Constructions on second order connections

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Abstract. We classify all $\mathcal{FM}_{m,n}$-natural operators $D : J^2 \rightsquigarrow J^2 V^A$ transforming second order connections $\Gamma : Y \to J^2 Y$ on a fibred manifold $Y \to M$ into second order connections $D(\Gamma) : V^A Y \to J^2 V^A Y$ on the vertical Weil bundle $V^A Y \to M$ corresponding to a Weil algebra $A$.

0. Introduction. An $r$th order connection on a fibred manifold $Y \to M$ is a section $\Gamma : Y \to J^r Y$ of the $r$-jet prolongation $J^r Y \to Y$ of $Y \to M$ (see [5]). In [6], we studied the problem how a first order connection $\Gamma : Y \to J^1 Y$ on $Y \to M$ induces a first order connection $D(\Gamma) : V^A Y \to J^1 V^A Y$ on the vertical Weil bundle $V^A Y \to M$ corresponding to a Weil algebra $A$. In the present paper we study the similar problem of how a second order connection $\Gamma : Y \to J^2 Y$ on a fibred manifold $Y \to M$ can induce a second order connection $D(\Gamma) : V^A Y \to J^2 V^A Y$ on $V^A Y \to M$. This problem corresponds to the classification of $\mathcal{FM}_{m,n}$-natural operators $D : J^2 \rightsquigarrow J^2 V^A$ in the sense of [5], where $\mathcal{FM}_{m,n}$ is the category of fibred manifolds with $n$-dimensional fibres and $m$-dimensional bases and their fibred local diffeomorphisms. We prove that the set of all $\mathcal{FM}_{m,n}$-natural operators $D : J^2 \rightsquigarrow J^2 V^A$ forms a $\dim A$-dimensional affine space and we explicitly describe this affine space. Thus we obtain a quite different result than the one from [6], where it is proved that there is only one $\mathcal{FM}_{m,n}$-natural operator $D : J^1 \rightsquigarrow J^1 V^A$.

All manifolds and maps are of class $C^\infty$.

1. The main result. The general concept of natural operators is described in [5]. In particular, an $\mathcal{FM}_{m,n}$-natural operator $D : J^r \rightsquigarrow J^r V^A$ transforming $r$th order general connections $\Gamma$ on $\mathcal{FM}_{m,n}$-objects $Y \to M$ to $r$th order connections $D(\Gamma)$ on the vertical Weil bundle $V^A Y \to M$
corresponding to a Weil algebra $A$ is a family of $\mathcal{FM}_{m,n}$-invariant regular operators (functions) $\mathcal{D}: \text{Con}^r(Y \to M) \to \text{Con}^r(V^A Y \to M)$ from the space $\text{Con}^r(Y \to M)$ of all $r$th order connections on $Y \to M$ into the space $\text{Con}^r(V^A Y \to M)$ of all $r$th order connections on $V^A Y \to M$ for $\mathcal{FM}_{m,n}$-objects $Y \to M$. By [6], any $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D}: J^1 \rightsquigarrow J^1 V^A$ is equal to the well-known $A$-vertical prolongation operator $\mathcal{V}^A: J^1 \rightsquigarrow J^1 V^A$. We have the following examples of $\mathcal{FM}_{m,n}$-natural operators $\mathcal{D}: J^2 \rightsquigarrow J^2 V^A$.

**Example 1.** Given a general second order connection $\Gamma: Y \to J^2 Y$ on $Y \to M$ we define a second order general connection $\mathcal{V}^A,2\Gamma$ on $V^A Y \to M$ by $\mathcal{V}^A,2\Gamma = (\kappa^{A,2})_Y \circ \mathcal{V}^A \Gamma: V^A Y \to J^2 V^A Y$, where $(\kappa^{A,2})_Y: V^A J^2 Y \to J^2 V^A Y$ is the canonical exchange isomorphism [5], [1]. The correspondence $\mathcal{V}^A,2: J^2 \rightsquigarrow J^2 V^A$ is the $\mathcal{FM}_{m,n}$-natural operator in question.

