_\nu(q)), \qquad \max(\beta_\infty(q), \underline{\tau}_\nu(q)) \le \underline{\tau}_{\mu_\infty}(q).$$

Proof. Fix any $q \in \mathbb{R}$ and $n \in \mathbb{N}$. Let $(T_m, t_m)_{m=1}^\infty : \mathbb{R}^d \to \mathbb{R}^d$ be a sequence of contracting similarities such that $\lim_{m\to\infty} t_m = 1$ and

$$T_m(S_i K_\infty) \subset \operatorname{int} S_i K_\infty, \quad T_m(C) \subset \operatorname{int} C.$$

Define

$$K_m := \bigcup_{i=1}^n T_m(S_i(K_\infty)) \cup T_m(C). \tag{5.1}$$

We start by showing the inequality "$\ge$". To do this, first observe that Theorem 5.1 implies that

$$I_{\mu_\infty}(q,r) = \sum_{i=1}^\infty p_i^q F_i(q,r) + p^q F_C(q,r) \ge \sum_{i=1}^n p_i^q F_i(q,r).$$

Hence,

$$I_{\mu_\infty|K_m}(q,r) \ge \sum_{i=1}^n p_i^q F_{i,\mu_\infty,K_m}(q,r).$$

From conditions (IW2) and (IW3) of the IWIOSC, we conclude that, for every $m \in \mathbb{N}$, the sets $(S_1K_m, \ldots, S_nK_m, C)$ are pairwise disjoint. Let

$$r_m = \min\big\{\min_{i\in\{1,\ldots,n\}} \inf_{j\ne i} \operatorname{dist}(S_iK_m, S_jK_\infty), \min_{i\in\{1,\ldots,n\}} \operatorname{dist}(S_iK_m, C)\big\}.$$

Then, for all $0 < r < r_m$,

$$I_{\mu_\infty|K_m}(q,r) \ge \sum_{i=1}^n p_i^q I_{\mu_\infty|K_m}(q, r/r_i)$$

because $J_{i,\mu_\infty,K_m}(S_i x, r) = 0$ for $0 < r < r_m$. Hence, from Theorem 5.2,

$$\underline{\tau}_{\mu_\infty|K_m}(q) \ge \beta_n(q).$$

Because $(\underline{\tau}_{\mu_\infty|K_m}(q))_{m\in\mathbb{N}}$ is monotonic and tends to $\underline{\tau}_{\mu_\infty|K_{\infty|n}}(q)$, we have

$$\underline{\tau}_{\mu_\infty|K_{\infty|n}}(q) \geq \beta_n(q), \qquad n \in \mathbb{N}.$$

Furthermore, $(\underline{\tau}_{\mu_\infty|K_{\infty|n}}(q))_{n\in\mathbb{N}}$ is monotonic and tends to $\underline{\tau}_{\mu_\infty}(q)$, so

$$\overline{\tau}_{\mu_\infty}(q) \geq \underline{\tau}_{\mu_\infty}(q) = \lim_{n\to\infty} \underline{\tau}_{\mu_\infty|K_{\infty|n}}(q) \geq \lim_{n\to\infty} \beta_n(q) = \beta_\infty(q).$$

To show that $\overline{\tau}_{\mu_\infty}(q) \geq \overline{\tau}_\nu(q)$ and $\underline{\tau}_{\mu_\infty}(q) \geq \underline{\tau}_\nu(q)$, from Theorem 5.1, observe that

$$I_{\mu_\infty}(q,r) \geq p^q F_C(q,r).$$

Define $C_m := T_m(C)$. By the above,

$$I_{\mu_\infty}(q,r) \geq p^q F_{C,\nu,C_m}(q,r).$$

From condition (IW3), for every $m \in \mathbb{N}$, we have $r_m = \inf_{i\in\mathbb{N}} \operatorname{dist}(S_iK_\infty, C_m) > 0$. Thus, for all $0 < r < r_m$,

$$I_{\mu_\infty}(q,r) \geq p^q I_{\nu|C_m}(q,r),$$

as $J_{C,\mu_\infty,C_m}(x,r) = 0$ for $0 < r < r_m$. Hence,

$$\overline{\tau}_{\mu_\infty}(q) \geq \overline{\tau}_{\nu|C_m}(q), \qquad \underline{\tau}_{\mu_\infty}(q) \geq \underline{\tau}_{\nu|C_m}(q).$$

The sequences $(\overline{\tau}_{\nu|C_m}(q))_{m\in\mathbb{N}}$, $(\underline{\tau}_{\nu|C_m}(q))_{m\in\mathbb{N}}$ are monotonic and converge, respectively, to $\overline{\tau}_\nu(q)$, $\underline{\tau}_\nu(q)$, so

$$\overline{\tau}_{\mu_\infty}(q) \geq \overline{\tau}_\nu(q), \qquad \underline{\tau}_{\mu_\infty}(q) \geq \underline{\tau}_\nu(q).$$

The proof of the inequality "$\geq$" is now finished.

