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On subextension and approximation of plurisubharmonic functions with given boundary values

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Abstract. Our aim in this article is the study of subextension and approximation of plurisubharmonic functions in $\mathcal{E}_\chi(\Omega, H)$, the class of functions with finite χ-energy and given boundary values. We show that, under certain conditions, one can approximate any function in $\mathcal{E}_\chi(\Omega, H)$ by an increasing sequence of plurisubharmonic functions defined on strictly larger domains.

1. Introduction. The purpose of this paper is to study subextension and approximation of plurisubharmonic functions with given boundary values. Let $\Omega \subset \hat{\Omega}$ be domains in $\mathbb{C}^n$ and let $\mathcal{PSH}(\Omega)$ denote the cone of plurisubharmonic functions (psh for short) on Ω, and $\mathcal{PSH}^-(\Omega)$ the subclass of negative functions. A function $\hat{u} \in \mathcal{PSH}(\hat{\Omega})$ is said to be a *subextension* of u if $\hat{u}(z) \le u(z)$ for all $z \in \Omega$. In [E], El Mir gave an example of a plurisubharmonic function on the unit bidisc for which the restriction to any smaller bidisc admits no subextension to the whole space. In contrast with this negative answer, some important results have been proved by many authors on hyperconvex domains for functions belonging to classes of psh functions, called energy classes, introduced by U. Cegrell (see Section 1 for detailed definitions). In the class $\mathcal{F}(\Omega)$, $\mathcal{E}_p$ $(p > 0)$, $\mathcal{E}_\chi(\Omega)$ and $\mathcal{E}^\psi(\Omega)$ the problem has been studied by Cegrell, Zeriahi and Kołodziej (see [CZ] and [CKZ]), P. H. Hiep (see [Hi]), Benelkourchi (see [B2]), and Hai and Long (see [HL]) respectively. The problem of subextension of psh functions with boundary values was studied in the class $\mathcal{F}(\Omega, H)$ by Czyż and Hed (see [CzH]) and by Hed (see [H]).

Another problem, which is considered in Cegrell classes, is the approximation of plurisubharmonic functions on a domain Ω by plurisubharmonic functions defined on a neighborhood of $\bar{\Omega}$. In [B1], under certain conditions

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on Ω, Benelkourchi proved that any function u in the class $\mathcal{F}^a(\Omega)$ can be approximated by an increasing sequence $(u_j)_j$ of plurisubharmonic functions defined on larger domains $(\Omega_j)_j$ with $u_j \in \mathcal{F}^a(\Omega_j)$. This result was generalized to the class $\mathcal{F}(\Omega)$ by Cegrell and Hed (see [CH]) and recently to the class $\mathcal{E}_\chi(\Omega)$ by Benelkourchi (see [B2]). The case of functions with given boundary values was studied in the class $\mathcal{F}(\Omega, H)$ by Hed (see [H]).

Inspired and motivated by the research going on in this area, we give a subextension theorem for plurisubharmonic functions in the class $\mathcal{E}_\chi(\Omega, H)$ of functions with finite χ-energy and given boundary values; this will be used to prove the following theorem on approximation of psh functions, which is the main motivation of this paper.

Main Theorem 1.1. *Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex domain and $\{\Omega_j\}$ be a decreasing sequence of hyperconvex domains containing Ω such that $\lim_{j\to\infty} \operatorname{cap}_{\Omega_j}(K) = \operatorname{cap}_\Omega(K)$ for all compact subsets $K \subset \Omega$. Let $G \in \mathcal{MPSH}^-(\Omega_1) \cap \mathcal{C}(\overline{\Omega})$ and $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a nondecreasing function such that $\chi(-t) < 0$ for all $t > 0$. Then for every $u \in \mathcal{E}_\chi(\Omega, H)$ such that*

$$\int_\Omega -\chi(u-H)(dd^c u)^n < \infty,$$

where $H = G_{|\Omega}$, there exists a nondecreasing sequence of functions $u_j \in \mathcal{E}_\chi(\Omega_j, G_{|\Omega_j})$ such that $\lim_{j\to\infty} u_j(z) = u(z)$ for all $z \in \Omega$.

The rest of the paper is organized as follows. In Section 2, we recall the Cegrell classes. In Section 3, we give some properties of the energy classes $\mathcal{E}_\chi(\Omega, H)$. Finally, in Section 4, subextension of functions from $\mathcal{E}_\chi(\Omega, H)$ is studied and our main theorem is proved.

2. Preliminaries. In this section, we summarize some basic properties and related definitions which are essential in the following discussions.

A bounded domain Ω is *hyperconvex* if there exists ϱ in $\mathcal{PSH}^-(\Omega) \cap \mathcal{C}(\Omega)$ such that $\{z \in \Omega;\ \varrho(z) < -c\} \Subset \Omega$ for all $c > 0$.

The following classes of plurisubharmonic functions, on which the complex Monge–Ampère operator $(dd^c\,\cdot\,)^n$ is well defined, were introduced by U. Cegrell in [C1] and [C2].

We denote by $\mathcal{E}_0(\Omega)$ the set of negative and bounded psh functions φ on Ω which tend to zero at the boundary and satisfy $\int_\Omega (dd^c\varphi)^n < \infty$.

