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Uniqueness of meromorphic functions and differential polynomials sharing one value with finite weight

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Abstract. This paper deals with the uniqueness problem for meromorphic functions sharing one value with finite weight. Our results generalize those of Fang, Hong, Bhoosnurmath and Dyavanal.

1. Introduction and main results. Let $f(z)$ be a non-constant meromorphic function in the whole complex plane. We shall use the following standard notations of value distribution theory:

$$T(r,f),\ m(r,f),\ N(r,f),\ \overline{N}(r,f),\ \ldots$$

(see Hayman [8], Yang [16] and Yi and Yang [18]). We denote by $S(r,f)$ any quantity satisfying

$$S(r,f)=o(T(r,f))$$

as $r\to+\infty$, possibly outside a set of finite measure. For $a\in\mathbb{C}\cup\{\infty\}$, we define

$$\Theta(a,f)=1-\overline{\lim_{r\to\infty}}\,\frac{\overline{N}(r,a;f)}{T(r,f)}.$$

For $a\in\mathbb{C}\cup\{\infty\}$ and k a positive integer, we denote by $N(r,a;f\,|\!=\!1)$ the counting function of simple a-points of f, and by $N(r,a;f\,|\!\le k)$ (resp. $N(r,a;f\,|\!\ge k)$) the counting functions of those a-points of f whose multiplicities are not greater (resp. less) than k where each a-point is counted according to its multiplicity (see [8]). The functions $\overline{N}(r,a;f\,|\!\le k)$ and $\overline{N}(r,a;f\,|\!\ge k)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Set

$$N_k(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f\,|\!\ge 2)+\cdots+\overline{N}(r,a;f\,|\!\ge k).$$

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We define

$$\delta_k(a,f) = 1 - \varlimsup_{r\to\infty} \frac{N_k(r,a;f)}{T(r,f)}.$$

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions on $\mathbb{C}$. If for some $a \in \mathbb{C}\cup\{\infty\}$ the roots of $f(z)-a$ and $g(z)-a$ (if $a=\infty$, the roots of $f(z)-a$ and $g(z)-a$ are poles of $f(z)$ and $g(z)$ respectively) coincide in locations and multiplicities, we say that $f(z)$ and $g(z)$ *share the value a CM* (counting multiplicities), and if they coincide in locations only, we say that $f(z)$ and $g(z)$ *share a IM* (ignoring multiplicities).

Yang and Hua [15] proved the following result.

THEOREM A ([15]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, $n \geq 11$ an integer, and $a \in \mathbb{C}-\{0\}$. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share the value a CM, then either $f(z) = dg(z)$ for some $(n+1)$th root of unity d, or*

$$g(z) = c_1e^{cz} \quad \textit{and} \quad f(z) = c_2e^{-cz}$$

where c, c_1, and c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -a^2$.

Using the same argument as in [15], Fang and Hong [6] proved the following result.

THEOREM B ([6]). *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and $n \geq 11$ an integer. If the functions $[f^n(z)(f(z)-1)]f'(z)$ and $[g^n(z)(g(z)-1)]g'(z)$ share the value 1 CM, then $f(z) \equiv g(z)$.*

To state the next result, we require the following definition.

DEFINITION 1.1 ([10, 11]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$, we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g *share the value a with weight k*.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f-a$ with multiplicity m $(\leq k)$ if and only if it is a zero of $g-a$ with multiplicity m $(\leq k)$; and z_0 is a zero of $f-a$ with multiplicity $m(> k)$ if and only if it is a zero of $g-a$ with multiplicity n $(> k)$ where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k; clearly if f, g share (a,k), then f, g share (a,p) for all integers p with $0 \leq p \leq k$. Also, we note that f, g share a value a IM or CM if and only if they share $(a,0)$ or (a,∞), respectively.

With the notion of weighted sharing of values the following results improving Theorem A are proved in [4].

Theorem C ([4]). *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and $n \geq 9$ an integer. If*

$$E_2(1, f^n(z)(f(z)-1)]f'(z)) = E_2(1, [g^n(z)(g(z)-1)]g'(z)),$$

then $f(z) \equiv g(z)$.

Theorem D ([4]). *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and $n \geq 17$ an integer. If $[f^n(z)(f(z)-1)]f'(z)$ and $[g^n(z)(g(z)-1)]g'(z)$ share $(1,0)$, then $f(z) \equiv g(z)$.*

W. C. Lin and H. X. Yi [13] and Fang [5] obtained some unicity theorems corresponding to Theorem B.

Theorem E ([13]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions with $\Theta(\infty, f) > 2/(n+1)$ for some $n \geq 12$. If $[f^n(z)(f(z)-1)]f'(z)$ and $[g^n(z)(g(z)-1)]g'(z)$ share 1 CM, then $f(z) \equiv g(z)$.*

Theorem F ([5]). *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, k be positive integers with $n > 2k+8$. If $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

Bhoosnurmath and Dyavanal proved the following theorem.

Theorem G ([3]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k+13$. If $\Theta(\infty, f) > 3/(n+1)$, and if $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*

Now one may ask the following question which is the motivation for this paper.

Question. In Theorems E, F and G, can the nature of sharing 1 CM be further relaxed?

We now state the following three main results of this paper.