To give the next such example we need some preparation. Let $\Gamma: Y \to J^2 Y$ be a second order connection on $Y \to M$ with first order underlying connection $\Gamma^0: Y \to J^1 Y$. Let $\Gamma^0 \ast \Gamma^0 := J^1 \Gamma^0 \circ \Gamma^0: Y \to J^2 Y$ be the second order semi-holonomic Ehresmann prolongation of $\Gamma^0$ and $C^{(2)}: J^2 Y \to J^2 Y$ be the well-known symmetrization of second order semi-holonomic jets [4], [3]. Then $(\Gamma^0)^2 := C^{(2)} \circ (\Gamma^0 \ast \Gamma^0): Y \to J^2 Y$ is another second order connection on $Y \to M$ with the same underlying first order connection $\Gamma^0$. Since $J^2 Y \to J^1 Y$ is an affine bundle with corresponding vector bundle $S^2 T^* M \otimes VY$ over $J^1 Y$, we have the difference tensor field $\mathcal{E}(\Gamma) := \Gamma - (\Gamma^0)^2: Y \to S^2 T^* M \otimes VY$. Using this tensor, we construct the next example.

**Example 2.** For any $a \in A$ we have a tensor field $\mathcal{E}^a(\Gamma): V^A Y \to S^2 T^* M \otimes V V^A Y$ given by $\mathcal{E}^a(\Gamma)(X_1, X_2) := J_a \circ V^A(\mathcal{E}(\Gamma)(X_1, X_2))$, where $J_a: V V^A Y \to V V^A Y$ is a canonical “affinor” defined fibre-wise from the canonical affinor $J_a: TT^* N \to TT^* N$, and $V^A(\mathcal{E}(\Gamma)(X_1, X_2))$ is the flow prolongation of the vertical vector field $\mathcal{E}(\Gamma)(X_1, X_2)$ to $V^A Y$ for any vector fields $X_1, X_2$ on $M$. Since $J^2 V^A Y \to J^1 V^A Y$ is an affine bundle with the corresponding vector bundle $S^2 T^* M \otimes V V^A Y$ over $J^1 V^A Y$, we can define a second order connection $\mathcal{D}^a(\Gamma): V^A,2\Gamma + \mathcal{E}^a(\Gamma)$ on $V^A Y \to M$. The correspondence $\mathcal{D}^a: J^2 \rightsquigarrow J^2 V^A$ is an $\mathcal{FM}_{m,n}$-natural operator.

The main result of the paper is the following classification theorem.

**Theorem 1.** Every $\mathcal{FM}_{m,n}$-natural operators $\mathcal{D}: J^2 \rightsquigarrow J^2 V^A$ is $\mathcal{D}^a: J^2 \rightsquigarrow J^2 V^A$ for some $a \in A$.

The proof of Theorem 1 will occupy the rest of the paper. We prove three propositions. In Proposition 1, we show that any $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D}: J^2 \rightsquigarrow J^2 V^A$ is of finite order. In Proposition 2, we observe that for any $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D}: J^2 \rightsquigarrow J^2 V^A$ the under-
lying first order connection $D(\Gamma)^0$ of $D(\Gamma)$ on $V^A Y \to M$ is equal to the connection $V^A \Gamma^0$, where $\Gamma^0$ is the underlying first order connection of the second order connection $\Gamma : Y \to J^2 Y$ on $Y \to M$. Thus we have the difference $\mathcal{FM}_{m,n}$-natural operator $\mathcal{E} : J^2 \rightsquigarrow S^2 T^* \otimes VV^A$ given by $\mathcal{E}(\Gamma) = D(\Gamma) - V^A \Gamma : V^A Y \to S^2 T^* M \otimes VV^A Y$. In Proposition 3, we prove that the vector space (over $\mathbb{R}$) of all $\mathcal{FM}_{m,n}$-natural operators $\mathcal{E} : J^2 \rightsquigarrow S^2 T^* \otimes VV^A$ is of dimension $\leq \dim_{\mathbb{R}} A$. Then Theorem 1 follows by a dimension argument.

1. Finite order. We start the proof of Theorem 1 from the following proposition.

**Proposition 1.** Any $\mathcal{FM}_{m,n}$-natural operator $D$ transforming second order general connections $\Gamma$ on $Y \to M$ into second order general connections $D(\Gamma)$ on $V^A Y \to M$ is of finite order.