We say that a negative function is in the class $\mathcal{F}(\Omega)$ if there exists a decreasing sequence $(u_j)_j$ in $\mathcal{E}_0(\Omega)$ which converges pointwise to u on Ω and

$$\sup_{j\in\mathbb{N}} \int_\Omega (dd^c u_j)^n < \infty.$$

We denote by $\mathcal{F}^a(\Omega)$ the set of functions u in $\mathcal{F}(\Omega)$ such that $(dd^c\,\cdot\,)^n$ vanishes on all pluripolar sets.

A negative function u is in the class $\mathcal{E}(\Omega)$ if for all $z \in \Omega$, there exists a neighborhood $w \ni z$ and a decreasing sequence $(u_j)_j$ in $\mathcal{E}_0(\Omega)$ which converges pointwise to u on w and

$$\sup_{j\in\mathbb{N}} \int_\Omega (dd^c u_j)^n < \infty.$$

For each $p > 0$ define $\mathcal{E}_p(\Omega)$ (resp. $\mathcal{F}_p(\Omega)$) to be the class of functions $\varphi \in \mathcal{PSH}^-(\Omega)$ such that there exists a decreasing sequence $(u_j)_j$ in $\mathcal{E}_0(\Omega)$ which converges to u such that $\sup_{j\in\mathbb{N}} \int_\Omega (-u_j)^p (dd^c u_j)^n < \infty$ (resp. $\sup_{j\in\mathbb{N}} \int_\Omega (1+(-u_j)^p)(dd^c u_j)^n < \infty$).

Let $\psi \in \mathcal{PSH}^-(\Omega)$ with $\psi \not\equiv 0$. We denote by $\mathcal{E}^\psi(\Omega)$ the set of functions $u \in \mathcal{PSH}^-(\Omega)$ such that there is a decreasing sequence $(u_j)_j$ in $\mathcal{E}_0(\Omega)$ which converges pointwise to u on Ω and

$$\sup_{j\in\mathbb{N}} \int_\Omega (-\psi)(dd^c u_j)^n < \infty.$$

We have $\mathcal{E}^\psi(\Omega) \subset \mathcal{E}(\Omega)$.

A *fundamental sequence* $(\Omega_j)_j$ of Ω is a sequence of strictly pseudoconvex domains such that $\Omega_j \Subset \Omega_{j+1} \Subset \Omega$ for every j , and $\bigcup_j \Omega_j = \Omega$. Let $u \in \mathcal{E}(\Omega)$ and

$$u_{\Omega_j} := \sup\{\varphi \in \mathcal{PSH}(\Omega);\ \varphi \le u \text{ on } \Omega \setminus \Omega_j\}.$$

We have $u_{\Omega_j} \in \mathcal{E}(\Omega)$ and $(u_{\Omega_j})_j$ is an increasing sequence.

Define $\tilde{u} := (\lim_j u_{\Omega_j})^*$; then $\tilde{u} \in \mathcal{E}(\Omega)$ and $(dd^c \tilde{u})^n = 0$. We define

$$\mathcal{N}(\Omega) := \{u \in \mathcal{E}(\Omega);\ \tilde{u} = 0\}.$$

The above definitions imply that $\mathcal{E}_0(\Omega) \subset \mathcal{F}^a(\Omega) \subset \mathcal{F}(\Omega) \subset \mathcal{N}(\Omega) \subset \mathcal{E}(\Omega)$.

Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a nondecreasing function. We consider the set $\mathcal{E}_\chi(\Omega)$ of plurisubharmonic functions of finite χ-weighted Monge–Ampère energy. These are the functions $u \in \mathcal{PSH}(\Omega)$ such that there exists a decreasing sequence $(u_j)_j$ in $\mathcal{E}_0(\Omega)$ with limit u and

$$\sup_{j\in\mathbb{N}} \int_\Omega -\chi \circ u_j (dd^c u_j)^n < \infty.$$

When χ is bounded and $\chi(0) \neq 0$, then $\mathcal{E}_\chi(\Omega) = \mathcal{F}(\Omega)$; and when $\chi(t) = -(-t)^p$ (resp. $\chi(t) = -1 - (-t)^p$), then $\mathcal{E}_\chi(\Omega) = \mathcal{E}_p(\Omega)$ (resp. $\mathcal{E}_\chi(\Omega) = \mathcal{F}_p(\Omega)$).

Let $\mathcal{K}(\Omega) \in \{\mathcal{E}_0(\Omega), \mathcal{F}(\Omega), \mathcal{N}(\Omega), \mathcal{E}_\chi(\Omega)\}$ and $H \in \mathcal{E}(\Omega)$. We say that a plurisubharmonic function u defined on Ω is in $\mathcal{K}(\Omega, H)$ if there exists a function $\varphi \in \mathcal{K}(\Omega)$ such that $H \ge u \ge H + \varphi$.

Finally, we denote by $\mathcal{M}(\Omega)$ the set of psh functions u in $\mathcal{E}(\Omega)$ with $(dd^c u)^n = 0$.

