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Global uniqueness results for fractional partial hyperbolic differential equations with state-dependent delay

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Dedicated to Professor John Graef for his seventieth birthday

Abstract. We investigate the existence and uniqueness of solutions of hyperbolic fractional order differential equations with state-dependent delay by using a nonlinear alternative of Leray–Schauder type due to Frigon and Granas for contraction maps on Fréchet spaces.

1. Introduction. In this paper we provide sufficient conditions for the global existence and uniqueness of solutions of some class of fractional order partial hyperbolic differential equations. Firstly, we present a global existence and uniqueness of solutions to the fractional order initial value problem (IVP for short)

$$({}^cD_0^r u)(t,x) = f(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}), \qquad (t,x)\in J, \tag{1.1}$$

$$u(t,x)=\phi(t,x), \qquad (t,x)\in \tilde{J}, \tag{1.2}$$

$$u(t,0)=\varphi(t), \qquad u(0,x)=\psi(x), \qquad (t,x)\in J, \tag{1.3}$$

where $\varphi(0)=\psi(0)$, $J:=[0,\infty)\times[0,\infty)$, $\tilde{J}:=[-\alpha,\infty)\times[-\beta,\infty)\setminus[0,\infty)\times[0,\infty)$, $\alpha,\beta>0$, ${}^cD_0^r$ is the standard Caputo fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $f: J\times C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n)\to\mathbb{R}^n$, $\rho_1: J\times C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n)\to[-\alpha,\infty)$, $\rho_2: J\times C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n)\to[-\beta,\infty)$ are given functions, $\phi:\tilde{J}\to\mathbb{R}^n$ is a given continuous function with

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$\phi(t,0) = \varphi(t)$, $\phi(0,x) = \psi(x)$ for each $(t,x) \in J$, $\varphi, \psi : [0,\infty) \to \mathbb{R}^n$ are given absolutely continuous functions and $C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n) := C$ is the Banach space of continuous functions on $[-\alpha,0]\times[-\beta,0]$ with norm

$$\|u\|_C = \sup\{\|u(s,\tau)\| : (s,\tau) \in [-\alpha,0]\times[-\beta,0]\}.$$

We denote by $u_{(t,x)}$ the element of $C([-\alpha,\infty)\times[-\beta,\infty),\mathbb{R}^n)$ defined by

$$u_{(t,x)}(s,\tau) = u(t+s, x+\tau), \qquad (s,\tau) \in [-\alpha,0]\times[-\beta,0];$$

here $u_{(t,x)}(\cdot,\cdot)$ represents the history of the state from time $t-\alpha$ up to the present time t and from time $x-\beta$ up to the present time x.

Next we consider the following system of partial neutral hyperbolic differential equations of fractional order:

$$(1.4)\qquad {}^cD_0^r[u(x,y) - g(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))})] = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}), \qquad (x,y) \in J,$$

$$(1.5)\qquad u(x,y) = \phi(x,y), \qquad (x,y) \in \tilde{J},$$

$$(1.6)\qquad u(x,0) = \varphi(x), \qquad u(0,y) = \psi(y), \qquad (x,y) \in J,$$

where f, ρ_1, ρ_2, ϕ, φ, ψ are as in problem (1.1)–(1.3) and $g : J\times C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n) \to \mathbb{R}^n$ is a given continuous function.

The third result deals with the existence of solutions to fractional order partial differential equations

$$(1.7)\qquad ({}^cD_0^r u)(x,y) = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}), \qquad (x,y) \in J,$$

$$(1.8)\qquad u(x,y) = \phi(x,y), \qquad (x,y) \in \tilde{J}',$$

$$(1.9)\qquad u(x,0) = \varphi(x), \qquad u(0,y) = \psi(y), \qquad (x,y) \in J,$$

where φ and ψ are as in problem (1.1)–(1.3), $\tilde{J}' := (-\infty,\infty)\times(-\infty,\infty) \setminus [0,\infty)\times[0,\infty)$, $f : J\times\mathcal{B} \to \mathbb{R}$, $\rho_1 : J\times\mathcal{B} \to \mathbb{R}$, $\rho_2 : J\times\mathcal{B} \to \mathbb{R}$ are given functions, $\phi : \tilde{J}' \to \mathbb{R}^n$ is a given continuous function with $\phi(t,0) = \varphi(t)$, $\phi(0,x) = \psi(x)$ for each $(t,x) \in J$, and $\mathcal{B}$, called the phase space, will be specified in Section 4.

Finally we consider the following initial value problem for partial neutral functional differential equations:

$$(1.10)\qquad {}^cD_0^r[u(x,y) - g(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))})] = f(x,y,u_{(\rho_1(x,y,u_{(x,y)}),\rho_2(x,y,u_{(x,y)}))}), \qquad (x,y) \in J,$$

$$(1.11)\qquad u(x,y) = \phi(x,y), \qquad (x,y) \in \tilde{J}',$$

$$(1.12)\qquad u(x,0) = \varphi(x), \qquad u(0,y) = \psi(y), \qquad (x,y) \in J,$$

where f, ρ_1, ρ_2, ϕ, φ, ψ are as in problem (1.7)–(1.9) and $g : J\times\mathcal{B} \to \mathbb{R}^n$ is a given continuous function.

The idea of fractional calculus and fractional order differential equations has been a subject of interest not only among mathematicians, but also among physicists and engineers. Indeed, we can find numerous applications in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [BDST, Hi, P1, T]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas et al. [ABN], Lakshmikantham et al. [LLV], Miller and Ross [MR], Samko et al. [SKM], and Podlubny [P1], the papers of Abbas and Benchohra [AB1, AB2], Belarbi et al. [BBO], Benchohra et al. [BHNO], Kilbas and Marzan [KM], Vityuk and Golushkov [VG], Vityuk and Mykhailenko [VM], and the references therein.

