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Constructions on second order connections

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Abstract. We classify all $\mathcal{FM}_{m,n}$-natural operators $\mathcal{D} : J^2 \rightsquigarrow J^2V^A$ transforming second order connections $\Gamma : Y \to J^2Y$ on a fibred manifold $Y \to M$ into second order connections $\mathcal{D}(\Gamma) : V^AY \to J^2V^AY$ on the vertical Weil bundle $V^AY \to M$ corresponding to a Weil algebra A.

0. Introduction. An *rth order connection* on a fibred manifold $Y \to M$ is a section $\Gamma : Y \to J^rY$ of the r-jet prolongation $J^rY \to Y$ of $Y \to M$ (see [5]). In [6], we studied the problem how a first order connection $\Gamma : Y \to J^1Y$ on $Y \to M$ induces a first order connection $\mathcal{D}(\Gamma) : V^AY \to J^1V^AY$ on the vertical Weil bundle $V^AY \to M$ corresponding to a Weil algebra A. In the present paper we study the similar problem of how a second order connection $\Gamma : Y \to J^2Y$ on a fibred manifold $Y \to M$ can induce a second order connection $\mathcal{D}(\Gamma) : V^AY \to J^2V^AY$ on $V^AY \to M$. This problem corresponds to the classification of $\mathcal{FM}_{m,n}$-natural operators $\mathcal{D} : J^2 \rightsquigarrow J^2V^A$ in the sense of [5], where $\mathcal{FM}_{m,n}$ is the category of fibred manifolds with n-dimensional fibres and m-dimensional bases and their fibred local diffeomorphisms. We prove that the set of all $\mathcal{FM}_{m,n}$-natural operators $\mathcal{D} : J^2 \rightsquigarrow J^2V^A$ forms a $\dim_{\mathbb{R}} A$-dimensional affine space and we explicitly describe this affine space. Thus we obtain a quite different result than the one from [6], where it is proved that there is only one $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D} : J^1 \rightsquigarrow J^1V^A$.

All manifolds and maps are of class C^∞.

1. The main result. The general concept of natural operators is described in [5]. In particular, an $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D} : J^r \rightsquigarrow J^rV^A$ transforming rth order general connections Γ on $\mathcal{FM}_{m,n}$-objects $Y \to M$ to rth order connections $\mathcal{D}(\Gamma)$ on the vertical Weil bundle $V^AY \to M$

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corresponding to a Weil algebra A is a family of $\mathcal{FM}_{m,n}$-invariant regular operators (functions) $\mathcal{D} : \mathrm{Con}^r(Y \to M) \to \mathrm{Con}^r(V^AY \to M)$ from the space $\mathrm{Con}^r(Y \to M)$ of all rth order connections on $Y \to M$ into the space $\mathrm{Con}^r(V^AY \to M)$ of all rth order connections on $V^AY \to M$ for $\mathcal{FM}_{m,n}$-objects $Y \to M$. By [6], any $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D} : J^1 \rightsquigarrow J^1V^A$ is equal to the well-known A-vertical prolongation operator $\mathcal{V}^A : J^1 \rightsquigarrow J^1V^A$. We have the following examples of $\mathcal{FM}_{m,n}$-natural operators $\mathcal{D} : J^2 \rightsquigarrow J^2V^A$.

Example 1. Given a general second order connection $\Gamma : Y \to J^2Y$ on $Y \to M$ we define a second order general connection $\mathcal{V}^{A,2}\Gamma$ on $V^AY \to M$ by $\mathcal{V}^{A,2}\Gamma = (\kappa^{A,2})_Y \circ V^A\Gamma : V^AY \to J^2V^AY$, where $(\kappa^{A,2})_Y : V^AJ^2Y \to J^2V^AY$ is the canonical exchange isomorphism [5], [1]. The correspondence $\mathcal{V}^{A,2} : J^2 \rightsquigarrow J^2V^A$ is the $\mathcal{FM}_{m,n}$-natural operator in question.