Theorem 1.2. *Let $f(z), g(z)$ be two nonconstant meromorphic functions, and let n, k be two positive integers with $n \geq 8k+18$. If $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$, and if $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share $(1,0)$, then $f \equiv g$.*

Theorem 1.3. *Let $f(z), g(z)$ be two nonconstant meromorphic functions, and let n, k be two positive integers with $n \geq 7k+23/2$. If $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$, and if $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share $(1,1)$, then $f \equiv g$.*

Theorem 1.4. *Let $f(z), g(z)$ be two nonconstant meromorphic functions, and let n, k be two positive integers with $n \geq 5k+11$. If $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$, and if $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share $(1,2)$, then $f \equiv g$.*

When f and g are two entire functions we can similarly get the following results.

Corollary 1.5. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, k be two positive integers with $n > 4k + 11$. If $[f^n(z)(f(z) - 1)]^{(k)}$ and $[g^n(z)(g(z) - 1)]^{(k)}$ share $(1, 0)$, then $f \equiv g$.*

Corollary 1.6. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, k be two positive integers with $n > 4k + 9$. If $[f^n(z)(f(z) - 1)]^{(k)}$ and $[g^n(z)(g(z) - 1)]^{(k)}$ share $(1, 1)$, then $f \equiv g$.*

Corollary 1.7. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, k be two positive integers with $n > 3k + 7$. If $[f^n(z)(f(z) - 1)]^{(k)}$ and $[g^n(z)(g(z) - 1)]^{(k)}$ share $(1, 2)$, then $f \equiv g$.*

Now we explain some definitions and notations which are used in the paper.

Definition 1.8 ([2, 18]). When f and g share 1 IM, we denote by $\overline{N}_L(r, 1; f)$ the counting function of the 1-points of f whose multiplicities are greater than the multiplicites of the corresponding 1-points of g, where each zero is counted only once; similarly, we have $\overline{N}_L(r, 1; g)$. We also denote by $N_{11}(r, 1; f)$ the counting function of common simple a-points of f and g; and $\overline{N}_E^{(2}(r, 1; f)$ denotes the counting function of those multiplicity 1-points of f and g, each point in these counting functions is counted only once. In the same way, one can define $N_{11}(r, 1; g), \overline{N}_E^{(2}(r, 1; g)$.

In addition, let z_0 be the zeros of $f - 1$ with multiplicity p and zeros of $g - 1$ with multiplicity q. We denote by $\overline{N}_{f>k}(r, 1; g)$ the reduced counting function of those zeros of $f - 1$ and $g - 1$ such that $p > q = k$; $\overline{N}_{g>k}(r, 1; f)$ is defined analogously.

Definition 1.9 ([10, 11]). Let f, g share the value 1 IM. We denote by $\overline{N}_*(r, 1; f, g)$ the reduced counting function of those 1-points of f whose multiplicities differ from the multiplicities of the corresponding 1-points of g. Clearly $\overline{N}_*(r, 1; f, g) \equiv \overline{N}_*(r, 1; g, f)$ and $\overline{N}_*(r, 1; f, g) = \overline{N}_L(r, 1; f) + \overline{N}_L(r, 1; g)$.

2. Some lemmas. For the proof of our results we need the following lemmas.

Lemma 2.1 ([8]). *Let f be a nonconstant meromorphic function, k a positive integer, and c a nonzero finite complex number. Then*

$$T(r, f) \leq \overline{N}(r, f) + N(r, 0; f) + N(r, c; f^{(k)}) - N(r, 0; f^{(k+1)}) + S(r, f),$$

$$T(r, f) \leq \overline{N}(r, f) + N_{k+1}(r, 0; f) + \overline{N}(r, c; f^{(k)}) - N_0(r, 0; f^{(k+1)}) + S(r, f),$$

where $N_0(r, 0; f^{(k+1)})$ *only counts those points such that* $f^{(k+1)} = 0$ *but* $f(f^{(k)} - c) \neq 0$.

LEMMA 2.2 ([8]). *Let* f *be a meromorphic function and* α_1, α_2 *be two meromorphic functions such that* $T(r, \alpha_i) = S(r, f)$ $(i = 1, 2)$. *Then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}(r, \alpha_1(z); f) + \overline{N}(r, \alpha_2(z); f) + S(r, f).$$

LEMMA 2.3 ([14]). *Let* f *be a nonconstant meromorphic function and* $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, *where* $a_0, a_1, \ldots, a_n$ *are constants and* $a_n \neq 0$. *Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

LEMMA 2.4 ([1]). *Let* f *be a nonconstant meromorphic function and* k *be a positive integer. Then*

$$N_2(r, 1/f^{(k)}) \leq k\overline{N}(r, f) + N_{k+2}(r, 1/f) + S(r, f).$$

LEMMA 2.5 ([7]). *Let* f *be a nonconstant entire function and* $k \geq 2$ *be a positive integer. If* $f(z)f^{(k)}(z) \neq 0$, *then* $f = e^{az+b}$, *where* $a \neq 0, b$ *are constants.*

LEMMA 2.6 ([11]). *Let* F *and* G *be two nonconstant meromorphic functions sharing* $(1, 0)$, *and* $H \not\equiv 0$. *Then*

$$N_{11}(r, 1; F) \leq N(r, H) + S(r, F) + S(r, G),$$

where $H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$.