Moreover, complicated situations in which the delay depends on the unknown functions have been studied in recent years (see for instance [ABB, RW, W, WB] and the references therein). Over the past several years it has become apparent that equations with state-dependent delay arise also in several areas such as classical electrodynamics [DN1], population models [B], models of commodity price fluctuations [BM, Ma], and models of blood cell production [MM]. Existence results, among other things, were derived recently for various classes of functional differential equations when the delay depends on the solution. We refer the reader to the papers by Ait Dads and Ezzinbi [AE], Győri and Hartung [GH], Hartung [H1, H2, H3], and Hernández et al. [HPL]. In [DN], the authors considered a class of semilinear functional fractional order differential equations with state-dependent delay. By means of the Banach contraction principle and the nonlinear alternative of Leray–Schauder type, Abbas et al. [ABV] gave some existence as well as uniqueness results for each of our problems on a bounded domain. Vityuk et al. [VG, VM] considered two classes of Darboux problems for partial differential equations involving the Riemann–Liouville derivative. Człapiński [C1, C2] considered the Darboux problem for some integer order fractional differential equations with infinite delay.

Motivated by the previous papers, we consider the existence of solutions for problems (1.1)–(1.3), (1.4)–(1.6), (1.7)–(1.9), and (1.10)–(1.12). Our analysis is based upon the nonlinear alternative of Leray–Schauder type due to Frigon–Granas type for contraction maps on Fréchet spaces [FG] and a fractional version of Gronwall's inequality. We look for sufficient conditions ensuring existence of solutions for each of our problems. The present results extend those considered with finite and/or infinite constant delay on bounded domains in [AB1, AB2, ABN, ABV], and those for constant delay and integer order derivative in [C1, C2].

2. Preliminaries. In this section, we introduce the notations, definitions, and preliminary facts which are used throughout this paper. Let $p \in \mathbb{N}$ and $J_0 = [0,p] \times [0,p]$. By $C(J_0,\mathbb{R})$ we denote the Banach space of all continuous functions from J_0 into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(t,x)\in J_0} \|w(t,x)\|,$$

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$.

As usual, $AC(J_0,\mathbb{R})$ denotes the space of absolutely continuous functions from J_0 into $\mathbb{R}^n$, and $L^1(J_0,\mathbb{R})$ is the space of Lebesgue-integrable functions $w : J_0 \to \mathbb{R}^n$ with the norm

$$\|w\|_{L^1} = \int_0^p\int_0^p \|w(t,x)\|\, dt\, dx.$$

Definition 2.1 ([VG]). Let $r = (r_1,r_2) \in (0,\infty)\times(0,\infty)$, $\theta = (0,0)$ and $u \in L^1(J_0,\mathbb{R}^n)$. The *left-sided mixed Riemann–Liouville integral* of order r of u is defined by

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} u(s,\tau)\, d\tau\, ds.$$

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x),$$
$$(I_\theta^\sigma u)(t,x) = \int_0^t\int_0^x u(s,\tau)\, d\tau\, ds \quad \text{for almost all } (t,x) \in J_0,$$

where $\sigma = (1,1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)\times(0,\infty)$ when $u \in L^1(J_0,\mathbb{R}^n)$. Note also that if $u \in C(J_0,\mathbb{R}^n)$, then $I_\theta^r u \in C(J_0,\mathbb{R}^n)$, and

$$(I_\theta^r u)(t,0) = (I_\theta^r u)(0,x) = 0, \quad (t,x) \in J_0.$$

Example 2.2. Let $\lambda,\omega \in (-1,\infty)$ and $r = (r_1,r_2) \in (0,\infty)\times(0,\infty)$. Then

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2} \quad \text{for almost all } (t,x) \in J_0.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in (0,1]\times(0,1]$. Denote by $D_{tx}^2 := \partial^2/\partial t\partial x$ the mixed second order partial derivative.

Definition 2.3 ([VG]). Let $r \in (0,1]\times(0,1]$ and $u \in L^1(J_0,\mathbb{R}^n)$. The *mixed fractional Riemann–Liouville derivative* of order r of u is defined by

$$D_\theta^r u(t,x) = (D_{tx}^2 I_\theta^{1-r} u)(t,x),$$

and the *Caputo fractional-order derivative* of order r of u is defined by

$$({}^cD_0^r u)(t,x) = \left(I_\theta^{1-r} \frac{\partial^2}{\partial t\partial x} u\right)(t,x).$$

The case $\sigma = (1,1)$ is included and we have

$$(D^\sigma_\theta u)(t,x) = ({}^cD^\sigma_\theta u)(t,x) = (D^2_{tx} u)(t,x) \quad \text{for almost all } (t,x)\in J_0.$$

EXAMPLE 2.4. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$. Then

$$D^r_\theta t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)}\, t^{\lambda-r_1} x^{\omega-r_2} \quad \text{for almost all } (t,x)\in J_0.$$

We will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

LEMMA 2.5 ([He]). *Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(\cdot,\cdot)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that*

$$\upsilon(t,x) \le \omega(t,x) + c\int_0^t\int_0^x \frac{\upsilon(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}}\, d\tau\, ds,$$

then there exists a constant $\delta = \delta(r_1,r_2)$ such that

$$\upsilon(t,x) \le \omega(t,x) + \delta c\int_0^t\int_0^x \frac{\omega(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}}\, d\tau\, ds$$

for every $(t,x) \in J$.

3. Some properties in Fréchet spaces. Let X be a Fréchet space with a family $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ of seminorms. We assume that

$$\|u\|_1 \le \|u\|_2 \le \cdots \quad \text{for every } u \in X.$$

Let $Y \subset X$. We say that Y is *bounded* if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \le \overline{M}_n \quad \text{for all } v \in Y.$$

To X we associate a sequence $\{(X^n, \|\cdot\|_n)\}$ of Banach spaces as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by: $u \sim_n v$ if and only if $\|u-v\|_n = 0$ for $u,v \in X$. We denote by $X^n = (X/\sim_n, \|\cdot\|_n)$ the quotient space. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $u \in X$, we denote by $[u]_n$ the equivalence class of u in X^n and we define $Y^n = \{[u]_n : u \in Y\}$. We denote by $\overline{Y^n}$, $\mathrm{int}_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, interior and boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more information about this subject we refer the reader to [FG].

DEFINITION 3.1. Let X be a Fréchet space. A function $N : X \to X$ is said to be a *contraction* if for each $n \in \mathbb{N}$ there exists $k_n \in [0,1)$ such that

$$\|N(u) - N(v)\|_n \le k_n\|u - v\|_n \quad \text{for all } u,v \in X.$$

THEOREM 3.2 (Nonlinear alternative of Leray–Schauder type due to Frigon–Granas [FG]). *Let X be a Fréchet space and $Y \subset X$ a closed subset in X. Let $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements holds:*

(a) *the operator N has a unique fixed point;*
(b) *there exist $\lambda \in [0,1)$, $n \in \mathbb{N}$ and $u \in \partial_n Y^n$ such that*
$$\|u - \lambda N(u)\|_n = 0.$$

4. Global results for the finite delay case. In this section we present a global existence and uniqueness result for problem (1.1)–(1.3).