To give the next such example we need some preparation. Let $\Gamma : Y \to J^2Y$ be a second order connection on $Y \to M$ with first order underlying connection $\Gamma^0 : Y \to J^1Y$. Let $\Gamma^0 * \Gamma^0 := J^1\Gamma^0 \circ \Gamma^0 : Y \to \bar{J}^2Y$ be the second order semi-holonomic Ehresmann prolongation of Γ^0 and $C^{(2)} : \bar{J}^2Y \to J^2Y$ be the well-known symmetrization of second order semi-holonomic jets [4], [3]. Then $(\Gamma^0)^2 := C^{(2)} \circ (\Gamma^0 * \Gamma^0) : Y \to J^2Y$ is another second order connection on $Y \to M$ with the same underlying first order connection Γ^0. Since $J^2Y \to J^1Y$ is an affine bundle with corresponding vector bundle $S^2T^*M \otimes VY$ over J^1Y, we have the difference tensor field $\mathcal{E}(\Gamma) := \Gamma - (\Gamma^0)^2 : Y \to S^2T^*M \otimes VY$. Using this tensor, we construct the next example.

Example 2. For any $a \in A$ we have a tensor field $\mathcal{E}^a(\Gamma) : V^AY \to S^2T^*M \otimes VV^AY$ given by $\mathcal{E}^a(\Gamma)(X_1, X_2)) := J_a \circ \mathcal{V}^A(\mathcal{E}(\Gamma)(X_1, X_2))$, where $J_a : VV^AY \to VV^AY$ is a canonical "affinor" defined fibre-wise from the canonical affinor $J_a : TT^AN \to TT^AN$, and $\mathcal{V}^A(\mathcal{E}(\Gamma)(X_1, X_2))$ is the flow prolongation of the vertical vector field $\mathcal{E}(\Gamma)(X_1, X_2)$ to V^AY for any vector fields X_1, X_2 on M. Since $J^2V^AY \to J^1V^AY$ is an affine bundle with the corresponding vector bundle $S^2T^*M \otimes VV^AY$ over J^1V^AY, we can define a second order connection $\mathcal{D}^a(\Gamma) : \mathcal{V}^{A,2}\Gamma + \mathcal{E}^a(\Gamma)$ on $V^AY \to M$. The correspondence $\mathcal{D}^a : J^2 \rightsquigarrow J^2V^A$ is an $\mathcal{FM}_{m,n}$-natural operator.

The main result of the paper is the following classification theorem.

Theorem 1. *Every $\mathcal{FM}_{m,n}$-natural operators $\mathcal{D} : J^2 \rightsquigarrow J^2V^A$ is $\mathcal{D}^a : J^2 \rightsquigarrow J^2V^A$ for some $a \in A$.*

The proof of Theorem 1 will occupy the rest of the paper. We prove three propositions. In Proposition 1, we show that any $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D} : J^2 \rightsquigarrow J^2V^A$ is of finite order. In Proposition 2, we observe that for any $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D} : J^2 \rightsquigarrow J^2V^A$ the underlying first order connection $\mathcal{D}(\Gamma)^0$ of $\mathcal{D}(\Gamma)$ on $V^AY \to M$ is equal to the connection $\mathcal{V}^A\Gamma^0$, where Γ^0 is the underlying first order connection of the second order connection $\Gamma : Y \to J^2Y$ on $Y \to M$. Thus we have the difference $\mathcal{FM}_{m,n}$-natural operator $\mathcal{E} : J^2 \rightsquigarrow S^2T^* \otimes VV^A$ given by $\mathcal{E}(\Gamma) = \mathcal{D}(\Gamma) - \mathcal{V}^{A,2}\Gamma : V^AY \to S^2T^*M \otimes VV^AY$. In Proposition 3, we prove that the vector space (over $\mathbb{R}$) of all $\mathcal{FM}_{m,n}$-natural operators $\mathcal{E} : J^2 \rightsquigarrow S^2T^* \otimes VV^A$ is of dimension $\le \dim_{\mathbb{R}} A$. Then Theorem 1 follows by a dimension argument.