LEMMA 2.7 ([1]). *Let* F *and* G *be two nonconstant meromorphic functions sharing* $(1, 0)$, *and* $H \not\equiv 0$. *Then*

$$\begin{aligned} N(r, \infty; H) \leq{} & \overline{N}(r, \infty; F \mid \geq 2) + \overline{N}(r, \infty; G \mid \geq 2) + \overline{N}(r, 0; F \mid \geq 2) \\ & + \overline{N}(r, 0; G \mid \geq 2) + N_0(r, 0; F') + N_0(r, 0; G') \\ & + \overline{N}_*(r, 1; F, G) + S(r, F) + S(r, G), \end{aligned}$$

where $N_0(r, 0; F')$ *is the reduced counting function of those zeros of* F' *which are not zeros of* $F(F-1)$, *and* $N_0(r, 0; G')$ *is similarly defined.*

LEMMA 2.8 ([1]). *Let* F *and* G *be two meromorphic functions sharing* $(1, 0)$. *Then*

$$\overline{N}_{G>1}(r, 1; F) \leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) - N_0(r, 0; G') + S(r, G),$$
$$\overline{N}_{F>1}(r, 1; G) \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) - N_0(r, 0; F') + S(r, F).$$

LEMMA 2.9 ([17]). *Let* F *and* G *be two meromorphic functions sharing* $(1, 0)$. *Then*

$$\overline{N}_L(r, 1; F) \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + S(r, F),$$
$$\overline{N}_L(r, 1; G) \leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + S(r, G).$$

LEMMA 2.10 ([1]). *Let f, g share $(1,1)$. Then*

$$\overline{N}_{f>2}(r,1;g) \le \tfrac{1}{2}\overline{N}(r,0;f) + \tfrac{1}{2}\overline{N}(r,\infty;f) - \tfrac{1}{2}\overline{N}_0(r,0;f') + S(r,f),$$
$$\overline{N}_{g>2}(r,1;f) \le \tfrac{1}{2}\overline{N}(r,0;g) + \tfrac{1}{2}\overline{N}(r,\infty;g) - \tfrac{1}{2}\overline{N}_0(r,0;g') + S(r,g).$$

LEMMA 2.11 ([11]). *If two nonconstant meromorphic functions f, g share $(1,2)$ then*

$$\overline{N}_0(r,0;g') + \overline{N}(r,1;g\,|\ge 2) + \overline{N}_*(r,1;f,g) \le \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + S(r,g).$$

LEMMA 2.12. *Let f and g be two meromorphic functions, and let k be a positive integer. Suppose that $f^{(k)}$ and $g^{(k)}$ share $(1,l)$ $(l = 0,1,2)$.*

(i) *If $l = 0$ and*

$$(1)\qquad \begin{aligned} \Delta_1 = {} & (2k+4)\min\{\Theta(\infty,f),\Theta(\infty,g)\} \\ & + (2k+3)\min\{\Theta(\infty,f),\Theta(\infty,g)\} + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) \\ & + \delta_{k+2}(0,f) + \delta_{k+2}(0,g) + \min\{\Theta(0,f),\Theta(0,g)\} \\ > {} & 4k+11, \end{aligned}$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

(ii) *If $l = 1$ and*

$$(2)\qquad \begin{aligned} \Delta_2 = {} & (k+5/2)\min\{\Theta(\infty,f),\Theta(\infty,g)\} \\ & + (2k+2)\min\{\Theta(\infty,f),\Theta(\infty,g)\} + \delta_{k+1}(0,f) \\ & + \delta_{k+1}(0,g) + \delta_{k+2}(0,f) + \delta_{k+2}(0,g) \\ > {} & 3k+15/2, \end{aligned}$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

(iii) *If $l = 2$ and*

$$(3)\qquad \begin{aligned} \Delta_3 = {} & (k+2)\min\{\Theta(\infty,f),\Theta(\infty,g)\} \\ & + (k+2)\min\{\Theta(\infty,f),\Theta(\infty,g)\} \\ & + \min\{\delta_{k+1}(0,f),\delta_{k+1}(0,g)\} \\ & + \delta_{k+2}(0,f) + \delta_{k+2}(0,g) \\ > {} & 2k+6, \end{aligned}$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

Proof. Let $F = f^{(k)}$, $G = g^{(k)}$ and

$$H \equiv \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

(i) If $l = 0$, then F, G share $(1,0)$. Suppose $H \not\equiv 0$. Then we have

$$(4)\qquad N_{11}(r,1;F) \le \overline{N}(r,0;H) \le N(r,H) + S(r,F) + S(r,G).$$

So using Lemmas 2.6–2.9 and (4), we get

$$
\begin{aligned}
(5)\quad & \overline{N}(r,1;F)+\overline{N}(r,1;G) \\
&\le N_{11}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G) \\
&\quad+\overline{N}^{(2}_E(r,1;F)+\overline{N}(r,1;G) \\
&\le N_{11}(r,1;F)+N(r,1;G)-\overline{N}_L(r,1;G) \\
&\quad+\overline{N}_{G>1}(r,1;F)+\overline{N}_{F>1}(r,1;G) \\
&\le \overline{N}(r,\infty;F\,|\,{\ge}2)+\overline{N}(r,\infty;G\,|\,{\ge}2)+\overline{N}(r,0;F\,|\,{\ge}2) \\
&\quad+\overline{N}(r,0;G\,|\,{\ge}2)+\overline{N}_*(r,1;F,G)+T(r,G)-m(r,1;G) \\
&\quad+O(1)-\overline{N}_L(r,1;G)-\overline{N}_{F>1}(r,1;G)+\overline{N}_{G>1}(r,1;F) \\
&\quad+N_0(r,0;F')+N_0(r,0;G')+S(r,F)+S(r,G) \\
&\le N_2(r,0;F)+N_2(r,0;G)+N_2(r,F)+N_2(r,G) \\
&\quad+T(r,G)+\overline{N}(r,0;F)+\overline{N}(r,F)+N_0(r,0;F') \\
&\quad+N_0(r,0;G')+S(r,F)+S(r,G).
\end{aligned}
$$