For each $p \in \mathbb{N}$ we set
$$C_p = C([-\alpha,p] \times [-\beta,p], \mathbb{R}^n)$$
and we define seminorms in $C_0 := C([-\alpha,\infty) \times [-\beta,\infty), \mathbb{R}^n)$ by
$$\|u\|_p = \sup\{\|u(t,x)\| : -\alpha \le t \le p,\ -\beta \le x \le p\}.$$
Then C_0 is a Fréchet space with the family $\{\|\cdot\|_p\}$ of seminorms.

Let us start by defining what we mean by a solution of problem (1.1)–(1.3).

DEFINITION 4.1. A function $u \in C_0$ is said to be a *solution of* (1.1)–(1.3) if u satisfies equations (1.1) and (1.3) on J and the condition (1.2) on $\tilde{J}$.

For the existence of solutions of (1.1)–(1.3), we need the following lemma:

LEMMA 4.2. *A function $u \in C_0$ is a solution of problem* (1.1)–(1.3) *if and only if u satisfies the equation*
$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \times f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\,d\tau\,ds$$
for all $(t,x) \in J$ and the condition (1.2) *on $\tilde{J}$, where*
$$z(t,x) = \varphi(t) + \psi(x) - \varphi(0).$$

Set
$$\begin{aligned}\mathcal{R} &:= \mathcal{R}_{(\rho_1^-,\rho_2^-)} \\ &= \{(\rho_1(s,t,u),\rho_2(s,t,u)) : (s,t,u) \in J \times C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n), \\ &\qquad \rho_i(s,t,u) \le 0,\ i=1,2\}.\end{aligned}$$
We always assume that
$$\rho_1 : J \times C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n) \to [-\alpha,\infty),$$
$$\rho_2 : J \times C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n) \to [-\beta,\infty)$$
are continuous and $(s,t) \mapsto u_{(s,t)}$ is continuous from $\mathcal{R}$ into C.

Now, we present sufficient conditions for the existence and uniqueness of a solution of problem (1.1)–(1.3).

THEOREM 4.3. *Assume that:*

(H1) $f : J \times C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n) \to \mathbb{R}^n$ *is continuous.*
(H2) *For each* $p \in \mathbb{N}$, *there exists* $\ell_p \in C(J_0, \mathbb{R}^n)$ *such that for each* $(t, x) \in J_0$,

$$\|f(t, x, u) - f(t, x, v)\| \le \ell_p(t, x)\|u - v\|_C \quad \textit{for all } u, v \in C.$$

If

$$\frac{\ell_p^* p^{r_1+r_2}}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} < 1, \tag{4.1}$$

where $\ell_p^* = \sup_{(t,x)\in J_0} \ell_p(t, x)$, *then there exists a unique solution of* (1.1)–(1.3) *on* $[-\alpha, \infty) \times [-\beta, \infty)$.

Proof. We transform (1.1)–(1.3) into a fixed point problem. Consider the operator $N : C_0 \to C_0$ defined by

$$(Nu)(t, x) = \begin{cases} \phi(t, x), & (t, x) \in \tilde{J}, \\ z(t, x) + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t\int_0^x (t - s)^{r_1-1}(x - \tau)^{r_2-1} & \\ \qquad \times f(s, \tau, u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\, d\tau\, ds, & (t, x) \in J. \end{cases}$$

Let u be a possible solution of the equation $u = \lambda N(u)$ for some $0 < \lambda < 1$. Then for each $(t, x) \in J_0$,

$$u(t, x) = \lambda z(t, x) + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t - s)^{r_1-1}(x - \tau)^{r_2-1} \times f(s, \tau, u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\, d\tau\, ds.$$

This implies by (H2) that

$$\begin{aligned} \|u(t, x)\| \le \lambda\|z(t, x)\| &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t - s)^{r_1-1}(x - \tau)^{r_2-1} \\ &\times \|f(s, \tau, u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}) - f(s, \tau, 0)\|\, d\tau\, ds \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t - s)^{r_1-1}(x - \tau)^{r_2-1}\|f(s, \tau, 0)\|\, d\tau\, ds \end{aligned}$$

$$\leq \|z(t,x)\| + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_p(s,\tau)\|u_{(s,\tau)}\|_C \, d\tau \, ds,$$

where $f^* = \sup_{(s,\tau)\in J_0} \|f(s,\tau,0)\|$. We define

$$y(t,x) = \sup\{\|u(s,\tau)\| : -\alpha \leq s \leq t, \ -\beta \leq \tau \leq x\}, \qquad 0 \leq t \leq p, \ 0 \leq x \leq p.$$

Let $(t^*,x^*) \in [-\alpha,t]\times[-\beta,x]$ be such that $y(t,x) = \|u(t^*,x^*)\|$. If $(t^*,x^*) \in J_0$, then by the above inequality, for $(t,x)\in J_0$,

$$\|u(t,x)\| \leq \|z(t,x)\| + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_p(s,\tau)y(s,\tau) \, d\tau \, ds. \tag{4.2}$$

If $(t^*,x^*) \in \tilde{J}$, then $y(t,x) = \|\phi\|_C$ and the previous inequality holds. By (4.2) we obtain

$$\begin{aligned}\|y(t,x)\| &\leq \|z(t,x)\| + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_p(s,\tau)y(s,\tau) \, d\tau \, ds \\ &\leq \|z(t,x)\| + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad + \frac{\ell_p^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}y(s,\tau) \, d\tau \, ds,\end{aligned}$$

and Lemma 2.5 implies that there exists a positive constant $\delta = \delta(r_1,r_2)$ such that

$$\begin{aligned}y(t,x) \leq &\left[\|z\|_p + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right] \times \left[1 + \frac{\delta \ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right] \\ &:= M_p.\end{aligned}$$

Then from (4.2) we have

$$\|u\|_p \leq \|z\|_p + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{M_p \ell_p^*}{\Gamma(r_1+1)\Gamma(r_2+1)} := M_p^*.$$