**Proof.** (See also the proof of Proposition 3 in [6].) This follows from the proof of Proposition 23.7 in [5], which can be generalized to our situation in the following way. Let $x^i, y^j$ ($i = 1, \ldots, m$, $j = 1, \ldots, n$) be the usual fibre coordinates on $\mathbb{R}^{m,n}$, the trivial bundle $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$. Let $x^i, y^j_{\alpha} \Gamma^{x_i y_j}$ for $\alpha \in (\mathbb{N} \cup \{0\})^m$ with $0 \leq |\alpha| \leq 2$ be the induced coordinates on $J^2 \mathbb{R}^{m,n}$. Consider the map $\varphi_{a,b} : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$, $\varphi_{a,b}(x,y) = (ax, by)$. Fix some $r \in \mathbb{N}$ and choose $a = b^{-r}$, $0 < b < 1$ arbitrary. Hence for every multiindex $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1$ includes all the derivatives with respect to the base coordinates while $\alpha_2$ those with respect to the fibre coordinates, and for every second order general connection $\Gamma$ on $\mathbb{R}^{m,n}$,

$$|\partial^{\alpha_1 + \alpha_2}(y_{\beta} \circ \varphi_{a,b} \Gamma)(0,0)| = b^r(|\beta| + |\alpha_1| + 1 - |\alpha_2|)|\partial^{\alpha_1 + \alpha_2}(y_{\beta} \circ \Gamma)(0,0)|$$

for all $|\beta| = 1, 2$, and so for all $|\alpha| \leq r$ we get

$$|\partial^{\alpha}(\varphi_{a,b} \Gamma)(0,0)| \leq b|\partial^{\alpha} \Gamma(0,0)|,$$

where $|\partial^{\alpha} \Gamma(0,0)| = \sum_{j=1}^n \sum_{|\beta| = 1, 2} |\partial^{\alpha}(y_{\beta} \circ \Gamma)(0,0)|$. On the other hand, there is a compact subset $K \subset (V^A \mathbb{R}^{m,n})_{0,0} = T^A_0 \mathbb{R}^n \Gamma(0,0)$ such that for any $z \in (V^A \mathbb{R}^{m,n})_{0,0}$ we will have $V^A \varphi_{a,b}(z) \in K$ for sufficiently small $b$. Hence Corollary 23.4 in [5] implies our assertion. $\blacksquare$

2. An underlying connection. Given a second order general connection $\Gamma : Y \to J^2 Y$ on $Y \to M$ we denote by $\Gamma^0 : Y \to J^1 Y$ the underlying first order general connection on $Y \to M$.

**Proposition 2.** Let $D$ be an $\mathcal{FM}_{m,n}$-natural operator transforming second order general connections $\Gamma$ on $Y \to M$ into second order general con-
transforming second order connections \( \Gamma \) on \( V^A Y \to M \). Then
\[
(D(\Gamma))^0 = (V^A)^2 \Gamma^0
\]
for any second order general connection \( \Gamma \) on \( Y \to M \).

**Proof.** Let \( x^i, y^j, y^j_\alpha \) be as in the proof of Proposition 1. Let \( v^l \) be a coordinate system on \( A^n \). Then on \( J^0_0(\mathbb{R}^m, A^n) \) we have the induced coordinates \( v^l, v^l_k \), where \( l = 1, \ldots, \dim A^n \), \( k = 1, \ldots, m \). Let \( \Gamma \) be a second order general connection on \( \mathbb{R}^{m,n} \). We will study \( (D(\Gamma))^0_w \in (J^1_0 V^A \mathbb{R}^{m,n})_0 = J^1_0(\mathbb{R}^m, A^n) \) for \( w \in (V^A \mathbb{R}^{m,n})_{(0,0)} = (T^A\mathbb{R}^n)_0 \).