Since $F=f^{(k)}$ and $G=g^{(k)}$, from Lemma 2.4 and (5) we obtain

$$
\begin{aligned}
(6)\quad & \overline{N}(r,1;F)+\overline{N}(r,1;G) \\
&\le N_2(r,0;f^{(k)})+N_2(r,0;g^{(k)})+N_2(r,f^{(k)})+N_2(r,g^{(k)}) \\
&\quad+T(r,g^{(k)})+\overline{N}(r,0;f^{(k)})+\overline{N}(r,f^{(k)})+N_0(r,0;f^{(k+1)}) \\
&\quad+N_0(r,0;g^{(k+1)})+S(r,f)+S(r,g) \\
&\le (2k+3)\overline{N}(r,f)+(2k+2)\overline{N}(r,g)+\overline{N}(r,0;f)+T(r,g) \\
&\quad+N_{k+2}(r,0;f)+N_{k+2}(r,0;g)+N_0(r,0;f^{(k+1)}) \\
&\quad+N_0(r,0;g^{(k+1)})+S(r,f)+S(r,g).
\end{aligned}
$$

And from Lemma 2.1, we have

$$
\begin{aligned}
(7)\quad & T(r,f)\le \overline{N}(r,f)+N_{k+1}(r,0;f)+\overline{N}(r,c;f^{(k)}) \\
&\qquad\qquad -N_0(r,0;f^{(k+1)})+S(r,f), \\
(8)\quad & T(r,g)\le \overline{N}(r,g)+N_{k+1}(r,0;g)+\overline{N}(r,c;g^{(k)}) \\
&\qquad\qquad -N_0(r,0;g^{(k+1)})+S(r,g).
\end{aligned}
$$

Thus, from (4)–(8) (let $c=1$) we get

$$
\begin{aligned}
T(r,f)+T(r,g) &\le (2k+4)\overline{N}(r,f)+(2k+3)\overline{N}(r,g)+\overline{N}(r,0;f) \\
&\quad+T(r,g)+N_{k+1}(r,0;f)+N_{k+1}(r,0;g) \\
&\quad+N_{k+2}(r,0;f)+N_{k+2}(r,0;g)+S(r,f)+S(r,g).
\end{aligned}
$$

This becomes

$$T(r,f) \le (2k+4)\overline{N}(r,f) + (2k+3)\overline{N}(r,g) + \overline{N}(r,0;f) + N_{k+1}(r,0;f) + N_{k+1}(r,0;g) + N_{k+2}(r,0;f) + N_{k+2}(r,0;g) + S(r,f) + S(r,g). \tag{9}$$

Without loss of generality, we suppose that there exists a set I of infinite measure such that $T(r,g) \le T(r,f)$ for $r \in I$. Hence

$$\begin{aligned} T(r,f) \le [&4k+12-(2k+4)\Theta(\infty,f)-(2k+3)\Theta(\infty,g) \\ &-\delta_{k+1}(0,f)-\delta_{k+1}(0,g)-\delta_{k+2}(0,f)-\delta_{k+2}(0,g) \\ &-\Theta(0,f)+\varepsilon]T(r,f)+S(r,f) \end{aligned}$$

for $r \in I$ and $0 < \varepsilon < \Delta_1 - 4k - 11$, that is, $\{\Delta_1 - (4k+11) - \varepsilon\}T(r,f) \le S(r,f)$, so $\Delta_1 - (4k+11) \le 0$, contrary to hypothesis.

Therefore, we have $H \equiv 0$. Then

$$\frac{f^{(k+2)}}{f^{(k+1)}} - \frac{2f^{(k+1)}}{f^{(k)}-1} \equiv \frac{g^{(k+2)}}{g^{(k+1)}} - \frac{2g^{(k+1)}}{g^{(k)}-1}.$$

From this equation we get

$$g^{(k)} = \frac{(b+1)f^{(k)} + (a-b-1)}{bf^{(k)} + (a-b)}, \tag{10}$$

where $a\ (\neq 0), b$ are two constants.

Now, we consider three cases as follows.

(i)$_1$ Suppose $b \neq 0, -1$. If $a-b-1 \neq 0$, then by (10) we know that

$$\overline{N}\left(r, \frac{a-b-1}{b+1}; f^{(k)}\right) = \overline{N}(r,0;g^{(k)}).$$

By Lemma 2.1 we have

$$\begin{aligned} T(r,f) &\le \overline{N}(r,f) + N_{k+1}(r,0;f) + \overline{N}(r,c;f^{(k)}) \\ &\quad - N_0(r,0;f^{(k+1)}) + S(r,f) \\ &\le \overline{N}(r,f) + N_{k+1}(r,0;f) + \overline{N}\left(r, \frac{a-b-1}{b+1}; f^{(k)}\right) + S(r,f) \\ &\le \overline{N}(r,f) + N_{k+1}(r,0;f) + k\overline{N}(r,g) \\ &\quad + \overline{N}(r,0;g) + S(r,f) \\ &\le (2k+4)\overline{N}(r,f) + (2k+3)\overline{N}(r,g) + \overline{N}(r,0;f) \\ &\quad + N_{k+1}(r,0;f) + N_{k+1}(r,0;g) + N_{k+2}(r,0;f) \\ &\quad + N_{k+2}(r,0;g) + S(r,f) + S(r,g). \end{aligned}$$

Hence, by (1) we deduce that $T(r,f) \le S(r,f)$ for $r \in I$, a contradiction.