3. The energy class $\mathcal{E}_\chi(\Omega, H)$. In the following, $\chi : \mathbb{R}^- \to \mathbb{R}^-$ is a nondecreasing function such that $t_\chi = 0$, where $t_\chi := \sup\{t > 0;\ \chi(-t) = 0\}$. From [B2], we have $\bigcup_{\chi, t_\chi = 0} \mathcal{E}_\chi(\Omega) \subsetneq \mathcal{N}(\Omega)$.

LEMMA 3.1. *Let $H \in \mathcal{M}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ and $u \in \mathcal{N}(\Omega, H)$. Then there is a constant $C > 0$ such that for all v in $\mathcal{N}(\Omega, H)$ with $u \le v$, we have*

$$\int_\Omega -\chi(v - H)(dd^c v)^n \le C \int_\Omega -\chi(u - H)(dd^c u)^n. \tag{3.1}$$

Proof. We remark that

$$\int_\Omega -\chi(u - H)(dd^c u)^n = 0 \quad \text{and} \quad \int_\Omega -\chi(u - H)(dd^c u)^n = \infty$$

are trivial cases, so we can assume that $0 < \int_\Omega -\chi(u - H)(dd^c u)^n < \infty$.

From [ACCP, Proposition 2.5], there exists a decreasing sequence $(v_j)_j$ in $\mathcal{N}(\Omega, H) \cap \mathcal{C}(\bar{\Omega})$ that converges pointwise to v on Ω. We can choose $(v_j)_j$ such that $\int_\Omega (dd^c v_j)^n < \infty$ for all $j \ge 0$. Indeed, let $(\Omega_j)_j$ be a fundamental sequence of Ω and put $\varphi_j := \sup\{\varphi \in \mathcal{N}(\Omega, H);\ \varphi \le v$ on Ω_j and $\varphi \le H$ on $\Omega\}$. Then $(\varphi_j)_j$ is a decreasing sequence that converges pointwise to v and $\int_\Omega (dd^c \varphi_j)^n < \infty$ for all $j \ge 0$. Therefore from [ACCP, Proposition 2.5 and Lemma 3.3], for every $j \ge 0$ there exists a decreasing sequence $(\varphi_j^k)_j$ in $\mathcal{N}(\Omega, H) \cap \mathcal{C}(\bar{\Omega})$ that converges pointwise to φ_j on Ω and $\int_\Omega (dd^c \varphi_j^k)^n < \infty$ for all $k \ge 0$. We can extract from it a subsequence satisfying the desired conditions.

Now, we have

$$\begin{aligned}
\int_{(v_j - H \le -s)} (dd^c v_j)^n &= \int_\Omega (dd^c v_j)^n - \int_{(v_j - H > -s)} (dd^c v_j)^n \\
&= \int_\Omega (dd^c \max(v_j, H - s))^n - \int_{(v_j > H - s)} (dd^c \max(v_j, H - s)^n \\
&= \int_{(v_j - H \ge -s)} (dd^c \max(v_j, H - s))^n \\
&\le \Big(s - \inf_{\bar{\Omega}} H\Big)^n \operatorname{cap}_\Omega(v_j - H \le -s).
\end{aligned}$$

Let $a = -\inf_{\bar{\Omega}} H$. Then

$$\begin{aligned}
\int_\Omega -\chi(v - H)(dd^c v)^n &\le \lim_{j\to\infty} \int_\Omega -\chi(v_j - H)(dd^c v_j)^n \\
&= \lim_{j\to\infty} \int_0^\infty \chi'(-t) \int_{(v_j - H < -s)} (dd^c v_j)^n
\end{aligned}$$

$$\leq \lim_{j\to\infty} \int_0^\infty \chi'(-t)(t+a)^n \operatorname{cap}_\Omega(v_j - H < -t)\,dt$$
$$= \int_0^\infty \chi'(-t)(t+a)^n \operatorname{cap}_\Omega(v - H < -t)\,dt$$
$$\leq \int_0^\infty \chi'(-t)(t+a)^n \operatorname{cap}_\Omega(u - H < -t)\,dt.$$

On the other hand, from [B3, Lemma 3.3] we have

$$t^n \operatorname{cap}_\Omega(u - H < -s - t) \leq \int_{(u-H<-s)} (dd^c u)^n, \qquad \forall s,t > 0.$$