We fix an arbitrary \( w \) as above. By Proposition 1, \( E \) is of finite order \( q \). So, we can assume that \( y^j_\alpha \circ \Gamma \) is a polynomial of degree \( q \) for any \( j, \alpha \) as above, i.e. \( y^j_\alpha \circ \Gamma(x, y) = \sum \Gamma^j_{\alpha,\beta,q} x^\beta y^q \) for \( (x, y) \in \mathbb{R}^{m,n} \), where the sum is over all \( \beta \in (\mathbb{N} \cup \{0\})^m \) and \( q \in (\mathbb{N} \cup \{0\})^n \) with \( |\beta| + |q| \leq q \), and \( \Gamma^j_{\alpha,\beta,q} \) are real numbers determined by \( \Gamma \). Moreover, we have \( y^j_{(0)} \circ \Gamma(x, y) = y^j \).

We identify \( \Gamma \) with \((\Gamma^j_{\alpha,\beta,q})\). Using the invariance of \( D \) with respect to the base homotheties \( t \text{id}_{\mathbb{R}^m} \times \text{id}_{\mathbb{R}^n} \) we obtain the homogeneity conditions
\[
v^l \circ (D(t^{[\alpha]+|\beta|}) \Gamma^j_{\alpha,\beta,q})_w^0 = v^l \circ (D(\Gamma^j_{\alpha,\beta,q}))_w^0
\]
and
\[
v^l_k \circ (D(t^{[\alpha]+|\beta|}) \Gamma^j_{\alpha,\beta,q})_w^0 = tv^l_k \circ (D(\Gamma^j_{\alpha,\beta,q}))_w^0.
\]
Then by the homogeneous function theorem, \( (D(\Gamma))^0_w \) is independent of \( \Gamma^j_{\alpha,\beta,q} \) for \( |\alpha| = 2 \). This means that \( (D(\Gamma))^0 \) over \((0,0) \in \mathbb{R}^m \times \mathbb{R}^n \) depends on a finite jet of \( \Gamma^0 \) at \((0,0) \) only. Then we have a well-defined \( \mathcal{F}\mathcal{M}_{m,n}\)-natural operator \( D^0 \) by \( D^0(\tilde{\Gamma}) = (D(\Gamma))^0 \) for any first order general connection \( \tilde{\Gamma} \) on \( Y \to M \), where \( \Gamma \) is a second order general connection on \( Y \to M \) with \( \Gamma^0 = \tilde{\Gamma} \). By the above-mentioned result of [6], \( D^0 = V^A \). This implies the equality in the proposition. \( \blacksquare \)

**3. The main difficulty.** The main difficulty in the proof of Theorem 1 is to establish the following proposition.

**Proposition 3.** The vector space over \( \mathbb{R} \) of all \( \mathcal{F}\mathcal{M}_{m,n}\)-natural operators sending second order general connections \( \Gamma \) on \( Y \to M \) into tensor fields \( \mathcal{E}(\Gamma) : V^A Y \to S^2 T^* M \otimes VV^A Y \) is of dimension \( \leq \dim \mathbb{R} A \).

To prove Proposition 3 we need some lemmas.

Let \( x^i, y^j, y^j_\alpha \) and \( v^l \) be as in the proof of Proposition 2. We can of course assume that the \( v^l \) are obtained as follows. We choose a basis \( a_1, \ldots, a_K \) of \( A \) over \( \mathbb{R} \). Let \((a_1, 0, \ldots, 0), \ldots, (a_K, 0, \ldots, 0), (0, a_{K+1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, a_{nK})\) be the corresponding basis of \( A^n \). Then \( v^l, l = 1, \ldots, \dim A^n \), is the basis dual to the last one. Let \( \mathcal{E} \) be an \( \mathcal{F}\mathcal{M}_{m,n}\)-natural operator transforming second order connections \( \Gamma \) on \( Y \to M \) into tensor fields
\[ E(\Gamma) : V^A Y \rightarrow S^2 T^* M \otimes V V^A Y. \] We denote the order of \( E \) by \( q \) (\( q \) is finite by Proposition 1).

**Lemma 1.** If
\[
\langle E(\Gamma)_w, u \circ u \rangle = 0 \in V_w V^A \mathbb{R}^{m,n} = T_w T^A \mathbb{R}^n = T_w A^n
\]
for all \( w \in (V^A \mathbb{R}^{m,n})_{(0,0)} = T^A_0 \mathbb{R}^n \), all \( u \in T^A_0 \mathbb{R}^m \) and all second order general connections \( \Gamma \) on \( \mathbb{R}^{m,n} \), then \( E = 0 \).