Since $\|u_{(t,x)}\|_C \leq y(t,x)$ for every $(t,x)\in J_0$, we have

$$\|u\|_p \leq \max(\|\phi\|_C, M_p^*) := R_p.$$

Set

$$U = \{u \in C_0 : \|u\|_p \leq R_p + 1 \text{ for all } p \in \mathbb{N}\}.$$

We shall show that $N : U \to C_p$ is a contraction map. Indeed, consider $v, w \in U$. Then for each $(t,x) \in J_0$, we have

$$
\begin{aligned}
&\|(Nv)(t,x) - (Nw)(t,x)\| \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\!\!\int_0^x |(t-s)^{r_1-1}|\,|(x-\tau)^{r_2-1}| \\
&\quad \times \|f(s,\tau,v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}) - f(s,\tau,w_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\|\,d\tau\,ds \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\!\!\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_p \|v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))} \\
&\qquad\qquad - w_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\|_C\,d\tau\,ds \\
&\le \frac{\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\|v-w\|_p.
\end{aligned}
$$

Thus

$$\|N(v) - N(w)\|_p \le \frac{\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\|v-w\|_p.$$

Hence by (4.2), $N : U \to C_p$ is a contraction. From the choice of U, there is no $u \in \partial_n U^n$ such that $u = \lambda N(u)$ with $\lambda \in (0,1)$. From Theorem 3.2, we deduce that N has a unique fixed point u in U, which is a solution to problem (1.1)–(1.3). ∎

Now, we present a global existence and uniqueness result for (1.4)–(1.6).

Definition 4.4. A function $u \in C_0$ is said to be a *solution of* (1.4)–(1.6) if u satisfies equations (1.4) and (1.6) on J and the condition (1.5) on $\tilde{J}$.

Let $f \in L^1(J,\mathbb{R}^n)$ and $g \in AC(J,\mathbb{R}^n)$ and consider the linear problem

$$ {}^cD_0^r[u(t,x) - g(t,x)] = f(t,x), \qquad (t,x) \in J, \tag{4.3}$$

$$ u(t,0) = \varphi(t), \quad u(0,x) = \psi(x), \qquad (t,x) \in J, \tag{4.4}$$

with $\varphi(0) = \psi(0)$.

For the existence of solutions of (1.4)–(1.6), we need the following lemma. Its proof is left to the reader.

Lemma 4.5. *A function $u \in AC(J,\mathbb{R}^n)$ is a solution of problem* (4.3)–(4.4) *if and only if*

$$
\begin{aligned}
u(t,x) &= z(t,x) + g(t,x) - g(t,0) - g(0,x) + g(0,0) \\
&\quad + I_0^r(f)(t,x), \qquad (t,x) \in J,
\end{aligned} \tag{4.5}
$$

where $z(t,x) = \varphi(t) + \psi(x) - \varphi(0)$.

As a consequence of Lemma 4.5 we have the following auxiliary result.

COROLLARY 4.6. *A function $u \in C_0$ is a solution of* (1.4)–(1.6) *if and only if u satisfies the equation*

$$u(t,x) = z(t,x) + g(t,x,u_{(t,x)}) - g(t,0,u_{(t,0)}) - g(0,x,u_{(0,x)}) + g(0,0,u_{(0,0)}) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \times f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\,d\tau\,ds$$

for all $(t,x) \in J$ and the condition (1.5) *on $\tilde{J}$.*

THEOREM 4.7. *Assume* (H1)–(H2) *and the following hypothesis holds:*

(H3) *For each $p \in \mathbb{N}$, there exists a constant c_p with $0 < c_p < 1/4$ such that for each $(t,x) \in J_0$,*

$$\|g(t,x,u) - g(t,x,v)\| \le c_p\|u-v\| \quad \text{for all } u,v \in C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n).$$

If

$$4c_p + \frac{\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{4.6}$$

then there exists a unique solution of (1.4)–(1.6) *on $[-\alpha,\infty)\times[-\beta,\infty)$.*

Proof. We transform (1.4)–(1.6) into a fixed point problem. Consider the operator $N_1 : C_0 \to C_0$ defined by

$$(N_1u)(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) + g(t,x,u_{(t,x)}) \\ \quad - g(t,0,u_{(t,0)}) - g(0,x,u_{(0,x)}) + g(0,0,u) \\ \quad + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ \quad \times f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\,d\tau\,ds, & (t,x) \in J, \end{cases}$$

In order to apply the nonlinear alternative, we shall obtain a priori estimates for solutions of the integral equation

$$u(t,x) = \lambda[z(t,x) + g(t,x,u_{(t,x)}) - g(t,0,u_{(t,0)}) - g(0,x,u_{(0,x)}) + g(0,0,u)] + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \times f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\,d\tau\,ds,$$

for some $\lambda \in (0,1)$. Then using (H1)–(H2) and (4.2) we get

$$\begin{aligned}\|u(t,x)\| \le \|z(t,x)\| &+ \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \|g(t,x,u_{(t,x)})\| \\ &+ \|g(t,0,u_{(t,0)})\| + \|g(0,x,u_{(0,x)})\| + \|g(0,0,u_{(0,0)})\| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \ell_p(s,\tau) y(s,\tau)\,d\tau\,ds.\end{aligned}$$

Hence

$$\begin{aligned}\|u(t,x)\| \le \|z(t,x)\| &+ \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + 4c_p y(t,x) \\ &+ \|g(t,x,0)\| + \|g(t,0,0)\| + \|g(0,x,0)\| + \|g(0,0,0)\| \\ &+ \frac{\ell_p^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} y(s,\tau)\,d\tau\,ds,\end{aligned} \tag{4.7}$$

Inserting (4.7) in the definition of $y(t,x)$, we get

$$\begin{aligned}y(t,x) = {} & \frac{1}{1-4c_p}\left[\|z(t,x)\| + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + 4g^*\right] \\ &+ \frac{\ell_p'^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} y(s,\tau)\,d\tau\,ds,\end{aligned}$$

where $\ell_p'^* = \ell_p^*/(1-4c_p)$ and $g^* = \sup_{(s,\tau)\in J_0} \|g(s,\tau,0)\|$.