Therefore, for all $k \in \mathbb{N}^*$ we obtain

$$\int_0^\infty \chi'(-t)(t+a)^n \operatorname{cap}_\Omega(u - H < -t)\,dt$$
$$= \int_{-\frac{a}{1+k}}^\infty \big(s + \tfrac{s+a}{k} + a\big)^n \chi'(-s - \tfrac{s+a}{k}) \operatorname{cap}_\Omega\big(u - H < -s - \tfrac{s+a}{k}\big)\big(1 + \tfrac{1}{k}\big)\,ds$$
$$= (1+k)^n \int_{-\frac{a}{1+k}}^\infty \chi'\big(-s - \tfrac{s+a}{k}\big)\big(\tfrac{s+a}{k}\big)^n \operatorname{cap}_\Omega\big(u - H < -s - \tfrac{s+a}{k}\big)\big(1 + \tfrac{1}{k}\big)\,ds$$
$$\leq (1+k)^n \int_{-\frac{a}{1+k}}^\infty \chi'\big(-s - \tfrac{s+a}{k}\big) \int_{(u-H<-s)} (dd^c u)^n \big(1 + \tfrac{1}{k}\big)\,ds$$
$$= (1+k)^n \int_{-\frac{a}{1+k}}^\infty \chi'\big(-s - \tfrac{s+a}{k}\big) \int_{((1+\frac{1}{k})(u-H)-\frac{a}{k}<-s-\frac{s+a}{k})} (dd^c u)^n \big(1 + \tfrac{1}{k}\big)\,ds$$
$$= (1+k)^n \int_0^\infty \chi'(-t) \int_{((1+\frac{1}{k})(u-H)-\frac{a}{k}<-t)} (dd^c u)^n\,dt$$
$$= (1+k)^n \int_\Omega -\chi\big(\big(1 + \tfrac{1}{k}\big)(u - H) - \tfrac{a}{k}\big)(dd^c u)^n.$$

Since

$$\lim_{k\to\infty} \int_\Omega -\chi\big(\big(1 + \tfrac{1}{k}\big)(u - H) - \tfrac{a}{k}\big)(dd^c u)^n = \int_\Omega -\chi(u - H)\big)(dd^c u)^n,$$

there exists k_0 such that

$$\int_\Omega -\chi\big(\big(1 + \tfrac{1}{k}\big)(u - H) - \tfrac{a}{k}\big)(dd^c u)^n \leq 2 \int_\Omega -\chi(u - H)(dd^c u)^n, \qquad \forall k \geq k_0.$$

It follows that

$$\int_{\Omega} -\chi(v-H)(dd^c v)^n \le \int_0^{\infty} \chi'(-t)(t+a)^n \operatorname{cap}_{\Omega}(u-H<-t)dt$$
$$\le 2(1+k_0)^n \int_{\Omega} -\chi(u-H)(dd^c u)^n. \blacksquare$$

PROPOSITION 3.2. *Let $H \in \mathcal{M}(\Omega) \cap \mathcal{C}(\bar{\Omega})$.*

(i) *If $u \in \mathcal{E}_{\chi}(\Omega, H)$ with $\int_{\Omega} -\chi(u-H)(dd^c u)^n < \infty$, then there exists a decreasing sequence $(u_j)_j$ in $\mathcal{E}_0(\Omega, H)$ that converges pointwise to u and $\sup_j \int_{\Omega} -\chi(u_j-H)(dd^c u_j)^n < \infty$.*
(ii) *Conversely, if there exists a decreasing sequence $(u_j)_j$ in $\mathcal{E}_0(\Omega, H)$ that converges pointwise to u and $\sup_j \int_{\Omega} -\chi(u_j-H)(dd^c u_j)^n < \infty$, then $u \in \mathcal{E}_{\chi}(\Omega, H)$ and $\int_{\Omega} -\chi(u-H)(dd^c u)^n < \infty$.*

Proof. Assume that $u \in \mathcal{E}_{\chi}(\Omega, H)$. From [ACCP, Proposition 2.5] there exists a decreasing sequence $(u_j) \subset \mathcal{E}_0(\Omega, H)$ that converges pointwise to u. Since $u \le u_j$ and $\int_{\Omega} -\chi(u-H)(dd^c u)^n < \infty$, from Lemma 3.1 we deduce that $\int_{\Omega} -\chi(u_j-H)(dd^c u_j)^n \le C \int_{\Omega} -\chi(u-H)(dd^c u)^n$ for all j, hence $\sup_j \int_{\Omega} -\chi(u-H)(dd^c u_j)^n < \infty$.

Conversely, assume that $(u_j) \subset \mathcal{E}_0(\Omega, H)$ is a decreasing sequence converging pointwise to u and

$$\sup_j \int_{\Omega} -\chi(u_j-H)(dd^c u_j)^n < \infty. \tag{3.2}$$

From the upper semicontinuity of $u-H$, $-\chi(u-H)(dd^c u)^n$ is bounded from above by any cluster point of the bounded sequence $-\chi(u_j-H)(dd^c u_j)^n$. Therefore $\int_{\Omega} -\chi(u-H)(dd^c u)^n < \infty$.

On the other hand, for all j, put

$$\psi_j := \sup\{\varphi \in \mathrm{PSH}(\Omega);\ \varphi + H \le u_j\}.$$

It is clear that $\psi_j \in \mathcal{E}_0(\Omega)$ and $(\psi_j)_j$ is a decreasing sequence. Note that if u_j is continuous then:

(1) $(dd^c \psi_j)^n = 0$ on $\{\psi_j + H < u_j\}$,
(2) $(dd^c \psi_j)^n \le (dd^c u_j)^n$ on Ω.