If $a-b-1=0$, then by (10) we know $g^{(k)}=((b+1)f^{(k)})/(bf^{(k)}+1)$. Obviously,

$$\overline{N}(r,1/b;f^{(k)})=\overline{N}(r,g^{(k)}).$$

By Lemma 2.1 we have

$$\begin{aligned}T(r,f)&\le \overline{N}(r,f)+N_{k+1}(r,0;f)+\overline{N}(r,c;f^{(k)})\\&\quad-N_0(r,0;f^{(k+1)})+S(r,f)\\&\le \overline{N}(r,f)+N_{k+1}(r,0;f)+\overline{N}(r,1/b;f^{(k)})+S(r,f)\\&\le \overline{N}(r,f)+N_{k+1}(r,0;f)+\overline{N}(r,g)\\&\quad+N_{k+1}(r,0;f)+N_{k+1}(r,0;g)+N_{k+2}(r,0;f)\\&\quad+N_{k+2}(r,0;g)+S(r,f)+S(r,g).\end{aligned}$$

Hence, by (1) we deduce that $T(r,f)\le S(r,f)$ for $r\in I$, a contradiction.

$(\mathrm{i})_2$ Suppose $b=-1$. Then (10) becomes $g^{(k)}=a/(a+1-f^{(k)})$.

If $a+1\ne 0$, then $\overline{N}(r,a+1;f^{(k)})=\overline{N}(r,g^{(k)})$, and we can deduce a contradiction as in $(\mathrm{i})_1$.

If $a+1=0$, then $f^{(k)}g^{(k)}\equiv 1$.

$(\mathrm{i})_3$ Suppose $b=0$. Then (10) becomes $g^{(k)}=(f^{(k)}+a-1)/a$.

If $a-1\ne 0$, then $\overline{N}(r,1-a;f^{(k)})=\overline{N}(r,0;g^{(k)})$, and again we deduce a contradiction as in $(\mathrm{i})_1$.

If $a-1=0$, then $f^{(k)}\equiv g^{(k)}$. From this, we obtain

$$f=g+p(z),$$

where $p(z)$ is a polynomial, so $T(r,f)=T(r,g)+S(r,f)$. If $p(z)\not\equiv 0$, then by Lemma 2.2, we have

$$\begin{aligned}T(r,f)&\le \overline{N}(r,f)+\overline{N}(r,0;f)+\overline{N}(r,p;f)+S(r,f)\\&\le \overline{N}(r,f)+\overline{N}(r,0;f)+\overline{N}(r,0;g)+S(r,f).\end{aligned}$$

Hence,

$$T(r,f)\le\{3-[\Theta(\infty,f)+\Theta(0,g)+\Theta(0,f)]+\varepsilon\}T(r,f)+S(r,f),$$

where $0<\varepsilon<(2k+3)[1-\Theta(\infty,f)]+(2k+3)[1-\Theta(\infty,g)+1-\delta_{k+1}(0,f)+1-\delta_{k+2}(0,f)+1-\delta_{k+2}(0,g)$. Therefore

$$\{\Delta_1-4k-11\}T(r,f)\le S(r,f).$$

Hence, by (1), we deduce that $T(r,f)\le S(r,f)$ for $r\in I$, a contradiction.

Therefore, we conclude that $p(z)\equiv 0$, that is, $f\equiv g$.

(ii) If $l = 1$, then F, G share $(1,1)$. By Lemma 2.10, (5) becomes

$$\begin{aligned}
&\overline{N}(r,1;F) + \overline{N}(r,1;G) \\
&\qquad \le N_{11}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) \\
&\qquad\quad + \overline{N}_E^{(2}(r,1;F) + \overline{N}(r,1;G) \\
&\qquad \le N_{11}(r,1;F) + N(r,1;G) - \overline{N}_L(r,1;G) \\
&\qquad\quad - \overline{N}_L(r,1;F) + \overline{N}_{F>2}(r,1;G) \\
&\qquad \le \overline{N}(r,\infty;F\,|\,{\ge}2) + \overline{N}(r,\infty;G\,|\,{\ge}2) + \overline{N}(r,0;F\,|\,{\ge}2) \\
&\qquad\quad + \overline{N}_*(r,1;F,G) + T(r,G) - m(r,1;G) + O(1) \\
&\qquad\quad - \overline{N}_L(r,1;G) + \overline{N}(r,0;G\,|\,{\ge}2) - \overline{N}_L(r,1;F) \\
&\qquad\quad + \tfrac{1}{2}\overline{N}(r,0;F) + \tfrac{1}{2}\overline{N}(r,\infty;F) + N_0(r,0;F') \\
&\qquad\quad + N_0(r,0;G') + S(r,F) + S(r,G).
\end{aligned}$$