1. Finite order. We start the proof of Theorem 1 from the following proposition.

PROPOSITION 1. *Any $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D}$ transforming second order general connections Γ on $Y \to M$ into second order general connections $\mathcal{D}(\Gamma)$ on $V^AY \to M$ is of finite order.*

Proof. (See also the proof of Proposition 3 in [6].) This follows from the proof of Proposition 23.7 in [5], which can be generalized to our situation in the following way. Let x^i, y^j ($i = 1, \dots, m$, $j = 1, \dots, n$) be the usual fibre coordinates on $\mathbb{R}^{m,n}$, the trivial bundle $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$. Let x^i, y^j_α for $\alpha \in (\mathbb{N} \cup \{0\})^m$ with $0 \le |\alpha| \le 2$ be the induced coordinates on $J^2\mathbb{R}^{m,n}$. Consider the map $\varphi_{a,b} : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$, $\varphi_{a,b}(x, y) = (ax, by)$. Fix some $r \in \mathbb{N}$ and choose $a = b^{-r}$, $0 < b < 1$ arbitrary. Hence for every multiindex $\alpha = \alpha_1 + \alpha_2$, where α_1 includes all the derivatives with respect to the base coordinates while α_2 those with respect to the fibre coordinates, and for every second order general connection Γ on $\mathbb{R}^{m,n}$,

$$|\partial^{\alpha_1+\alpha_2}(y^j_\beta \circ \varphi^*_{a,b}\Gamma)(0,0)| = b^{r(|\beta|+|\alpha_1|)+1-|\alpha_2|}|\partial^{\alpha_1+\alpha_2}(y^j_\beta \circ \Gamma)(0,0)|$$

for all $|\beta| = 1, 2$, and so for all $|\alpha| \le r$ we get

$$|\partial^\alpha(\varphi^*_{a,b}\Gamma)(0,0)| \le b|\partial^\alpha\Gamma(0,0)|,$$

where $|\partial^\alpha\Gamma(0,0)| = \sum_{j=1}^n \sum_{|\beta|=1,2} |\partial^\alpha(y^j_\beta \circ \Gamma)(0,0)|$. On the other hand, there is a compact subset $K \subset (V^A\mathbb{R}^{m,n})_{(0,0)} = T^A_0\mathbb{R}^n$ (K is a compact neighbourhood of $z_0 = j^A0$) such that for any $z \in (V^A\mathbb{R}^{m,n})_{(0,0)}$ we will have $V^A\varphi_{a,b}(z) \in K$ for sufficiently small b. Hence Corollary 23.4 in [5] implies our assertion. ∎

2. An underlying connection. Given a second order general connection $\Gamma : Y \to J^2Y$ on $Y \to M$ we denote by $\Gamma^0 : Y \to J^1Y$ the underlying first order general connection on $Y \to M$.

PROPOSITION 2. *Let $\mathcal{D}$ be an $\mathcal{FM}_{m,n}$-natural operator transforming second order general connections Γ on $Y \to M$ into second order general con-*

nections $\mathcal{D}(\Gamma)$ on $V^AY \to M$. Then

$$(\mathcal{D}(\Gamma))^0 = (\mathcal{V}^{A,2}\Gamma)^0$$

for any second order general connection Γ on $Y \to M$.

Proof. Let x^i, y^j, y^j_α be as in the proof of Proposition 1. Let v^l be a coordinate system on A^n. Then on $J^1_0(\mathbb{R}^m, A^n)$ we have the induced coordinates v^l, v^l_k, where $l = 1, \dots, \dim A^n$, $k = 1, \dots, m$. Let Γ be a second order general connection on $\mathbb{R}^{m,n}$. We will study $(\mathcal{D}(\Gamma))^0_w \in (J^1V^A\mathbb{R}^{m,n})_0 = J^1_0(\mathbb{R}^m, A^n)$ for $w \in (V^A\mathbb{R}^{m,n})_{(0,0)} = (T^A\mathbb{R}^n)_0$.