**Proof.** This is an immediate consequence of the invariance of \( E \) with respect to charts. ■

Using the invariance of \( E \) with respect to \( \mathcal{F}M_{m,n} \)-maps of the form \( \varphi \times \text{id}_{\mathbb{R}^n} \) for linear isomorphisms \( \varphi : \mathbb{R}^m \rightarrow \mathbb{R}^m \), we have

**Lemma 2.** If
\[
\langle E(\Gamma)_w, u_0 \circ u_0 \rangle = 0 \in T_w A^n
\]
for all \( w \in T^A_0 \mathbb{R}^n \) and all second order general connections \( \Gamma \) on \( \mathbb{R}^{m,n} \), where \( u_0 := \frac{\partial}{\partial x^1}|_{0} \), then \( E = 0 \).

Define
\[
\Phi^l_w(\Gamma) := d_w v^l(\langle E(\Gamma)_w, u_0 \circ u_0 \rangle) \in \mathbb{R}
\]
for all \( w \in T^A_0 \mathbb{R}^n \), all second order connections \( \Gamma \) on \( \mathbb{R}^{m,n} \) and \( l = 1, \ldots, \text{dim } A^n \).

**Lemma 3.** If \( \Phi^l_w(\Gamma) = 0 \) for all \( w \) and \( \Gamma \) as above and \( l = 1, \ldots, \text{dim } A \), then \( E = 0 \).

**Proof.** Because of the invariance of \( E \) with respect to permutations of the fibred coordinates, from the assumption of the lemma we deduce that \( \Phi^l_w(\Gamma) = 0 \) for all \( w \) and \( \Gamma \) as above and \( l = \text{dim } A^n \). Then \( \langle E(\Gamma)_w, u_0 \circ u_0 \rangle = 0 \) for all \( w \) and \( \Gamma \) as, and Lemma 2 ends the proof. ■

Because of the order of \( E \) we can assume that in the above lemmas we have
\[
y^j_\alpha \circ \Gamma(x,y) = \sum \Gamma^j_{\alpha,\beta,\varrho} x^\beta y^\varrho
\]
for all \((x,y) \in \mathbb{R}^{m,n}\), where the sum is over all \( \beta \in (\mathbb{N} \cup \{0\})^m \) and \( \varrho \in (\mathbb{N} \cup \{0\})^n \) with \(|\beta| + |\varrho| \leq q\), and \( \Gamma^j_{\alpha,\beta,\varrho} \) are real numbers determined by \( \Gamma \). Moreover, we have \( y^j_{(0)} \circ \Gamma(x,y) = y^j \).

We identify \( \Gamma \) with \((\Gamma^j_{\alpha,\beta,\varrho})\). Using the invariance of \( E \) with respect to the base homotheties \((t^1 x^1, \ldots, t^m x^m, y^1, \ldots, y^n)\) for \( t^j > 0 \), we get the homogeneity condition
\[
(t^1)^2 \Phi^l_w(\Gamma^j_{\alpha,\beta,\varrho}) = \Phi^l_w(t^{\alpha + \beta} \Gamma^j_{\alpha,\beta,\varrho}).
\]
Then by the homogeneous function theorem we can write  

\[ \Phi^l_w(\Gamma) = \sum a^\rho_j \Gamma^j_{(2,0,...,0),(0),\rho} + \sum b^\rho_j \Gamma^j_{(1,0,...,0),(1,0,...,0),\rho} + \sum c^\rho_{j_1,j_2} \Gamma^j_{(1,0,...,0),(0),\rho_1} \Gamma^j_{(1,0,...,0),(0),\rho_2} \]

for some uniquely determined real numbers \( a^\rho_j = a^\rho_{j,l}(w) \), \( b^\rho_j = b^\rho_{j,l}(w) \) and \( c^\rho_{j_1,j_2} = c^\rho_{j_1,j_2,l}(w) \) (smoothly depending on \( w \)), where the first sum is over all \( j = 1,\ldots,n \) and all \( \rho \in (\mathbb{N} \cup \{0\})^n \) with \( |\rho| \leq q \), the second sum is over all \( j = 1,\ldots,n \) and all \( \rho \in (\mathbb{N} \cup \{0\})^n \) with \( |\rho| \leq q - 1 \) and the third sum is over all \((q_1,j_1) \leq (q_2,j_2) \) for \( j_1,j_2 = 1,\ldots,n \) and \( q_1,q_2 \in (\mathbb{N} \cup \{0\})^n \) with \( |q_1| \leq q \) and \( |q_2| \leq q \) (here \( \leq \) means an ordering). Of course \((2,0,\ldots,0),(1,0,\ldots,0),(0) \in (\mathbb{N} \cup \{0\})^m \).