The first statement follows from [BT, Corollary 9.2]. For the second, we have $(dd^c \psi_j)^n \le (dd^c u_j)^n$ on the open set $\{\psi_j + H < u_j\}$. To show that is true on $A = \{\psi_j + H = u_j\}$, we proceed as in [CH, proof of Lemma 2.1]. Let $K \subset A$ be a compact set; then $K \subset \{\psi_j + H + \varepsilon > u_j\}$. It follows from

[BGZ, Theorem 2.2] that

$$\int_K (dd^c\psi_j)^n = \int_K \mathbf{1}_{\{\psi_j+H+\varepsilon>u_j\}}(dd^c\psi_j)^n \le \int_K \mathbf{1}_{\{\psi_j+H+\varepsilon>u_j\}}(dd^c(\psi_j+H))^n$$
$$= \int_K \mathbf{1}_{\{\psi_j+H+\varepsilon>u_j\}}(dd^c\max(\psi_j+H+\varepsilon,u_j))^n$$
$$\le \int_K (dd^c\max(\psi_j+H+\varepsilon,u_j))^n.$$

Since $\max(\psi_j+H+\varepsilon,u_j), \max(\psi_j+H,u_j) \in \mathcal{E}(\Omega)$, it follows from [C3, Lemma 3.2] that the measure $(dd^c\max(\psi_j+H+\varepsilon,u_j))^n$ converges to $(dd^cu_j)^n$ in the weak* topology. The characteristic function $\mathbf{1}_K$ can be approximated by a decreasing sequence of continuous functions φ_k that are bounded from above. Then from Lebesgue's dominated convergence theorem, we have

$$\limsup_{\varepsilon\to 0}\int_\Omega \mathbf{1}_K(dd^c\max(\psi_j+H+\varepsilon,u_j))^n$$
$$= \limsup_{\varepsilon\to 0}\Big(\lim_k \int_\Omega \varphi_k(dd^c\max(\psi_j+H+\varepsilon,u_j))^n\Big)$$
$$\le \limsup_{\varepsilon\to 0}\int_\Omega \varphi_k(dd^c\max(\psi_j+H+\varepsilon,u_j))^n = \int_\Omega \varphi_k(dd^cu_j)^n.$$

Since $\int_\Omega \varphi_k(dd^cu_j)^n \searrow \int_\Omega \mathbf{1}_K(dd^cu_j)^n$, we get $(dd^c\psi_j)^n \le (dd^cu_j)^n$ on A.

Now, fix $j\in\mathbb{N}$. We have $\int_\Omega -\chi(u_j-H)(dd^cu_j)^n < \infty$. Let $(u_j^k)_k \subset \mathcal{E}_0(\Omega,H)\cap\mathcal{C}(\bar\Omega)$ be a decreasing sequence that converges to u_j. Set

$$\psi_j^k := \sup\{\varphi\in \mathrm{PSH}(\Omega);\ \varphi+H\le u_j^k\}.$$

Then $(\psi_j^k)_k \subset \mathcal{E}_0(\Omega)$ and $\psi_j^k \searrow \psi_j$. Moreover from the above, $(dd^c\psi_j^k)^n \le (dd^cu_j^k)^n$ and $(dd^c\psi_j^k)^n = 0$ on $\{\psi_j^k+H<u_j^k\}$. It follows that

$$\begin{aligned}
\int_\Omega -\chi(\psi_j)(dd^c\psi_j)^n &\le \lim_{k\to\infty}\int_\Omega -\chi(\psi_j^k)(dd^c\psi_j^k)^n\\
&= \lim_{k\to\infty}\int_{\{\psi_j^k+H=u_j^k\}} -\chi(\psi_j^k)(dd^c\psi_j^k)^n\\
&= \lim_{k\to\infty}\int_{\{\psi_j^k+H=u_j^k\}} -\chi(u_j^k-H)(dd^c\psi_j^k)^n\\
&\le \lim_{k\to\infty}\int_{\{\psi_j^k+H=u_j^k\}} -\chi(u_j^k-H)(dd^cu_j^k)^n\\
&\le \lim_{k\to\infty}\int_\Omega -\chi(u_j^k-H)(dd^cu_j^k)^n\\
&\le C\int_\Omega -\chi(u-H)(dd^cu)^n \quad \text{(by (3.1) since } u\le u_j^k).
\end{aligned}$$

Therefore from (3.2), we get

$$\sup_j \int_\Omega -\chi\circ\psi_j(dd^c\psi_j)^n < \infty. \tag{3.3}$$

Since $(\psi_j)_j \subset \mathcal{E}_0(\Omega)$ is a decreasing sequence, and from (3.3), it follows that $\psi := \lim_j \psi_j \in \mathcal{E}_\chi(\Omega)$. On the other hand, for all $k\in\mathbb{N}$, we have $u_j \ge u_{j+k} \ge \psi_{j+k} + H$, hence $u_j \ge \psi + H$; it follows that $H \ge u \ge \psi + H$ and therefore $u \in \mathcal{E}_\chi(\Omega, H)$. ∎

4. Subextension and approximation

THEOREM 4.1. *Let $\Omega \Subset \hat{\Omega}$ be hyperconvex domains, $F \in \mathcal{E}(\Omega)\cap\mathcal{C}(\bar{\Omega})$ and $G \in \mathcal{M}(\hat{\Omega})$ with $G \le F$ on Ω. If $u \in \mathcal{E}_\chi(\Omega, F)$ and $\int_\Omega -\chi(u-F)(dd^cu)^n < \infty$, then there exists a function $\hat{u} \in \mathcal{E}_\chi(\hat{\Omega}, G)$ such that $\hat{u} \le u$ on Ω and $(dd^c\hat{u})^n \le \mathbf{1}_\Omega (dd^cu)^n$.*