We fix an arbitrary w as above. By Proposition 1, $\mathcal{E}$ is of finite order q. So, we can assume that $y^j_\alpha \circ \Gamma$ is a polynomial of degree q for any j, α as above, i.e. $y^j_\alpha \circ \Gamma(x,y) = \sum \Gamma^j_{\alpha,\beta,\varrho} x^\beta y^\varrho$ for $(x,y) \in \mathbb{R}^{m,n}$, where the sum is over all $\beta \in (\mathbb{N} \cup \{0\})^m$ and $\varrho \in (\mathbb{N} \cup \{0\})^n$ with $|\beta| + |\varrho| \le q$, and $\Gamma^j_{\alpha,\beta,\varrho}$ are real numbers determined by Γ. Moreover, we have $y^j_{(0)} \circ \Gamma(x,y) = y^j$. We identify Γ with $(\Gamma^j_{\alpha,\beta,\varrho})$. Using the invariance of $\mathcal{D}$ with respect to the base homotheties $t\,\mathrm{id}_{\mathbb{R}^m} \times \mathrm{id}_{\mathbb{R}^n}$ we obtain the homogeneity conditions

$$v^l \circ (\mathcal{D}(t^{|\alpha|+|\beta|}\Gamma^j_{\alpha,\beta,\varrho}))^0_w = v^l \circ (\mathcal{D}(\Gamma^j_{\alpha,\beta,\varrho}))^0_w$$

and

$$v^l_k \circ (\mathcal{D}(t^{|\alpha|+|\beta|}\Gamma^j_{\alpha,\beta,\varrho}))^0_w = t v^l_k \circ (\mathcal{D}(\Gamma^j_{\alpha,\beta,\varrho}))^0_w.$$

Then by the homogeneous function theorem, $(\mathcal{D}(\Gamma))^0_w$ is independent of $\Gamma^j_{\alpha,\beta,\varrho}$ for $|\alpha| = 2$. This means that $(\mathcal{D}(\Gamma))^0$ over $(0,0) \in \mathbb{R}^m \times \mathbb{R}^n$ depends on a finite jet of Γ^0 at $(0,0)$ only. Then we have a well-defined $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D}^0$ by $\mathcal{D}^0(\widetilde{\Gamma}) = (\mathcal{D}(\Gamma))^0$ for any first order general connection $\widetilde{\Gamma}$ on $Y \to M$, where Γ is a second order general connection on $Y \to M$ with $\Gamma^0 = \widetilde{\Gamma}$. By the above-mentioned result of [6], $\mathcal{D}^0 = \mathcal{V}^A$. This implies the equality in the proposition. ■

3. The main difficulty. The main difficulty in the proof of Theorem 1 is to establish the following proposition.

PROPOSITION 3. *The vector space over $\mathbb{R}$ of all $\mathcal{FM}_{m,n}$-natural operators sending second order general connections Γ on $Y \to M$ into tensor fields $\mathcal{E}(\Gamma) : V^AY \to S^2T^*M \otimes VV^AY$ is of dimension $\le \dim_{\mathbb{R}} A$.*

To prove Proposition 3 we need some lemmas.

Let x^i, y^j, y^j_α and v^l be as in the proof of Proposition 2. We can of course assume that the v^l are obtained as follows. We choose a basis $a_1, \dots, a_K$ of A over $\mathbb{R}$. Let $(a_1, 0, \dots, 0), \dots, (a_K, 0, \dots, 0)$, $(0, a_{K+1}, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, a_{nK})$ be the corresponding basis of A^n. Then v^l, $l = 1, \dots, \dim A^n$, is the basis dual to the last one. Let $\mathcal{E}$ be an $\mathcal{FM}_{m,n}$-natural operator transforming second order connections Γ on $Y \to M$ into tensor fields

$\mathcal{E}(\Gamma) : V^AY \to S^2T^*M \otimes VV^AY$. We denote the order of $\mathcal{E}$ by q (q is finite by Proposition 1).

LEMMA 1. *If*

$$\langle \mathcal{E}(\Gamma)_w, u \odot u\rangle = 0 \in V_wV^A\mathbb{R}^{m,n} = T_wT^A\mathbb{R}^n = T_wA^n$$

for all $w \in (V^A\mathbb{R}^{m,n})_{(0,0)} = T_0^A\mathbb{R}^n$, *all* $u \in T_0\mathbb{R}^m$ *and all second order general connections* Γ *on* $\mathbb{R}^{m,n}$, *then* $\mathcal{E} = 0$.