**Lemma 4.** Assume that all \( a^\rho_j \), all \( b^\rho_j \) and all \( c^\rho_{j_1,j_2} \) defined by (\*) are 0 for all \( w \in T^A_0 \mathbb{R}^n \) and all \( l = 1,\ldots,\dim A \). Then \( \mathcal{E} = 0 \).

**Proof.** This is obvious in view of the previous lemma.

**Lemma 5.** We have  

\[ a^\rho_j + b^\rho_j = 0 \]

for all \( j, \rho, w \) in question and \( l = 1,\ldots,\dim A \).

**Proof.** Fix \( \rho_0 \) and \( j_0 \). Choose \( \Gamma = (\Gamma^j_{\alpha,\beta,\rho}) \) such that \( \Gamma^j_{(1,0,...,0),(0),\rho_0} = 1 \), and \( \Gamma^j_{\alpha,\beta,\rho} = 0 \) for other \((j,\alpha,\beta,\rho) \) with \( |\alpha| \geq 1 \). Let \( \varphi = (x^1 + \frac{1}{2}(x^1)^2, x^2,\ldots,x^m,y^1,\ldots,y^n)^{-1} \). Using the invariance of \( \mathcal{E} \) with respect to \( \varphi \) we have  

\[ \Phi^l_w(\varphi_* \Gamma) = \Phi^l_w(\Gamma) \]

because \( \varphi \) preserves \( w, v^l \) and \( u_0 \odot w_0 \). Set  

\[ \Phi^l_w(\Gamma) = a. \]

We have \( J^q_{(0,0)}(\varphi_* \Gamma) = (\tilde{\Gamma}^j_{\alpha,\beta,\rho}) \), where  

\[
\tilde{\Gamma}^j_{(1,0,...,0),(0),\rho_0} = 1, \quad \tilde{\Gamma}^j_{(2,0,...,0),(0),\rho_0} = 1, \quad \tilde{\Gamma}^j_0_{(1,0,...,0),(1,0,...,0),\rho_0} = 1
\]

and other \( \tilde{\Gamma}^j_{\alpha,\beta,\rho} \) are zero for \( |\alpha| = 1,2 \). (Indeed,  

\[ \Gamma(z,y) = j^2_{z}(y + y_0(z^1 - z^1)e_{j_0}) \in J^2_z(\mathbb{R}^{m,n})_y, \]

where \( \{e_j\} \) is the canonical basis in \( \mathbb{R}^n \). Then  

\[ (\varphi_* \Gamma)(z,y) = j^2_{z}(y + y_0(x^1 - \frac{1}{2}(x^1)^2 - z^1 - \frac{1}{2}(z^1)^2)e_{j_0}). \]

This implies the formulas.) Then  

\[ \Phi^l_w(\varphi_* \Gamma) = a^\rho_0 + b^\rho_0 + a. \]

Hence \( a^\rho_0 + b^\rho_0 = 0 \).
LEMMA 6. Suppose that all \( a_j^e \) defined by \((*)\) are zero for any \( w \) in question and \( l = 1, \ldots, \dim A \). Then \( \mathcal{E} = 0 \).

Proof. By assumption and Lemma 5, all \( b_j^e \) are zero. Then it is sufficient to show that \( c_{j_1,j_2}^{e_1,e_2} = 0 \) for all \( e_1, e_2, j_1, j_2, w \) and \( l \) in question.