By Lemma 2.4, this becomes

$$\begin{aligned}
\overline{N}(r,1;F) + \overline{N}(r,1;G) &\le (k+3/2)\overline{N}(r,f) + (2k+1)\overline{N}(r,g) + T(r,g) \\
&\quad + N_{k+2}(r,0;f) + N_{k+2}(r,0;g) + N_0(r,0;f^{(k+1)}) \\
&\quad + N_0(r,0;g^{(k+1)}) + S(r,f) + S(r,g).
\end{aligned}$$

Then (9) becomes

$$\begin{aligned}
T(r,f) &\le (k+5/2)\overline{N}(r,f) + (2k+2)\overline{N}(r,g) + N_{k+1}(r,0;f) \\
&\quad + N_{k+1}(r,0;g) + N_{k+2}(r,0;f) + N_{k+2}(r,0;g) + S(r,f) + S(r,g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I of infinite measure such that $T(r,g) \le T(r,f)$ for $r \in I$. Hence

$$\begin{aligned}
T(r,f) &\le [3k + 17/2 - (k+5/2)\Theta(\infty,f) - (2k+2)\Theta(\infty,g) \\
&\quad - \delta_{k+1}(0,f) - \delta_{k+1}(0,g) - \delta_{k+2}(0,f) - \delta_{k+2}(0,g) + \varepsilon]T(r,f) + S(r,f)
\end{aligned}$$

for $r \in I$ and $0 < \varepsilon < \Delta_2 - 3k - 15/2$, that is,

$$\{\Delta_2 - (3k+15/2) - \varepsilon\}T(r,f) \le S(r,f),$$

so $\Delta_2 \le 3k + 15/2$, contrary to hypothesis.

Therefore, we have $H \equiv 0$, and using the same argument of (ii), we deduce that $p(z) \equiv 0$, that is, $f \equiv g$.

(iii) If $l = 2$, then F, G share $(1,2)$, and we see that $\overline{N}(r,1;F\,|\,{\ge}2) = \overline{N}(r,1;G\,|\,{\ge}2)$. By Lemmas 2.4, 2.5 and 2.11, we obtain

$$N(r,1;F\,|\!=\!1) \le \overline{N}(r,0;F\,|\!\ge\!2) + \overline{N}(r,0;G\,|\!\ge\!2) + \overline{N}(r,\infty;F\,|\!\ge\!2) + \overline{N}(r,\infty;G\,|\!\ge\!2) + \overline{N}(r,0;G) - \overline{N}(r,1;G\,|\!\ge\!2) + \overline{N}_0(r,0;F') + S(r,F) + S(r,G).$$

Since $\overline{N}(r,1;F) = N(r,1;F\,|\!=\!1) + \overline{N}(r,1;F\,|\!\ge\!2)$, by Lemmas 2.1 and 2.4 and $F = f^{(k)}$, $G = g^{(k)}$, we have

$$T(r,f) \le 2\overline{N}(r,f) + 2\overline{N}(r,g) + N_{k+1}(r,0;f) + \overline{N}(r,0;f^{(k)}\,|\!\ge\!2) + \overline{N}(r,0;g^{(k)}\,|\!\ge\!2) + \overline{N}(r,0;g^{(k)}) + S(r,f) + S(r,g).$$

Then we obtain

$$T(r,f) \le (k+2)\overline{N}(r,f) + (k+2)\overline{N}(r,g) + N_{k+1}(r,0;f) + N_{k+2}(r,0;f) + N_{k+2}(r,0;g) + S(r,f) + S(r,g).$$

Without loss of generality, we suppose that there exists a set I of infinite measure such that $T(r,g) \le T(r,f)$ for $r \in I$. Hence

$$T(r,f) \le [2k+7-(k+2)\Theta(\infty,f)-(k+2)\Theta(\infty,g)-\delta_{k+1}(0,f) - \delta_{k+2}(0,f)-\delta_{k+2}(0,g)+\varepsilon]T(r,f)+S(r,f)$$

for $r \in I$ and $0 < \varepsilon < \Delta_3 - 2k - 6$, that is, $\{\Delta_3 - (2k+6) - \varepsilon\}T(r,f) \le S(r,f)$, so $\Delta_3 \le 2k+6$, contrary to hypothesis.

Therefore, $H \equiv 0$, and using the same argument of (i), we deduce that $p(z) \equiv 0$, that is, $f \equiv g$.

This completes the proof of Lemma 2.12. ∎

3. Proof of Theorem 1.2. Let

$$F(z) = f^n(z)(f(z)-1) \quad \text{and} \quad G(z) = g^n(z)(g(z)-1).$$

We have

$$\Delta_1 = (2k+4)\min\{\Theta(\infty,F),\Theta(\infty,G)\} + (2k+3)\min\{\Theta(\infty,F),\Theta(\infty,G)\} + \min\{\Theta(0,F),\Theta(0,G)\} + \delta_{k+1}(0,F) + \delta_{k+1}(0,G) + \delta_{k+2}(0,F) + \delta_{k+2}(0,G).$$