Proof. Let $u \in \mathcal{E}_\chi(\Omega, F)$. Then $u \le F$ and since $F \in \mathcal{E}(\Omega)\cap\mathcal{C}(\bar{\Omega})$, it follows that there exists a decreasing sequence $(u_j)_j \subset \mathcal{E}_0(\Omega, F)\cap\mathcal{C}(\bar{\Omega})$ that converges pointwise to u on Ω (see [ACCP, Proposition 2.5]). Set $m = \inf_{z\in\overline{\Omega}} F(z)$ and let $\tilde{F}$ be the maximal function associated to F (see page 249). Then $m + \tilde{F} \le F \le \tilde{F}$, hence $F \in \mathcal{N}(\tilde{F})$. This implies that for all φ in $\mathcal{N}(F)$ we have $\varphi \in \mathcal{N}(\tilde{F})$. Hence, without loss of generality we can assume that $(dd^cF)^n = 0$. Since $\mathcal{E}_\chi(\Omega, F) \subset \mathcal{N}(\Omega, F)$, from Lemma 3.1 we have

$$\int_\Omega -\chi(u_j - F)(dd^cu_j)^n \le C\int_\Omega -\chi(u-F)(dd^cu)^n. \tag{4.1}$$

Since $u \in \mathcal{E}_\chi(\Omega, F)$, there exists $\varphi \in \mathcal{E}_\chi(\Omega)$ such that $\varphi + F \le u \le F$. Let $\hat{\varphi} \in \mathcal{E}_\chi(\hat{\Omega})$ be a subextension of φ (see [B2, Theorem 3.1]). Then $\hat{\varphi} + G \le G$ on $\hat{\Omega}$ and $\hat{\varphi} + G \le u$ on Ω, hence the set $\{\varphi \in \mathrm{PSH}(\hat{\Omega});\ \varphi \le G \text{ on } \hat{\Omega} \text{ and } \varphi \le u \text{ on } \Omega\}$ is not empty. Set

$$\hat{u} := \sup\{\varphi \in \mathcal{PSH}(\hat{\Omega});\ \varphi \le G \text{ on } \hat{\Omega} \text{ and } \varphi \le u \text{ on } \Omega\},$$

$$\hat{u}_j := \sup\{\varphi \in \mathcal{PSH}(\hat{\Omega});\ \varphi \le G \text{ on } \hat{\Omega} \text{ and } \varphi \le u_j \text{ on } \Omega\}.$$

Then $(\hat{u}_j)_j \searrow \hat{u}$, and from [H, Lemma 3.3] we have $\hat{u}_j \in \mathcal{E}_0(\hat{\Omega}, G)$, $(dd^c\hat{u}_j)^n \le (dd^cu_j)^n$ on Ω and $(dd^c\hat{u}_j)^n = 0$ on $[\hat{\Omega}\setminus\Omega]\cup\{\hat{u}_j < u_j\}$. Hence

$$\int_{\hat{\Omega}} -\chi(\hat{u}_j - G)(dd^c\hat{u}_j)^n \le \int_\Omega -\chi(\hat{u}_j - F)(dd^c\hat{u}_j)^n = \int_\Omega -\chi(u_j - F)(dd^c\hat{u}_j)^n$$
$$\le \int_\Omega -\chi(u_j - F)(dd^cu_j)^n.$$

Using (4.1), we get

$$\sup_j \int_{\hat{\Omega}} -\chi(\hat{u}_j - G)(dd^c\hat{u}_j)^n < \infty. \tag{4.2}$$

Therefore, Proposition 3.2 shows that $\hat{u} \in \mathcal{E}_\chi(\hat{\Omega}, G)$.

On the other hand, since $(\hat{u}_j)_j \searrow \hat{u}$ in $\mathcal{E}_\chi(\hat{\Omega}, G) \subset \mathcal{E}(\hat{\Omega})$, from [C3, Lemma 3.2] we have $(dd^c\hat{u})^n = \lim_j (dd^c\hat{u}_j)^n$. Also $(u_j)_j \searrow u$ in $\mathcal{E}_\chi(\Omega, F) \subset \mathcal{N}(\Omega, F)$, and from [ACCP, Corollary 3.4] it follows that $\lim_j \int_\Omega (dd^c u_j)^n = \int_\Omega (dd^c u)^n$, hence $\lim_j \mathbf{1}_\Omega (dd^c u_j)^n = \mathbf{1}_\Omega (dd^c u)^n$. Since $(dd^c\hat{u}_j)^n \le \mathbf{1}_\Omega (dd^c u_j)$, we conclude that $(dd^c\hat{u})^n \le \mathbf{1}_\Omega (dd^c u)^n$. ∎

Now, we are ready to prove our main theorem.