By Lemma 2.5, there exists a constant $\delta = \delta(r_1,r_2)$ such that

$$\begin{aligned}\|y\|_p \le {} & \frac{1}{1-4c_p}\left[\|z\|_p + \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + 4g^*\right] \\ &\times \left[1 + \frac{\delta \ell_p'^*}{\Gamma(r_1+1)\Gamma(r_2+1)}\right] =: D_p.\end{aligned} \tag{4.8}$$

Then from (4.7) and (4.8), we get

$$\begin{aligned}\|u\|_p \le \|z\|_p &+ \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + 4g^* \\ &+ 4c_p D_p + \frac{D_p \ell_p^*}{\Gamma(r_1+1)\Gamma(r_2+1)} =: D_p^*.\end{aligned}$$

Since $\|u_{(t,x)}\|_C \le y(t,x)$ for every $(t,x) \in J_0$, we have

$$\|u\|_p \le \max(\|\phi\|_C, D_p^*) =: R_p^*.$$

Set

$$U_1 = \{u \in C_0 : \|u\|_p \le R_p^* + 1 \text{ for all } p \in \mathbb{N}\}.$$

Clearly, U_1 is a closed subset of C_0. We shall show that $N_1 : U_1 \to C_p$ is a contraction map. Indeed, for $v, w \in U_1$ and $(t,x) \in J_0$, we have

$$\begin{aligned}
&\|(N_1 v)(t,x) - (N_1 w)(t,x)\| \\
&\le \|g(t,x,v_{(t,x)}) - g(t,x,w_{(t,x)})\| + \|g(t,0,v_{(t,0)}) - g(t,0,w_{(t,0)})\| \\
&\quad + \|g(0,x,v_{(0,x)}) - g(0,x,w_{(0,x)})\| + \|g(0,0,v) - g(0,0,w)\| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x |t-s|^{r_1-1} |x-\tau|^{r_2-1} \\
&\quad \times \|f(s,\tau,v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}) \\
&\qquad\qquad - f(s,\tau,w_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\| \, d\tau \, ds \\
&\le c_p(\|v_{(t,x)} - w_{(t,x)}\|_p + \|v_{(t,0)} - w_{(t,0)}\|_p + \|v_{(0,x)} - w_{(0,x)}\|_p + \|v-w\|_p) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x [(t-s)^{r_1-1}(x-\tau)^{r_2-1} \ell_p \|v_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))} \\
&\qquad\qquad - w_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))}\|_C \, d\tau \, ds \\
&\le 4c_p \|v-w\|_p \\
&\quad + \frac{\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \|v-w\|_p \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \, d\tau \, ds \\
&\le \left(4c_p + \frac{\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right) \|v-w\|_p.
\end{aligned}$$

Thus

$$\|N_1(v) - N_1(w)\|_p \le \left(4c_p + \frac{\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right) \|v-w\|_p.$$

Hence by (4.6), $N_1 : U_1 \to C_p$ is a contraction. From the choice of U_1, there is no $u \in \partial_n U_1^n$ such that $u = \lambda N_1(u)$ with $\lambda \in (0,1)$. By Theorem 3.2, we deduce that N_1 has a unique fixed point u in U_1, which is a solution to problem (1.4)–(1.6). ■

5. Global results for the infinite delay case

5.1. The phase space $\mathcal{B}$. The notion of phase space plays an important role in both the qualitative and quantitative theory of functional differential equations. A usual choice is a seminormed space satisfying suitable axioms, introduced by Hale and Kato (see [HK]). For further applications see for instance the books [HV, HMN] and their references.

For any $(t,x) \in J$ we denote $E_{(t,x)} := [0,t] \times \{0\} \cup \{0\} \times [0,x]$; in case $t = a,\ x = b$ we write simply E. $(\mathcal{B}, \|(\cdot,\cdot)\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty,0] \times (-\infty,0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms, adapted from those introduced by Hale and Kato for ordinary differential functional equations:

(A_1) If $y : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n$ is continuous on J and $y_{(t,x)} \in \mathcal{B}$ for all $(t, x) \in E$, then there are constants $H, K, M > 0$ such that for any $(t, x) \in J$ the following conditions hold:

(i) $y_{(t,x)}$ is in $\mathcal{B}$,
(ii) $\|y(t, x)\| \le H\|y_{(t,x)}\|_{\mathcal{B}}$,
(iii) $\|y_{(t,x)}\|_{\mathcal{B}} \le K \sup_{(s,\tau)\in[0,t]\times[0,x]} \|y(s, \tau)\| + M \sup_{(s,\tau)\in E_{(t,x)}} \|y_{(s,\tau)}\|_{\mathcal{B}}$.

(A_2) For the function $y(\cdot, \cdot)$ in (A_1), $y_{(t,x)}$ is a $\mathcal{B}$-valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Now, we present some examples of phase spaces [C1, C2].

Example 5.1. Let $\mathcal{B}$ be the set of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0]$, $\alpha, \beta \ge 0$, with the seminorm

$$\|\phi\|_{\mathcal{B}} = \sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s, \tau)\|.$$

Then $H = K = M = 1$. The quotient space $\widehat{\mathcal{B}} = \mathcal{B}/\|\cdot\|_{\mathcal{B}}$ is isometric to the space $C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n)$ of all continuous functions from $[-\alpha, 0] \times [-\beta, 0]$ into $\mathbb{R}^n$ with the supremum norm; this means that partial differential functional equations with finite delay are included in our axiomatic model.

Example 5.2. Let $\gamma \in \mathbb{R}$ and let C_γ be the set of all continuous functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ for which the limit $\lim_{\|(s,\tau)\|\to\infty} e^{\gamma(s+\tau)}\phi(s, \tau)$ exists, with the norm

$$\|\phi\|_{C_\gamma} = \sup_{(s,\tau)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(s+\tau)}\|\phi(s, \tau)\|.$$

Then $H = 1$ and $K = M = \max\{e^{-\gamma(a+b)}, 1\}$.