Since

$$\Theta(0,F) = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}(r,0;F)}{T(r,F)} = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}(r,0;f^n(f-1))}{(n+1)T(r,f)} = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}(r,0;f)+\overline{N}(r,1;f)}{(n+1)T(r,f)} \ge 1 - \overline{\lim_{r\to\infty}}\frac{2T(r,f)}{(n+1)T(r,f)},$$

we obtain

$$\Theta(0,F) \ge \frac{n-1}{n+1}. \tag{11}$$

Similarly,

$$\Theta(0,G) \geq \frac{n-1}{n+1}. \tag{12}$$

And since

$$\begin{aligned}\Theta(\infty,F) &= 1 - \varlimsup_{r\to\infty}\frac{\overline{N}(r,F)}{T(r,F)} = 1 - \varlimsup_{r\to\infty}\frac{\overline{N}(r,f^n(f-1))}{(n+1)T(r,f)}\\ &= 1 - \varlimsup_{r\to\infty}\frac{\overline{N}(r,f)}{(n+1)T(r,f)} \geq 1 - \varlimsup_{r\to\infty}\frac{T(r,f)}{(n+1)T(r,f)},\end{aligned}$$

it follows that

$$\Theta(\infty,F) \geq \frac{n}{n+1}. \tag{13}$$

Similarly,

$$\Theta(\infty,G) \geq \frac{n}{n+1}. \tag{14}$$

Next, by the definition of $N_k(r,a;f)$ we have

$$\begin{aligned}\delta_{k+1}(0,f) &= 1 - \varlimsup_{r\to\infty}\frac{N_{k+1}(r,0;f)}{T(r,f)} \geq 1 - \varlimsup_{r\to\infty}\frac{(k+1)\overline{N}(r,0;f)}{T(r,f)},\\ \delta_{k+1}(0,F) &= 1 - \varlimsup_{r\to\infty}\frac{N_{k+1}(r,0;f^n(f-1))}{T(r,F)} \geq 1 - \varlimsup_{r\to\infty}\frac{(k+1)\overline{N}(r,0;F)}{T(r,F)}.\end{aligned}$$

Therefore

$$\delta_{k+1}(0,F) \geq 1 - \varlimsup_{r\to\infty}\frac{(k+2)T(r,f)}{(n+1)T(r,f)} = 1 - \frac{k+2}{n+1}. \tag{15}$$

Similarly,

$$\delta_{k+1}(0,G) \geq 1 - \frac{k+2}{n+1} \tag{16}$$

and

$$\delta_{k+2}(0,F) \geq 1 - \frac{k+3}{n+1}, \qquad \delta_{k+2}(0,G) \geq 1 - \frac{k+3}{n+1}. \tag{17}$$

From (11)–(17), we get

$$\begin{aligned}\Delta_1 \geq (2k+4)\,\frac{n}{n+1} + (2k+3)\,\frac{n}{n+1} + \frac{n-1}{n+1}\\ + 2\left(1 - \frac{k+2}{n+1}\right) + 2\left(1 - \frac{k+3}{n+1}\right).\end{aligned}$$

Since $n > 8k+18$, we get $\Delta_1 > 4k+11$.

From the assumption of Theorem 1.2, we deduce that $F^{(k)} = [f^n(f-1)]^{(k)}$ and $G^{(k)} = [g^n(g-1)]^{(k)}$ share 1 IM and F, G satisfy the assumptions of Lemma 2.10. By that lemma, either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Next, we consider the following two cases:

CASE 1: $F^{(k)}G^{(k)} \equiv 1$, i.e.

$$[f^n(f-1)]^{(k)}[g^n(g-1)]^{(k)} \equiv 1. \tag{18}$$

(i) Let z_0 be a zero of f of order p. By (18), z_0 is a pole of g, say of order q; (18) yields $np-k = nq+q+k$, i.e., $n(p-q) = q+2k$, which implies that $p \geq q+1$ and $q+2k \geq n$. Thus

$$p \geq n-2k+1. \tag{19}$$

(ii) Let z_1 be a zero of $f-1$ of order p_1. Then it is a zero of $[f^n(f-1)]^{(k)}$ of order p_1-k and hence a pole of g, say of order q_1. By (18) we obtain $p_1-k = nq_1+q_1+k$, i.e., $p_1 = (n+1)q_1+2k$, so

$$p_1 \geq n+2k+1.$$

(iii) Let z_2 be a zero of f' of order p_2 that is not a zero of $f(f-1)$. Then z_2 is a pole of g, say of order q_2, and by (18) we obtain $p_2-(k-1) = nq_2+q_2+k$, i.e., $p_2 = (n+1)q_2+2k+1$, so

$$p_2 \geq n+2k.$$

We have similar results for the zeros of $[g^n(g-1)]^{(k)}$.

Thus we know that possible poles of g occur at (i) the zeros of f; (ii) the zeros of $f-1$; (iii) the zeros of f' that are not zeros of $f(f-1)$. Thus

$$\begin{aligned}\overline{N}(r,g) &\leq \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,0;f')\\ &\leq \frac{1}{n-2k+1}N(r,0;f)+\frac{1}{n+2k+1}N(r,1;f)+\frac{1}{n+2k}N(r,0;f').\end{aligned}$$

Since $n \geq 8k+18$, we get

$$\begin{aligned}\overline{N}(r,g) &\leq \frac{1}{6k+19}N(r,0;f)+\frac{1}{10k+19}N(r,1;f)+\frac{1}{10k+18}N(r,0;f')\\ &\leq \frac{1}{25}N(r,0;f)+\frac{1}{29}N(r,1;f)+\frac{1}{28}N(r,0;f')\\ &\leq \left(\frac{1}{25}+\frac{1}{29}+\frac{2}{28}\right)T(r,f)+S(r,f)\\ &\leq 0.1102\,T(r,f)+S(r,f).\end{aligned}$$