Fix \( e_1, e_2, j_1, j_2, l, w \). Let \( a, b \in \mathbb{R} \). Let \( \Gamma^0 \) be the trivial second order connection on \( \mathbb{R}^{m,n} \), given by

\[
\Gamma^0(z, y) = j_z^2(y) \in J_z^2(\mathbb{R}^{m,n})_y.
\]

Then by \((*)\) we have

\[
\Phi^l_w(\Gamma^0) = 0.
\]

Choose an \(\mathcal{FM}_{m,n}\)-map \(\psi: \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}\) given by

\[
\psi(x, y) = (x, y + ax^1y^{e_1}e_{j_1} + bx^1y^{e_2}e_{j_2}).
\]

Then

\[
\Phi^l_w(\psi \ast \Gamma^0) = 0
\]

because of the invariance of \(\mathcal{E}\), \(u_0\), \(w\) and \(v^l\) with respect to \(\psi\). Write \(\psi^{-1}(x, y) = (x, \tilde{y})\). Then

\[
(\psi \ast \Gamma^0)(z, y) = j_z^2(\tilde{y} + ax^1\tilde{y}^{e_1}e_{j_1} + bx^1\tilde{y}^{e_2}e_{j_2}).
\]

Then by \((*)\), \(\Phi^l_w(\psi \ast \Gamma^0)\) is a polynomial in \(a\) and \(b\) with the coefficient of \(ab\) equal to \(c_{j_1,j_2}^{e_1,e_2}\) as all \(b_j^e\) are zero. Therefore \(c_{j_1,j_2}^{e_1,e_2} = 0\). □

LEMMA 7. Suppose that all \(a_j^{(0)}\) defined by \((*)\) are zero for any \(w\) in question and \(l = 1, \ldots, \dim A\). Then \(\mathcal{E} = 0\).

Proof. For any \(\varrho \in (\mathbb{N} \cup \{0\})^n\) and \(j = 1, \ldots, n\), let \(\Gamma^{\varrho,j}\) be the second order connection on \(\mathbb{R}^{m,n}\) given by

\[
\Gamma^{\varrho,j}(z, y) = j_z^2(y + (x^1 - z^1)^2 y^{\varrho}e_j), \quad (z, y) \in \mathbb{R}^{m,n}.
\]

By \((*)\) and the assumption of the lemma we have

\[
d_w v^l \circ \langle \mathcal{E}(\Gamma^{(0),j})(w), u_0 \circ u_0 \rangle = \Phi^l_w(\Gamma^{(0),j}) = a_j^{(0),l}(w) = 0
\]

for any \(w \in (V_A^{\mathbb{R}^{m,n}})_{(0,0)}, j = 1, \ldots, n\) and \(l = 1, \ldots, \dim \mathbb{R} A\). Then by the invariance of \(\mathcal{E}\) with respect to the permutations of fibred coordinates we have

\[
d_w v^l \circ \langle \mathcal{E}(\Gamma^{(0),j})(w), u_0 \circ u_0 \rangle = 0
\]

for any \(w \in (V_A^{\mathbb{R}^{m,n}})_{(0,0)}, j = 1, \ldots, n\) and \(l = 1, \ldots, \dim \mathbb{R} A^n\). Therefore

\[
\langle \mathcal{E}(\Gamma^{(0),j})(w), u_0 \circ u_0 \rangle = 0
\]

for any \(w \in (V_A^{\mathbb{R}^{m,n}})_{(0,0)}\) and \(j = 1, \ldots, n\). Let \(\varrho \in (\mathbb{N} \cup \{0\})^n\), \(1 \leq |\varrho| \leq q, j = 1, \ldots, n\). Let \(\tau \in \mathbb{R}\) be sufficiently small. Consider an \(\mathcal{FM}_{m,n}\)-map
Let $\varphi^{0j} : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$, $\varphi^{0j}(x, y) = (x, y + \tau y^{0j+1}e_j)$ (defined near $(0, 0)$). We see that

$$(\varphi^{0j}_* \Gamma^{(0),j})(z, y) = j^2_z(y + (x^1 - z^1)^2 e_j + \tau (g_j + 1)(x^1 - z^1)^2 y^0 e_j + \cdots).$$