Example 5.3. Let $\alpha, \beta, \gamma \ge 0$ and let

$$\|\phi\|_{CL_\gamma} = \sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s, \tau)\| + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\gamma(s+\tau)}\|\phi(s, \tau)\|\, d\tau\, ds.$$

be the seminorm for the space CL_γ of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0]$ measurable on $(-\infty, -\alpha] \times (-\infty, 0] \cup (-\infty, 0] \times (-\infty, -\beta]$, and such that $\|\phi\|_{CL_\gamma} < \infty$. Then

$$H = 1, \qquad K = \int_{-\alpha}^{0}\int_{-\beta}^{0} e^{\gamma(s+\tau)}\, d\tau\, ds, \qquad M = 2.$$

5.2. Main results. In this section we present a global existence and uniqueness result for problem (1.7)–(1.9). Set

$$\Omega := \{u : \mathbb{R}^2 \to \mathbb{R}^n : u_{(t,x)} \in \mathcal{B} \text{ for } (t,x) \in E \text{ and } u|_J \in C(J, \mathbb{R}^n)\}.$$

DEFINITION 5.4. A function $u \in \Omega$ is said to be a *solution of* (1.7)–(1.9) if u satisfies equations (1.7) and (1.9) on J and the condition (1.8) on $\tilde{J}'$.

For each $p \in \mathbb{N}$, set

$$\begin{aligned} C'_p = \{u : (-\infty, p] \times (-\infty, p] \to \mathbb{R}^n : u_{(t,x)} \in \mathcal{B},\ u_{(t,x)} = 0 \text{ for } (t,x) \in E \\ \text{and } u|_{J_0} \in C(J_0, \mathbb{R}^n)\}, \end{aligned}$$

$$C'_0 = \{u \in \Omega : u_{(t,x)} = 0 \text{ for } (t,x) \in E\}.$$

On C'_0 we define the seminorms

$$\begin{aligned} \|u\|_p &= \sup_{(t,x)\in E} \|u_{(t,x)}\| + \sup_{(t,x)\in J_0} \|u(t,x)\| \\ &= \sup_{(t,x)\in J_0} \|u(t,x)\|, \qquad u \in C'_p. \end{aligned}$$

Then C'_0 is a Fréchet space with the family of seminorms $\{\|u\|_p\}$.

Set

$$\begin{aligned} \mathcal{R}' &:= \mathcal{R}'_{(\rho_1^-, \rho_2^-)} \\ &= \{(\rho_1(s,\tau,u), \rho_2(s,\tau,u)) : (s,\tau,u) \in J \times \mathcal{B},\ \rho_i(s,\tau,u) \le 0,\ i = 1,2\}. \end{aligned}$$

We always assume that $\rho_1 : J \times \mathcal{B} \to \mathbb{R}$ and $\rho_2 : J \times \mathcal{B} \to \mathbb{R}$ are continuous and the function $(s,\tau) \mapsto u_{(s,\tau)}$ is continuous from $\mathcal{R}'$ into $\mathcal{B}$.

We will need the following hypothesis:

(H_ϕ) There exists a continuous bounded function $L : \mathcal{R}'_{(\rho_1^-, \rho_2^-)} \to (0, \infty)$ such that

$$\|\phi_{(s,\tau)}\|_{\mathcal{B}} \le L(s,\tau)\|\phi\|_{\mathcal{B}} \quad \text{for any } (s,\tau) \in \mathcal{R}'.$$

We will make use of the following consequence of the phase space axioms [HPL, Lemma 2.1].

LEMMA 5.5. *If $u \in \Omega$, then*

$$\|u_{(s,\tau)}\|_{\mathcal{B}} = (M + L')\|\phi\|_{\mathcal{B}} + K \sup_{(\theta,\eta)\in[0,\max\{0,s\}]\times[0,\max\{0,\tau\}]} \|u(\theta,\eta)\|,$$

where $L' = \sup_{(s,\tau)\in\mathcal{R}'} L(s,\tau)$.

Further, we present sufficient conditions for the existence and uniqueness of a solution of problem (1.7)–(1.9).

THEOREM 5.6. *Assume* (H_ϕ) *and the following hypotheses hold:*

(H′1) $f : J \times \mathcal{B} \to \mathbb{R}^n$ *is continuous.*
(H′2) *For each* $p \in \mathbb{N}$, *there exists* $\ell_p \in C(J_0, \mathbb{R}^n)$ *such that for each* $(t,x) \in J_0$,

$$\|f(t,x,u) - f(t,x,v)\| \le \ell_p(t,x)\|u - v\|_{\mathcal{B}} \quad \textit{for all } u, v \in \mathcal{B}.$$

If

$$\frac{k\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{5.1}$$

where $\ell_p^* = \sup_{(t,x)\in J_0} \ell_p(t,x)$, *then there exists a unique solution* (1.7)–(1.9) *on* $(-\infty,\infty) \times (-\infty,\infty)$.

Proof. We transform (1.7)–(1.9) into a fixed point problem. Consider the operator $N' : \Omega \to \Omega$ defined by

$$(N'u)(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} & \\ \qquad \times f(s,\tau,u_{(\rho_1(s,\tau,u_{(s,\tau)}),\rho_2(s,\tau,u_{(s,\tau)}))})\, d\tau\, ds, & (t,x) \in J. \end{cases}$$

Let $v(\cdot,\cdot) : (-\infty,\infty) \times (-\infty,\infty) \to \mathbb{R}^n$ be defined by

$$v(t,x) = \begin{cases} z(t,x), & (t,x) \in J. \\ \phi(t,x), & (t,x) \in \tilde{J}, \end{cases}$$

Then $v_{(t,x)} = \phi$ for all $(t,x) \in E$.

For each $w \in C(J,\mathbb{R}^n)$ with $w(t,x) = 0$ for all $(t,x) \in E$ we define

$$\overline{w}(t,x) = \begin{cases} w(t,x), & (t,x) \in J, \\ 0, & (t,x) \in \tilde{J}. \end{cases}$$

If $u(\cdot,\cdot)$ satisfies the integral equation

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \times f(s,\tau,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))} + v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\, d\tau\, ds,$$

we can decompose $u(\cdot,\cdot)$ as

$$u(t,x) = \overline{w}(t,x) + v(t,x), \quad (t,x) \in J,$$

which implies

$$u_{(t,x)} = \overline{w}_{(t,x)} + v_{(t,x)}, \quad (t,x) \in J,$$

and the function $w(\cdot,\cdot)$ satisfies

$$w(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}$$
$$\times f(s,\tau,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}+v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\,d\tau\,ds.$$