Proof. This is an immediate consequence of the invariance of $\mathcal{E}$ with respect to charts. ∎

Using the invariance of $\mathcal{E}$ with respect to $\mathcal{FM}_{m,n}$-maps of the form $\varphi \times \mathrm{id}_{\mathbb{R}^n}$ for linear isomorphisms $\varphi : \mathbb{R}^m \to \mathbb{R}^m$, we have

LEMMA 2. *If*

$$\langle \mathcal{E}(\Gamma)_w, u_0 \odot u_0\rangle = 0 \in T_wA^n$$

for all $w \in T_0^A\mathbb{R}^n$ *and all second order general connections* Γ *on* $\mathbb{R}^{m,n}$, *where* $u_0 := \frac{\partial}{\partial x^1}_{|0}$, *then* $\mathcal{E} = 0$.

Define

$$\Phi_w^l(\Gamma) := d_wv^l(\langle \mathcal{E}(\Gamma)_w, u_0 \odot u_0\rangle) \in \mathbb{R}$$

for all $w \in T_0^A\mathbb{R}^n$, all second order connections Γ on $\mathbb{R}^{m,n}$ and $l = 1, \dots, \dim A^n$.

LEMMA 3. *If* $\Phi_w^l(\Gamma) = 0$ *for all* w *and* Γ *as above and* $l = 1, \dots, \dim A$, *then* $\mathcal{E} = 0$.

Proof. Because of the invariance of $\mathcal{E}$ with respect to permutations of the fibred coordinates, from the assumption of the lemma we deduce that $\Phi_w^l(\Gamma) = 0$ for all w and Γ as above and $l = \dim A^n$. Then $\langle \mathcal{E}(\Gamma)_w, u_0 \odot u_0\rangle = 0$ for all w and Γ as, and Lemma 2 ends the proof. ∎

Because of the order of $\mathcal{E}$ we can assume that in the above lemmas we have

$$y_\alpha^j \circ \Gamma(x, y) = \sum \Gamma_{\alpha,\beta,\varrho}^j x^\beta y^\varrho$$

for all $(x, y) \in \mathbb{R}^{m,n}$, where the sum is over all $\beta \in (\mathbb{N} \cup \{0\})^m$ and $\varrho \in (\mathbb{N} \cup \{0\})^n$ with $|\beta| + |\varrho| \le q$, and $\Gamma_{\alpha,\beta,\varrho}^j$ are real numbers determined by Γ. Moreover, we have $y_{(0)}^j \circ \Gamma(x, y) = y^j$.

We identify Γ with $(\Gamma_{\alpha,\beta,\varrho}^j)$. Using the invariance of $\mathcal{E}$ with respect to the base homotheties $(t^1x^1, \dots, t^mx^m, y^1, \dots, y^n)$ for $t^j > 0$, we get the homogeneity condition

$$(t^1)^2\Phi_w^l(\Gamma_{\alpha,\beta,\varrho}^j) = \Phi_w^l(t^{\alpha+\beta}\Gamma_{\alpha,\beta,\varrho}^j).$$

Then by the homogeneous function theorem we can write

$$(*)\qquad \Phi^l_w(\Gamma) = \sum a^\varrho_j \Gamma^j_{(2,0,\dots,0),(0),\varrho} + \sum b^\varrho_j \Gamma^j_{(1,0\dots,0),(1,0,\dots,0),\varrho} + \sum c^{\varrho_1,\varrho_2}_{j_1,j_2} \Gamma^{j_1}_{(1,0,\dots,0),(0),\varrho_1} \Gamma^{j_2}_{(1,0,\dots,0),(0),\varrho_2}$$

for some uniquely determined real numbers $a^\varrho_j = a^{\varrho,l}_j(w)$, $b^\varrho_j = b^{\varrho,l}_j(w)$ and $c^{\varrho_1,\varrho_2}_{j_1,j_2} = c^{\varrho_1,\varrho_2,l}_{j_1,j_2}(w)$ (smoothly depending on w), where the first sum is over all $j = 1,\dots,n$ and all $\varrho \in (\mathbb{N}\cup\{0\})^n$ with $|\varrho| \le q$, the second sum is over all $j = 1,\dots,n$ and all $\varrho \in (\mathbb{N}\cup\{0\})^n$ with $|\varrho| \le q-1$ and the third sum is over all $(\varrho_1, j_1) \le (\varrho_2, j_2)$ for $j_1, j_2 = 1,\dots,n$ and $\varrho_1, \varrho_2 \in (\mathbb{N}\cup\{0\})^n$ with $|\varrho_1| \le q$ and $|\varrho_2| \le q$ (here $\le$ means an ordering). Of course $(2,0,\dots,0), (1,0,\dots,0), (0) \in (\mathbb{N}\cup\{0\})^m$.