Then using the invariance of $\mathcal{E}$ with respect to $\varphi^{0j}$ we get

$$\langle \mathcal{E}(\varphi^{0j}_* \Gamma^{(0),j})(w), u_0 \otimes u_0 \rangle = 0$$

for all $w \in (V^A \mathbb{R}^{m,n})_{(0,0)}$. The left hand side of the last formula is a polynomial in $\tau$. The coefficient of $\tau = \tau^1$ in this polynomial is $(g_j + 1)\langle \mathcal{E}(\Gamma^{0,j})(w), u_0 \otimes u_0 \rangle$. Then $\langle \mathcal{E}(\Gamma^{0,j})(w), u_0 \otimes u_0 \rangle = 0$ for any $w \in (V^A \mathbb{R}^{m,n})_{(0,0)}$. Thus $a_j^0 = 0$ for all $l$ and $w$ in question. Then $\mathcal{E} = 0$ by Lemma 6.

**Lemma 8.** Suppose that all $a_j^{(0)}$ defined by (*) are zero for $l = 1, \ldots, \dim A$ and $w = 0 \in T_0^A \mathbb{R}^n$. Then $\mathcal{E} = 0$.

**Proof.** By (*) we have

$$\Phi^l_w(\Gamma^{(0),j}) = a_j^{(0)} = a_j^{(0),l}(w),$$

where $\Gamma^{(0),j}$ is defined in the proof of Lemma 7. Let $h_t = \text{id}_{\mathbb{R}^m} \times t \text{id}_{\mathbb{R}^n}$ be the fibre homotethy. Then $((h_t)_* \Gamma^{(0),j})(z, y) = j^2_z(y + t(x^1 - z^1)^2 e_j)$, and then by (*) we have

$$\Phi^l_{tw}((h_t)_* \Gamma^{(0),j}) = ta_j^{(0),l}(tw).$$

Hence by the invariance of $\mathcal{E}$ with respect to the fibre homotheties $h_t$ we get $\Phi^l_{tw}((h_t)_* \Gamma^{(0),j}) = t\Phi^l_w(\Gamma^{(0),j})$, and therefore

$$a_j^{(0),l}(w) = a_j^{(0),l}(tw).$$

Then putting $t \to 0$ and using the assumption of the lemma we see that $a_j^{(0)} = 0$ for any $l$ and $w$ in question. Then Lemma 7 ends the proof.

**Lemma 9.** Suppose that $a_1^{(0)} = 0$ for $w = 0 \in T_0^A \mathbb{R}^n$ and $l = 1, \ldots, \dim A$. Then $\mathcal{E} = 0$.

**Proof.** Let $a_t = (x^1, tx^2, \ldots, tx^m, y^1, \ldots, y^n)$ for $t \neq 0$ be $\mathcal{F}\mathcal{M}_m,n$-maps. Clearly, $((a_t)_* \Gamma^{(0),j})(y, z) = j^2_z(y + t(x^1 - z^1)^2 e_j)$ for $j = 2, \ldots, n$. Then by (*) and the invariance of $\mathcal{E}$ with respect to $a_t$ we get

$$a_j^{(0),l}(0) = ta_j^{(0),l}(0)$$

for $j = 2, \ldots, n$. Then $a_j^{(0),l}(0) = 0$ for $j = 1, \ldots, n$ (for $j = 1$ the equality holds by assumption). Now Lemma 8 ends the proof.

**Proof of Proposition 3.** This is an immediate consequence of Lemma 9.
Proof of Theorem 1. Let $\Gamma^{(0),1}$ be the second order connection on $\mathbb{R}^{m,n}$ as in the proof of Lemma 7. Let $E^a : J^2 \rightsquigarrow S^2T^* \otimes VV^A$ be the operators from Example 2 for $a \in A$. Let $(a_\nu)$ be a basis of $A$, and let $(v^l)$ correspond to $(a_\nu)$ (as at the beginning of Section 3). One can show by a standard argument that the numbers $\Phi^l_0(\Gamma^{(0),1}) = a^{(0),1}_l(0)$ (see (*)& for $E = E^{a_\nu}$ are proportional (with a non-zero coefficient) to the Kronecker delta $\delta^l_\nu$. Hence the operators $E^{a_\nu}$ are linearly independent. Theorem 1 is an immediate consequence of Propositions 1–3 by a dimension argument. ■

References


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