Let $P:C'_0\to C'_0$ be the operator defined by

$$(5.2)\qquad (Pw)(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}$$
$$\times f(s,\tau,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}+v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\,d\tau\,ds,\qquad (t,x)\in J.$$

Obviously, N' having a fixed point is equivalent to P having a fixed point, so we will prove that P has a fixed point. We shall use the nonlinear alternative of Leray–Schauder type due to Frigon and Granas. Let w be a possible solution of the equation $w=\lambda P(w)$ for some $0<\lambda<1$. This implies that for each $(t,x)\in J_0$,

$$w(t,x)=\frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}$$
$$\times f(s,\tau,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}+v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\,d\tau\,ds.$$

This implies by (H$'$2) that

$$(5.3)\qquad \|w(t,x)\|$$
$$\le\frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|f(s,\tau,\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}$$
$$+v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})-f(s,\tau,0)\|\,d\tau\,ds$$
$$+\frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|f(s,\tau,0)\|\,d\tau\,ds$$
$$\le\frac{f^*p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_p(s,\tau)$$
$$\times\|\overline{w}_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))}+v_{(\rho_1(s,t,u_{(s,t)}),\rho_2(s,t,u_{(s,t)}))})\|_{\mathcal{B}}\,d\tau\,ds,$$

where $f^*=\sup_{(s,\tau)\in J_0}\|f(s,\tau,0)\|$. Lemma 5.5 implies that

$$(5.4)\qquad \|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}}\le\|\overline{w}_{(s,\tau)}\|_{\mathcal{B}}+\|v_{(s,\tau)}\|_{\mathcal{B}}$$
$$\le K\sup\{w(\tilde{s},\tilde{\tau}):(\tilde{s},\tilde{\tau})\in[0,s]\times[0,\tau]\}$$
$$+(M+L')\|\phi\|_{\mathcal{B}}+K\|\phi(0,0)\|.$$

Let $y(s,\tau)$ be the right hand side of (5.4). Then

$$\|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}} \le y(t,x). \tag{5.5}$$

Therefore, from (5.3) and (5.5), we get

$$\begin{aligned}\|w(t,x)\| \le{}& \frac{f^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_p(s,\tau)y(s,\tau)\,d\tau\,ds.\end{aligned}$$

Using the above inequality and the definition of y, we have

$$\begin{aligned}y(t,x) \le{}& (M+L')\|\phi\|_{\mathcal{B}}+K\|\phi(0,0)\|+\frac{Kf^*p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\\ &+\frac{K\ell_p^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}y(s,t)\,d\tau\,ds.\end{aligned}$$

By Lemma 2.5, there exists a constant $\delta=\delta(r_1,r_2)$ such that

$$\begin{aligned}\|y\|_p \le{}& \left[(M+L')\|\phi\|_{\mathcal{B}}+K\|\phi(0,0)\|+\frac{Kf^*p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]\\ &\times\left[1+\delta\frac{K\ell_p^*}{\Gamma(r_1+1)\Gamma(r_2+1)}\right]=:\widetilde{M}_p.\end{aligned}$$

Then from (5.3) we have

$$\|w\|_p \le \frac{f^*p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}+\widetilde{M}_p\frac{\ell_p^*p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}=:\widetilde{M}_p^*.$$

Set

$$U'=\{w\in C_0:\|w\|_p\le \widetilde{M}_p^*+1 \quad \text{for all } p\in\mathbb{N}\}.$$

As in the previous argument, we can easily show that $P:U'\to C_p$ is a contraction map. By our choice of U', there is no $w\in\partial_n U'^n$ such that $w=\lambda P(w)$ with $\lambda\in(0,1)$. By Theorem 3.2, P has a unique fixed point w in U', which is a solution to problem (1.7)–(1.9). ∎

Now we present a global existence and uniqueness result for problem (1.10)–(1.12). Its proof is left to the reader.

THEOREM 5.7. *Assume* (H_ϕ), (H′1)–(H′2) *and the following hypothesis holds:*

(H′3) *For each $p\in\mathbb{N}$, there exists a constant c_p with $0<kc_p<1/4$ such that for any $(t,x)\in J_0$ we have*

$$\|g(t,x,u)-g(t,x,v)\|\le c_p(t,x)\|u-v\|_{\mathcal{B}} \quad \textit{for all } u,v\in\mathcal{B}.$$

If

$$4c_p + \frac{k\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1 \quad \text{for each } p \in \mathbb{N}, \tag{5.6}$$

then there exists a unique solution of (1.10)–(1.12) *on* $\mathbb{R}^2$.

6. Examples

Example 6.1. Consider the following fractional order partial hyperbolic functional differential equations with finite delay:

$$\begin{aligned}({}^cD_0^r u)(t,x) \\ = \frac{c_p}{e^{t+x+2}(1+|u(t-\sigma_1(u(t,x)),x-\sigma_2(u(t,x)))|)}, \quad (t,x)\in J,\end{aligned} \tag{6.1}$$

$$u(t,0)=t, \quad u(0,x)=x^2, \quad (t,x)\in J, \tag{6.2}$$

$$u(t,x)=t+x^2, \quad (t,x)\in \tilde{J}, \tag{6.3}$$

where $J:=[0,\infty)\times[0,\infty)$, $\tilde{J}:=[-1,\infty)\times[-2,\infty)\setminus[0,\infty)\times[0,\infty)$, $\sigma_1\in C(\mathbb{R},[0,1])$, $\sigma_2\in C(\mathbb{R},[0,2])$. Set

$$\begin{aligned}\rho_1(t,x,\varphi) &= t-\sigma_1(\varphi(0,0)), \quad (t,x,\varphi)\in J\times C([-1,0]\times[-2,0],\mathbb{R}),\\ \rho_2(t,x,\varphi) &= x-\sigma_2(\varphi(0,0)), \quad (t,x,\varphi)\in J\times C([-1,0]\times[-2,0],\mathbb{R}),\\ c_p &= \frac{\Gamma(r_1+1)\Gamma(r_2+1)}{p^{r_1+r_2}}, \quad p\in\mathbb{N}^*.\end{aligned}$$

$$f(t,x,\varphi)=\frac{c_p}{(e^{t+x+2})(1+|\varphi|)}, \quad (t,x)\in J,\ \varphi\in C([-1,0]\times[-2,0],\mathbb{R}).$$

For each $p\in\mathbb{N}^*$ and $\varphi,\overline{\varphi}\in C([-1,0]\times[-2,0],\mathbb{R})$ and $(t,x)\in J_0:=[0,p]\times[0,p]$ we have

$$|f(t,x,\varphi)-f(t,x,\overline{\varphi})| \le \frac{c_p}{e^2}\|\varphi-\overline{\varphi}\|_C.$$

Hence conditions (H1) and (H2) are satisfied with $\ell_p^* = c_p/e^2$. We shall show that condition (4.1) holds for all $p\in\mathbb{N}^*$. Indeed

$$\frac{\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{e^2} < 1,$$

which is satisfied for each $(r_1,r_2)\in(0,1]\times(0,1]$. Consequently, Theorem 4.3 implies that problem (6.1)–(6.3) has a unique solution defined on $[-1,\infty)\times[-2,\infty)$.