LEMMA 4. *Assume that all* a^ϱ_j, *all* b^ϱ_j *and all* $c^{\varrho_1,\varrho_2}_{j_1,j_2}$ *defined by* $(*)$ *are* 0 *for all* $w \in T^A_0\mathbb{R}^n$ *and all* $l = 1,\dots,\dim A$. *Then* $\mathcal{E} = 0$.

Proof. This is obvious in view of the previous lemma. ∎

LEMMA 5. *We have*

$$a^\varrho_j + b^\varrho_j = 0$$

for all j, ϱ, w *in question and* $l = 1,\dots,\dim A$.

Proof. Fix ϱ_0 and j_0. Choose $\Gamma = (\Gamma^j_{\alpha,\beta,\varrho})$ such that $\Gamma^{j_0}_{(1,0,\dots,0),(0),\varrho_0} = 1$, and $\Gamma^j_{\alpha,\beta,\varrho} = 0$ for other (j,α,β,ϱ) with $|\alpha| \ge 1$. Let $\varphi = (x^1 + \frac{1}{2}(x^1)^2, x^2,\dots,x^m, y^1,\dots,y^n)^{-1}$. Using the invariance of $\mathcal{E}$ with respect to φ we have

$$\Phi^l_w(\varphi_*\Gamma) = \Phi^l_w(\Gamma)$$

because φ preserves w, v^l and $u_0 \odot u_0$. Set

$$\Phi^l_w(\Gamma) = a.$$

We have $j^q_{(0,0)}(\varphi_*\Gamma) = (\widetilde{\Gamma}^j_{\alpha,\beta,\varrho})$, where

$$\widetilde{\Gamma}^{j_0}_{(1,0,\dots,0),(0),\varrho_0} = 1,\qquad \widetilde{\Gamma}^{j_0}_{(2,0,\dots,0),(0),\varrho_0} = 1,\qquad \widetilde{\Gamma}^{j_0}_{(1,0,\dots,0),(1,0,\dots,0),\varrho_0} = 1$$

and other $\widetilde{\Gamma}^j_{\alpha,\beta,\varrho}$ are zero for $|\alpha| = 1, 2$. (Indeed,

$$\Gamma(z,y) = j^2_z(y + y^{\varrho_0}(x^1 - z^1)e_{j_0}) \in J^2_z(\mathbb{R}^{m,n})_y,$$

where $\{e_j\}$ is the canonical basis in $\mathbb{R}^n$. Then

$$(\varphi_*\Gamma)(z,y) = j^2_z\bigg(y + y^{\varrho_0}\bigg(x^1 + \frac{1}{2}\,(x^1)^2 - z^1 - \frac{1}{2}\,(z^1)^2\bigg)e_{j_0}\bigg).$$

This implies the formulas.) Then

$$\Phi^l_w(\varphi_*\Gamma) = a^{\varrho_0}_{j_0} + b^{\varrho_0}_{j_0} + a.$$

Hence $a^{\varrho_0}_{j_0} + b^{\varrho_0}_{j_0} = 0$. ∎

Lemma 6. *Suppose that all a_j^ϱ defined by $(*)$ are zero for any w in question and $l = 1, \dots, \dim A$. Then $\mathcal{E} = 0$.*

Proof. By assumption and Lemma 5, all b_j^ϱ are zero. Then it is sufficient to show that $c_{j_1,j_2}^{\varrho_1,\varrho_2} = 0$ for all $\varrho_1, \varrho_2, j_1, j_2, w$ and l in question.