To establish the opposite inequality, let K_m denote the set (5.1). Define the sequences of finite measures

$$\mu_m(A) := \mu_\infty(A \cap K_m), \quad \nu_m(A) := \nu(A \cap C_m)$$

and

$$\mu_{\infty|n}(A) := \mu_\infty(A \cap K_{\infty|n}).$$

Then, from the IWIOSC and (4.1), we deduce that

$$\mu_m(A) = \sum_{i=1}^n p_i \mu_m \circ S_i^{-1}(A) + p\nu_m(A).$$

Note that $\operatorname{supp} \mu_m = K_m$. From the proofs of Lemma 5.1 and Theorem 5.1,

$$I_{\mu_\infty,\mu_m,K_m}(q,r) \leq \sum_{i=1}^n p_i^q F_{i,\mu_\infty,\mu_m,K_m}(q,r) + p^q F_{C,\nu,\nu_m,C_m}(q,r).$$

By (IW2) and (IW3), for every $m \in \mathbb{N}$, the sets $(S_1K_m, \ldots, S_nK_m, C)$ are pairwise disjoint. Let r_m be the minimum of

$$\Big\{\min_{i\in\{1,\ldots,n\}} \inf_{j\neq i} \operatorname{dist}(S_iK_m, S_jK_\infty), \min_{i\in\{1,\ldots,n\}} \operatorname{dist}(S_iK_m, C), \inf_{i\in\mathbb{N}} \operatorname{dist}(S_iK_\infty, C_m)\Big\}.$$

Then, for all $0 < r < r_m$,

$$I_{\mu_\infty,\mu_m,K_m}(q,r) \le \sum_{i=1}^{n} p_i^q I_{\mu_\infty,\mu_m,K_m}(q,r/r_i) + p^q I_{\nu,\nu_m,C_m}(q,r),$$

as $J_{i,\mu_\infty,K_m}(S_i x,r) = J_{C,\mu_\infty,C_m}(x,r) = 0$ for $0 < r < r_m$. Hence Theorem 5.3 implies that

$$\overline{\tau}_{\mu_\infty,\mu_m,K_m}(q) \le \max(\beta_n(q), \overline{\tau}_{\nu,\nu_m,C_m}(q)).$$

Because the sequences $(\overline{\tau}_{\mu_\infty,\mu_m,K_m}(q))_{m\in\mathbb{N}}$ and $(\overline{\tau}_{\nu,\nu_m,C_m}(q))_{m\in\mathbb{N}}$ are monotonic and converge, respectively, to $\overline{\tau}_{\mu_\infty,\mu_{\infty|n},K_{\infty|n}}(q)$ and $\overline{\tau}_\nu(q)$, we have

$$\overline{\tau}_{\mu_\infty,\mu_{\infty|n},K_{\infty|n}}(q) \le \max(\beta_n(q), \overline{\tau}_\nu(q)), \qquad n \in \mathbb{N}.$$

Furthermore, the sequence $(\overline{\tau}_{\mu_\infty,\mu_{\infty|n},K_{\infty|n}}(q))_{n\in\mathbb{N}}$ is also monotonic and converges to $\overline{\tau}_{\mu_\infty}(q)$. Hence,

$$\overline{\tau}_{\mu_\infty}(q) \le \max(\beta_\infty(q), \overline{\tau}_\nu(q)). \ \blacksquare$$

From the proof of Theorem 5.4, we immediately obtain the two corollaries below. In particular, Corollary 5.1, which is related to [15, Theorem 2.1], gives a partial answer to [15, Question 2.7]. Furthermore, in both corollaries, for all $q \in \mathbb{R}$, the estimates are much more accurate compared to those in [15, Theorem 2.1]. In a result that is similar to our main theorem, we even obtain an exact value for the upper L^q spectrum of an inhomogeneous self-similar measure.

COROLLARY 5.1. *Let μ be the inhomogeneous self-similar measure associated with $(S_1,\dots,S_N,p_1,\dots,p_N,p,\nu)$. Assume that the FWIOSC is satisfied. Then, for all $q \in \mathbb{R}$, we have*

$$\overline{\tau}_\mu(q) = \max(\beta(q), \overline{\tau}_\nu(q)), \qquad \max(\beta(q), \underline{\tau}_\nu(q)) \le \underline{\tau}_\mu(q).$$

COROLLARY 5.2. *Let μ be the inhomogeneous self-similar measure associated with $(S_1,\dots,S_N,p_1,\dots,p_N,p,\nu)$ and let K be the unique nonempty compact set satisfying $K = \bigcup_{i=1}^N S_i(K) \cup C$. Assume that the sets $(S_1K,\dots,S_NK,C)$ are pairwise disjoint. Then, for all $q \in \mathbb{R}$, we have*

$$\overline{\tau}_\mu(q) = \max(\beta(q), \overline{\tau}_\nu(q)), \qquad \max(\beta(q), \underline{\tau}_\nu(q)) \le \underline{\tau}_\mu(q).$$

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