Proof of Theorem 1.1. Let $(\Omega_j)_j$ be a decreasing sequence of hyperconvex domains containing Ω, G be a negative function in $\mathcal{M}(\Omega_1) \cap \mathcal{C}(\overline{\Omega})$, $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a nondecreasing function such that $\chi(-t) < 0$ for all $t > 0$ and let $u \in \mathcal{E}_\chi(\Omega, G_{|\Omega})$ be such that $\int_\Omega -\chi(u - H)(dd^c u)^n < \infty$ where $H = G_{|\Omega}$. Let $H_j = G_{|\Omega_j}$. Since $u \in \mathcal{E}_\chi(\Omega, H)$, there exists $\psi \in \mathcal{E}_\chi(\Omega)$ such that $H \ge u \ge \psi + H$.

From [B2, Theorem 4.1], there exists an increasing sequence $(\psi_j)_j$ such that $\psi_j \in \mathcal{E}(\Omega_j)$ and $\lim_j \psi_j(z) = \psi(z)$ for all $z \in \Omega$. Moreover, from Theorem 4.1, the functions u_j defined by

$$u_j := \sup\{\varphi \in \mathcal{PSH}(\Omega_j);\ \varphi \le H_j \text{ on } \Omega_j \text{ and } \varphi \le u \text{ on } \Omega\}$$

satisfy $u_j \in \mathcal{E}_\chi(\Omega_j, H_j)$, $u_j \le u$ and $(dd^c u_j)^n \le 1_\Omega (dd^c u)^n$. Since $\psi_j + H_j \le H_j$ and $\psi_j + H_j \le \psi + H \le u$ on Ω, we have $\psi_j + H_j \le u_j \le H_j$ for all j.

Let $h = (\lim_j u_j)^*$. From the above, $h \in \mathcal{E}_\chi(\Omega, H)$, and since $(u_j)_j$ is increasing, $(dd^c h)^n \le (dd^c u)^n$ (by the main theorem in [C4]). Since $\int_\Omega -\chi(u - H)(dd^c u)^n < \infty$, from [B2] there exists $\varphi \in \mathcal{PSH}^-(\Omega)$ such that $\int_\Omega -\varphi(dd^c u)^n < \infty$.

We claim that for all $\varphi \in \mathcal{PSH}^-(\Omega)$ such that $\int_\Omega -\varphi(dd^c u)^n < \infty$, we have $\int_\Omega -\varphi(dd^c h)^n = \int_\Omega -\varphi(dd^c u)^n$. Indeed, since $(dd^c h)^n \le (dd^c u)^n$, it follows that $\int_\Omega -\varphi(dd^c h)^n \le \int_\Omega -\varphi(dd^c u)^n$. On the other hand, since $h, u \in \mathcal{E}_\chi(\Omega, H) \subset \mathcal{N}(\Omega, H)$, $h \le u$ and $\int_\Omega -\varphi(dd^c h)^n < \infty$, from [ACCP, Lemma 3.3] we have $\int_\Omega -\varphi(dd^c u)^n \le \int_\Omega -\varphi(dd^c h)^n$, hence $\int_\Omega -\varphi(dd^c h)^n = \int_\Omega -\varphi(dd^c u)^n$. Finally from [B2, end of proof of Theorem 4.1], it follows that $h = u$ and the theorem is proved. ∎

Remark 4.2. Examples of domains Ω satisfying the condition of Theorem 1.1 are polydiscs, bounded pseudoconvex domains with $\mathcal{C}^1$-boundary and with a Stein neighborhood basis, and strictly pseudoconvex domains with $\mathcal{C}^2$-boundary (see [CH, Theorem 3.4 and Example 3.6]).

Remark 4.3. Let $H \in \mathcal{E}(\Omega)$ and $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a bounded nondecreasing function such that $\chi(-t) < 0$ for all $t > 0$. If $u \in \mathcal{F}(\Omega, H)$ and

$\int_\Omega (dd^c u)^n < \infty$, then $u \in \mathcal{E}_\chi(\Omega, H)$ and $\int_\Omega -\chi(u-H)(dd^c u)^n < \infty$. Hence our theorem generalizes Theorem 1.1 of [H].

EXAMPLE 4.4. In the following, we give an example of a function $v \in \mathcal{E}_\chi(\Omega, G)$ satisfying $\int_\Omega -\chi(v-G)(dd^c v)^n < \infty$ but $\int_\Omega (dd^c v)^n = \infty$.

Let $\mathrm{P} = \mathrm{D} \times \mathrm{D}$ be the unit polydisc in $\mathbb{C}^2$ and let $G : \mathrm{P} \to \mathbb{R}$ be defined by $G(z_1, z_2) = |z_2|^2 - 1$. Then $G \in \mathcal{PSH}(\mathcal{P}) \cap \mathcal{C}^\infty(\bar{\mathrm{P}})$ and $(dd^c G)^2 = 0$. For each $j \in \mathbb{N}$, define the function $u_j : \mathrm{P} \to \mathbb{R}$ by

$$u_j(z_1, z_2) = \max\left(\frac{1}{2^j}\log|z_1|, 2^j \log|z_2|, -\frac{1}{2^{3j}}\right).$$