By the second fundamental theorem and the above, we obtain

$$\begin{aligned}T(r,g) &\leq \overline{N}(r,0;g)+\overline{N}(r,1;g)+\overline{N}(r,g)+S(r,g)\\ &\leq \frac{1}{25}N(r,0;g)+\frac{1}{29}N(r,1;g)+0.1102\,T(r,f)\\ &\quad +S(r,f)+S(r,g)\\ &\leq 0.0745\,T(r,g)+0.1102\,T(r,f)+S(r,f)+S(r,g).\end{aligned} \tag{20}$$

Similarly,

$$T(r,f) \leq 0.0745\,T(r,f)+0.1102\,T(r,g)+S(r,f)+S(r,g). \tag{21}$$

Adding (20) and (21), we have

$$T(r,f)+T(r,g)\le 0.1848[T(r,f)+T(r,g)]+S(r,f)+S(r,g),$$

so

$$0.8152[T(r,f)+T(r,g)]\le S(r,f)+S(r,g), \tag{22}$$

which yields a contradiction.

CASE 2: Suppose $F\equiv G$, that is,

$$f^n(f-1)\equiv g^n(g-1).$$

We consider the following two cases.

(i) Suppose $h=f/g$ is a constant. If $h\equiv 1$, then $f\equiv g$. If $h\not\equiv 1$, we deduce that

$$g=\frac{1-h^n}{1-h^{n+1}}\quad\text{and}\quad f=\frac{h(1-h^n)}{(1-h^{n+1})}.$$

This is a contradiction because f,g are nonconstant.

(ii) Suppose $h=f/g$ is not a constant. Thus we get

$$g=\frac{h^n}{1+h+h^2+\cdots+h^n}-1.$$

Then by Nevanlinna's first fundamental theorem and Lemma 2.3,

$$\begin{aligned}T(r,g)&=T\left(r,\sum_{j=0}^{n}\frac{1}{h^j}\right)+S(r,h)=nT(r,1/h)+S(r,h)\\&=nT(r,h)+S(r,h).\end{aligned}$$

Now we note that a pole of h is not a pole of $h^n/(1+h+h^2+\cdots+h^n)-1$. So

$$\sum_{j=0}^{n}\overline{N}\left(r,\frac{1}{h-u_k}\right)\le\overline{N}(r,g),$$

where $u_k=\exp(2k\pi i/n)$ for $k=1,\dots,n$. By the second fundamental theorem we get

$$\begin{aligned}(n-2)T(r,h)&\le\sum_{k=1}^{n}\overline{N}\left(r,\frac{1}{h-u_k}\right)+S(r,h)\le\overline{N}(r,\infty;g)+S(r,h)\\&<(1-\Theta(\infty,g)+\varepsilon)T(r,g)+S(r,h)\\&=n(1-\Theta(\infty,g)+\varepsilon)T(r,h)+S(r,h)\end{aligned}$$

for all $\varepsilon>0$. Again putting $h_1=1/h$, noting that $T(r,h)=T(r,h_1)+O(1)$ and proceeding as above we get

$$(n-2)T(r,h)\le n(1-\Theta(\infty,f)+\varepsilon)T(r,h)+S(r,h)$$

for all $\varepsilon > 0$. Since $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$, there exists a δ (> 0) such that $\Theta(\infty, f) + \Theta(\infty, g) > \delta + 4/n$. Then

$$\begin{aligned} 2(n-2)T(r,h) &\le n(2 - \Theta(\infty, f) - \Theta(\infty, g) + 2\varepsilon)T(r,h) + S(r,h) \\ &< n(2 - 4/n - \delta + 2\varepsilon)T(r,h) + S(r,h), \end{aligned}$$

and so $(\delta - 2\varepsilon)T(r,h) \le S(r,h)$, which is a contradiction for $0 < 2\varepsilon < \delta$.

Therefore, $f \equiv g$ and so Theorem 1.2 is proved completely.

4. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. From (11)–(17), we have

$$\Delta_2 \ge \left(k + \frac{5}{2}\right)\frac{n}{n+1} + (2k+2)\frac{n}{n+1} + 2\left(1 - \frac{k+2}{n+1}\right) + 2\left(1 - \frac{k+3}{n+1}\right).$$

Since $n > 7k + 23/2$, we get $\Delta_2 > 3k + 15/2$.

Considering $F^{(k)} = [f^n(f-1)]^{(k)}$ and $G^{(k)} = [g^n(g-1)]^{(k)}$, by the assumptions of Theorem 1.2, $F^{(k)}$ and $G^{(k)}$ share $(1,1)$, and F and G satisfy the assumptions of Lemma 2.12; by that lemma, either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Using the same argument of Theorem 1.2, we deduce $f \equiv g$.

Proof of Theorem 1.4. From (11)–(17), we have

$$\Delta_3 \ge (k+2)\,\frac{n}{n+1} + (k+2)\,\frac{n}{n+1} + \left(1 - \frac{k+2}{n+2}\right) + 2\left(1 - \frac{k+3}{n+1}\right).$$

Since $n > 5k + 11$, we get $\Delta_3 > 2k + 6$.

Considering $F^{(k)} = [f^n(f-1)]^{(k)}$ and $G^{(k)} = [g^n(g-1)]^{(k)}$, by the assumptions of Theorem 1.2, $F^{(k)}$ and $G^{(k)}$ share $(1,2)$, and F and G satisfy the assumptions of Lemma 2.12; by that lemma, either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Using the same argument of Theorem 1.2, we deduce $f \equiv g$.

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