Example 6.2. Consider the following fractional order partial hyperbolic functional differential equations with infinite delay:

$$\begin{aligned}({}^cD_0^r u)(t,x) \\ = \frac{|u(t-\sigma_1(u(t,x)),x-\sigma_2(u(t,x)))|}{c_p e^{t+x}(1+|u(t-\sigma_1(u(t,x)),x-\sigma_2(u(t,x)))|)}, \quad (t,x)\in J,\end{aligned} \tag{6.4}$$

$$u(t,0) = t, \quad u(0,x) = x^2, \quad (t,x) \in J, \tag{6.5}$$

$$u(t,x) = t + x^2, \quad (t,x) \in \tilde{J} := \mathbb{R}^2 \setminus [0,\infty) \times [0,\infty), \tag{6.6}$$

where $\sigma_1 \in C(\mathbb{R},[0,1])$, $\sigma_2 \in C(\mathbb{R},[0,2])$ and

$$c_p = \frac{3p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)},$$

$$\mathcal{B}_\gamma = \Big\{u \in C((-\infty,0] \times (-\infty,0],\mathbb{R}) : \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)} u(\theta,\eta) \text{ exists } \in \mathbb{R}\Big\}.$$

The norm of $\mathcal{B}_\gamma$ is given by

$$\|u\|_\gamma = \sup_{(\theta,\eta)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(\theta+\eta)} |u(\theta,\eta)|.$$

Let

$$E := [0,1] \times \{0\} \cup \{0\} \times [0,1],$$

and let $u : (-\infty,1] \times (-\infty,1] \to \mathbb{R}$ be such that $u_{(t,x)} \in \mathcal{B}_\gamma$ for $(t,x) \in E$. Then

$$\begin{aligned}\lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)} u_{(t,x)}(\theta,\eta) &= \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta-t+\eta-x)} u(\theta,\eta)\\ &= e^{\gamma(t+x)} \lim_{\|(\theta,\eta)\|\to\infty} u(\theta,\eta) < \infty.\end{aligned}$$

Hence $u_{(t,x)} \in \mathcal{B}_\gamma$. Finally we will prove that

$$\begin{aligned}\|u_{(t,x)}\|_\gamma = K &\sup\{|u(s,\tau)| : (s,\tau) \in [0,t] \times [0,x]\}\\ &+ M \sup\{\|u_{(s,\tau)}\|_\gamma : (s,\tau) \in E_{(t,x)}\},\end{aligned}$$

where $K = M = 1$ and $H = 1$.

If $t + \theta \le 0$, $x + \eta \le 0$ we get

$$\|u_{(t,x)}\|_\gamma = \sup\{|u(s,\tau)| : (s,\tau) \in (-\infty,0] \times (-\infty,0]\},$$

and if $t + \theta \ge 0$, $x + \eta \ge 0$ then

$$\|u_{(t,x)}\|_\gamma = \sup\{|u(s,\tau)| : (s,\tau) \in [0,t] \times [0,x]\}.$$

Thus for all $(t+\theta, x+\eta) \in [0,1] \times [0,1]$, we get

$$\begin{aligned}\|u_{(t,x)}\|_\gamma = &\sup\{|u(s,\tau)| : (s,\tau) \in (-\infty,0] \times (-\infty,0]\}\\ &+ \sup\{|u(s,\tau)| : (s,\tau) \in [0,t] \times [0,x]\}.\end{aligned}$$

Then

$$\begin{aligned}\|u_{(t,x)}\|_\gamma = &\sup\{\|u_{(s,\tau)}\|_\gamma : (s,\tau) \in E\}\\ &+ \sup\{|u(s,\tau)| : (s,\tau) \in [0,t] \times [0,x]\}.\end{aligned}$$

$(\mathcal{B}_\gamma, \|\cdot\|_\gamma)$ is a Banach space. We conclude that $\mathcal{B}_\gamma$ is a phase space. Set

$$\begin{aligned}
\rho_1(t,x,\varphi) &= t - \sigma_1(\varphi(0,0)), &\quad (t,x,\varphi) \in J \times \mathcal{B}_\gamma,\\
\rho_2(t,x,\varphi) &= x - \sigma_2(\varphi(0,0)), &\quad (t,x,\varphi) \in J \times \mathcal{B}_\gamma,\\
f(t,x,\varphi) &= \frac{|\varphi|}{c_p e^{t+x}(1+|\varphi|)}, &\quad (t,x,\varphi) \in J \times \mathcal{B}_\gamma.
\end{aligned}$$

For all $\varphi, \overline{\varphi} \in \mathcal{B}_\gamma$ we have

$$|f(t,x,\varphi) - f(t,x,\overline{\varphi})| \le \frac{1}{c_p e^{t+x}} \|\varphi - \overline{\varphi}\|_{\mathcal{B}_\gamma}.$$

Hence condition (H′2) is satisfied with $\ell_p e^{t+x} = 1/(c_p e^{t+x})$. Since

$$\ell_p^* = \sup\left\{ \frac{1}{c_p e^{t+x}} : (t,x) \in J_0 \right\} \le \frac{1}{c}$$

and $K = 1$ we get

$$\frac{k \ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{3} < 1.$$

Hence condition (5.1) holds for each $(r_1, r_2) \in (0,1] \times (0,1]$ and all $p \in \mathbb{N}^*$. Consequently, Theorem 5.6 implies that problem (6.4)–(6.6) has a unique solution defined on $\mathbb{R}^2$.

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