Let $\psi \in \mathcal{PSH}^-(\mathrm{P})$. Since $2^j \max(\log|z_1|, \log|z_2|) \le u_j$ and

$$\int_{\mathrm{P}} -\psi(dd^c 2^j \max(\log|z_1|, \log|z_2|)^2 = -(2^j)^2(2\pi)^2\psi(0,0) < \infty,$$

from [ACCP, Lemma 3.3] we have

$$\int_{\mathrm{P}} -\psi(dd^c u_j)^n \le \int_{\mathrm{P}} -\psi(dd^c \max(\log|z_1|, \log|z_2|)^2 = -(2^j)^2(2\pi)^2\psi(0,0).$$

Hence $\int_{\mathrm{P}} -u_j(dd^c u_j)^2 \le (2\pi)^2/2^j$, and it follows from [C1, Lemma 3.9] that there exists a subsequence j_k such that the function $u = \sum_{k=1}^\infty u_{j_k}$ satisfies

$$\int_{\mathrm{P}} -u(dd^c u)^n < \infty.$$

Also, $\int_{\mathrm{P}} (dd^c u_{j_k})^2 = (2\pi)^2$; then $\int_{\mathrm{P}} (dd^c u)^2 \ge \int_{\mathrm{P}} (dd^c \sum_1^N u_{j_k})^2 \ge N(2\pi)^2$, hence $\int_\Omega (dd^c u)^n = \infty$.

Let $N \in \mathbb{N}$. We now prove that there exists a constant C independent of N such that $\int_{\mathrm{P}} -(\sum_{k=1}^N u_{j_k})(dd^c(\sum_{k=1}^N u_{j_k} + G))^2 \le C$. We have

$$I_1(N) = \int_{\mathrm{P}} -\Big(\sum_{k=1}^N u_{j_k}\Big)\Big(dd^c \sum_{k=1}^N u_{j_k}\Big)^2 < C_1,$$

$$\begin{aligned} I_2(N) &= 2\int_{\mathrm{P}} -\Big(\sum_{k=1}^N u_{j_k}\Big) dd^c\Big(\sum_{k=1}^N u_{j_k}\Big) \wedge dd^c G \\ &= 4i\int_{\mathrm{P}} -\Big(\sum_{k=1}^N u_{j_k}\Big) dd^c\Big(\sum_{k=1}^N u_{j_k}\Big) \wedge dz_2 \wedge d\bar{z}_2 \\ &= 32\int_{\mathrm{P}} -\Big(\sum_{k=1}^N u_{j_k}\Big) \frac{\partial^2}{\partial z_1 \partial \bar{z}_1}\Big(\sum_{k=1}^N u_{j_k}\Big)\, dV(z_1, z_2) \\ &\le 32\Big(\sum_{k=1}^N \frac{1}{2^{3j_k}}\Big) \int_{\mathrm{P}} \sum_{k=1}^N \frac{\partial^2 u_{j_k}}{\partial z_1 \partial \bar{z}_1} dV(z_1, z_2). \end{aligned}$$

Let $\varepsilon \in]0,1[$ and $\mathrm{D}_{\varepsilon,j_k} = \{t \in \mathrm{D};\ |t| \le r_{\varepsilon,j_k} = \min(1-\varepsilon,(1-\varepsilon)^{1/2^{2j_k}})\}$. Choose θ_1 and θ_2 in $\mathcal{C}_0^\infty(\mathrm{D})$ such that $0 \le \theta_1, \theta_2 \le 1$, $\theta_1 = 1$ on $\mathrm{D}(0, 1-\varepsilon)$, $\mathrm{supp}(\theta_2) \subset \mathrm{D}_{\varepsilon,j_k}$ and $\theta_2 = 1$ on $\mathrm{D}_{\varepsilon/2,j_k}$. For $z_2 \in \mathrm{D}_{\varepsilon,j_k}$ fixed, we have

$$\int_{\mathrm{D}} \theta_1(z_1) \frac{\partial^2 u_{j_k}}{\partial z \partial \bar{z}_1}\, dV(z_1) = \frac{8\pi}{2^{j_k}}.$$

It follows that

$$\int_{\mathrm{P}} \theta_1(z_1)\theta_2(z_2) \frac{\partial^2 u_{j_k}}{\partial z \partial \bar{z}_1}\, dV(z_1, z_2) \le C_2 \frac{1}{2^{j_k}},$$

for some constant C_2 independent of ε and j_k. By letting $\varepsilon \to 0^+$, we get $I_2(N) < 32C_2$. Since $\int_{\mathrm{P}} -(\sum_{j=1}^N u_{j_k})(dd^c(\sum_{j=1}^N u_{j_k} + G))^2 = I_1(N) + I_2(N)$, it follows that $\int_{\mathrm{P}} -u(dd^c(u+G))^n < \infty$. Moreover, $u \in \mathcal{E}_1(\mathrm{P}) = \mathcal{E}_\chi(\mathrm{P})$ where $\chi(-t) = -t$ for all $t > 0$. Consequently, $v = u + G \in \mathcal{E}_\chi(\mathrm{P}, G)$ and $\int_{\mathrm{P}} -\chi(v - G)(dd^c v)^n < \infty$. Finally, $\int_{\mathrm{P}} (dd^c u)^2 = \infty$ yields $\int_{\mathrm{P}} (dd^c v)^2 = \infty$.

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