Fix $\varrho_1, \varrho_2, j_1, j_2, l, w$. Let $a, b \in \mathbb{R}$. Let Γ^0 be the trivial second order connection on $\mathbb{R}^{m,n}$ given by

$$\Gamma^0(z, y) = j_z^2(y) \in J_z^2(\mathbb{R}^{m,n})_y.$$

Then by $(*)$ we have

$$\Phi_w^l(\Gamma^0) = 0.$$

Choose an $\mathcal{FM}_{m,n}$-map $\psi : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$ given by

$$\psi(x, y) = (x, y + ax^1 y^{\varrho_1} e_{j_1} + bx^1 y^{\varrho_2} e_{j_2}).$$

Then

$$\Phi_w^l(\psi_* \Gamma^0) = 0$$

because of the invariance of $\mathcal{E}$, u_0, w and v^l with respect to ψ. Write $\psi^{-1}(x, y) = (x, \widetilde{y})$. Then

$$(\psi_* \Gamma^0)(z, y) = j_z^2(\widetilde{y} + ax^1 \widetilde{y}^{\varrho_1} e_{j_1} + bx^1 \widetilde{y}^{\varrho_2} e_{j_2}).$$

Then by $(*)$, $\Phi_w^l(\psi_* \Gamma^0)$ is a polynomial in a and b with the coefficient of ab equal to $c_{j_1,j_2}^{\varrho_1,\varrho_2}$ as all b_j^ϱ are zero. Therefore $c_{j_1,j_2}^{\varrho_1,\varrho_2} = 0$. ■

Lemma 7. *Suppose that all $a_j^{(0)}$ defined by $(*)$ are zero for any w in question and $l = 1, \dots, \dim A$. Then $\mathcal{E} = 0$.*

Proof. For any $\varrho \in (\mathbb{N} \cup \{0\})^n$ and $j = 1, \dots, n$, let $\Gamma^{\varrho,j}$ be the second order connection on $\mathbb{R}^{m,n}$ given by

$$\Gamma^{\varrho,j}(z, y) = j_z^2(y + (x^1 - z^1)^2 y^\varrho e_j), \qquad (z, y) \in \mathbb{R}^{m,n}.$$

By $(*)$ and the assumption of the lemma we have

$$d_w v^l \circ \langle \mathcal{E}(\Gamma^{(0),j})(w), u_0 \odot u_0 \rangle = \Phi_w^l(\Gamma^{(0),j}) = a_j^{(0),l}(w) = 0$$

for any $w \in (V^A \mathbb{R}^{m,n})_{(0,0)}$, $j = 1, \dots, n$ and $l = 1, \dots, \dim_{\mathbb{R}} A$. Then by the invariance of $\mathcal{E}$ with respect to the permutations of fibred coordinates we have

$$d_w v^l \circ \langle \mathcal{E}(\Gamma^{(0),j})(w), u_0 \odot u_0 \rangle = 0$$

for any $w \in (V^A \mathbb{R}^{m,n})_{(0,0)}$, $j = 1, \dots, n$ and $l = 1, \dots, \dim_{\mathbb{R}} A^n$. Therefore

$$\langle \mathcal{E}(\Gamma^{(0),j})(w), u_0 \odot u_0 \rangle = 0$$

for any $w \in (V^A \mathbb{R}^{m,n})_{(0,0)}$ and $j = 1, \dots, n$. Let $\varrho \in (\mathbb{N} \cup \{0\})^n$, $1 \le |\varrho| \le q$, $j = 1, \dots, n$. Let $\tau \in \mathbb{R}$ be sufficiently small. Consider an $\mathcal{FM}_{m,n}$-map

$\varphi^{\varrho j\tau} : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$, $\varphi^{\varrho j\tau}(x,y) = (x, y + \tau y^{\varrho+1_j} e_j)$ (defined near $(0,0)$). We see that

$$(\varphi_*^{\varrho j\tau} \Gamma^{(0),j})(z,y) = j_z^2(y + (x^1 - z^1)^2 e_j + \tau(\varrho_j + 1)(x^1 - z^1)^2 y^\varrho e_j + \cdots).$$

Then using the invariance of $\mathcal{E}$ with respect to $\varphi^{\varrho j\tau}$ we get

$$\langle \mathcal{E}(\varphi_*^{\varrho j\tau} \Gamma^{(0),j})(w), u_0 \odot u_0 \rangle = 0$$

for all $w \in (V^A \mathbb{R}^{m,n})_{(0,0)}$. The left hand side of the last formula is a polynomial in τ. The coefficient of $\tau = \tau^1$ in this polynomial is $(\varrho_j + 1)\langle \mathcal{E}(\Gamma^{\varrho,j})(w), u_0 \odot u_0 \rangle$. Then $\langle \mathcal{E}(\Gamma^{\varrho,j})(w), u_0 \odot u_0 \rangle = 0$ for any $w \in (V^A \mathbb{R}^{m,n})_{(0,0)}$. Thus $a_j^\varrho = 0$ for all l and w in question. Then $\mathcal{E} = 0$ by Lemma 6. ■

Lemma 8. *Suppose that all $a_j^{(0)}$ defined by $(*)$ are zero for $l = 1, \ldots, \dim A$ and $w = 0 \in T_0^A \mathbb{R}^n$. Then $\mathcal{E} = 0$.*

Proof. By $(*)$ we have

$$\Phi_w^l(\Gamma^{(0),j}) = a_j^{(0)} = a_j^{(0),l}(w),$$

where $\Gamma^{(0),j}$ is defined in the proof of Lemma 7. Let $h_t = \mathrm{id}_{\mathbb{R}^m} \times t\, \mathrm{id}_{\mathbb{R}^n}$ be the fibre homothety. Then $((h_t)_* \Gamma^{(0),j})(z,y) = j_z^2(y + t(x^1 - z^1)^2 e_j)$, and then by $(*)$ we have

$$\Phi_{tw}^l((h_t)_* \Gamma^{(0),j}) = t a_j^{(0),l}(tw).$$

Hence by the invariance of $\mathcal{E}$ with respect to the fibre homotheties h_t we get $\Phi_{tw}^l((h_t)_* \Gamma^{(0),j}) = t\Phi_w^l(\Gamma^{(0),j})$, and therefore

$$a_j^{(0),l}(w) = a_j^{(0),l}(tw).$$

Then putting $t \to 0$ and using the assumption of the lemma we see that $a_j^{(0)} = 0$ for any l and w in question. Then Lemma 7 ends the proof. ■

Lemma 9. *Suppose that $a_1^{(0)} = 0$ for $w = 0 \in T_0^A \mathbb{R}^n$ and $l = 1, \ldots, \dim A$. Then $\mathcal{E} = 0$.*

Proof. Let $a_t = (x^1, tx^2, \ldots, tx^m, y^1, \ldots, y^n)$ for $t \neq 0$ be $\mathcal{FM}_{m,n}$-maps. Clearly, $((a_t)_* \Gamma^{(0),j})(y,z) = j_z^2(y + t(x^1 - z^1)^2 e_j)$ for $j = 2, \ldots, n$. Then by $(*)$ and the invariance of $\mathcal{E}$ with respect to a_t we get

$$a_j^{(0),l}(0) = t a_j^{(0),l}(0)$$

for $j = 2, \ldots, n$. Then $a_j^{(0),l}(0) = 0$ for $j = 1, \ldots, n$ (for $j = 1$ the equality holds by assumption). Now Lemma 8 ends the proof. ■

Proof of Proposition 3. This is an immediate consequence of Lemma 9. ■

Proof of Theorem 1. Let $\Gamma^{(0),1}$ be the second order connection on $\mathbb{R}^{m,n}$ as in the proof of Lemma 7. Let $\mathcal{E}^a : J^2 \rightsquigarrow S^2T^* \otimes VV^A$ be the operators from Example 2 for $a \in A$. Let (a_ν) be a basis of A, and let (v^l) correspond to (a_ν) (as at the beginning of Section 3). One can show by a standard argument that the numbers $\Phi_0^l(\Gamma^{(0),1}) = a_1^{(0),l}(0)$ (see $(*)$) for $\mathcal{E} = \mathcal{E}^{a_\nu}$ are proportional (with a non-zero coefficient) to the Kronecker delta δ_ν^l. Hence the operators $\mathcal{E}^{a_\nu}$ are linearly independent. Theorem 1 is an immediate consequence of Propositions 1–3 by a dimension argument. ∎

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