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The transmission problem with boundary conditions given by real measures

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Abstract. The unique solvability of the problem $\Delta u = 0$ in $G^+ \cup G^-$, $u_+ - au_- = f$ on ∂G^+, $n^+ \cdot \nabla u_+ - bn^+ \cdot \nabla u_- = g$ on ∂G^+ is proved. Here a, b are positive constants and g is a real measure. The solution is constructed using the boundary integral equation method.

1. Introduction. V. G. Maz'ya, J. Král and their collaborators studied the weak Neumann problem for the Laplace equation with boundary condition given by a real measure μ using the integral equation method (see [8], [9], [13]): The function u is a weak solution of the Neumann problem

$$\Delta u = 0 \quad \text{in } G,$$
$$\frac{\partial u}{\partial n} = \mu \quad \text{on } \partial G,$$

if u is a harmonic function in G, $|\nabla u| \in L^1(H)$ for each bounded open subset H of G, and for each infinitely differentiable function φ with compact support we have

$$\int_G \nabla u \cdot \nabla \varphi \, d\mathcal{H}_m = \int_{\partial G} \varphi \, d\mu.$$

For a given open set G with compact boundary ∂G they looked for a solution in the form of a single layer potential $\mathcal{U}\nu$ corresponding to a real measure ν on ∂G. They proved that one obtains an integral equation $\widetilde{T}\nu = \mu$ with a bounded linear operator $\widetilde{T}$ on the space of finite real measures on ∂G if and only if the set G has bounded cyclic variation. They restricted considerations to this case. (Note that open sets with piecewise-smooth boundary have bounded cyclic variation.) A necessary and sufficient condition for the solvability of the equation $\widetilde{T}\nu = \mu$ has been stated under the assumption

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that the essential spectral radius of the operator $\widetilde{T} - \frac{1}{2}I$, where I is the identity operator, is smaller than $\frac{1}{2}$ (see [14]). (Observe that this condition is fulfilled for G with boundary of class $C^{1+\alpha}$, for convex domains, for domains with piecewise-smooth boundary in $\mathbb{R}^3$ and for some class of domains with piecewise-smooth boundary in higher dimensional spaces; see [8], [22], [26], [14], [6]. We remark that such sets may not have Lipschitz boundary.) The solution of the integral equation $\widetilde{T}\nu = \mu$ was expressed in the form of a Neumann series first for convex domains by J. Král and I. Netuka (see [11], [8]) and later for general open sets by D. Medková (see [15], [16]). Similar results have been proved for the Robin problem for the Laplace equation (see [23]–[25], [15]).

A solution of the weak Neumann problem with homogeneous boundary condition may be nonconstant. This is evident for G unbounded because we have no restriction on the behaviour of a solution at infinity. For G bounded this is a bit surprising result, proved in [21, Example 2.1]. Uniqueness up to an additive constant was proved under the condition that a solution u is continuously extendible onto the closure of G and $u(x) \to 0$ as $|x| \to \infty$ (see [18, Theorem 2] and [19, Lemma 3]).

In this paper, the following transmission problem is studied using the integral equation method:

$$\begin{aligned} &\Delta u = 0 && \text{in } G^+ \cup G^-, \\ &u_+ - a u_- = f && \text{on } \partial G^+, \\ &\frac{\partial u_+}{\partial n^+} - b\,\frac{\partial u_-}{\partial n^+} = \mu && \text{on } \partial G^+. \end{aligned}$$

Here μ is a real measure on ∂G^+, and a, b are positive constants. To ensure the uniqueness of the solution we will suppose that u is continuously extendible onto the closure of G^+ and onto the closure of G^- and that $u(x) \to 0$ as $|x| \to \infty$. Here $u_\pm$ is the limit of u with respect to $G^\pm$. The first boundary condition is satisfied in the classical sense and the second one is satisfied in the weak sense. We suppose that G^+ is a bounded open set with bounded cyclic variation, and $G^- = \mathbb{R}^m \setminus \operatorname{cl} G^+$. (We do not suppose that G^+ and G^- are connected.) We look for a solution in the form $u = \mathcal{D}f + \mathcal{U}\nu$ in G^+, $u = (\mathcal{D}f + \mathcal{U}\nu)/a$ in G^-, where $\mathcal{D}f$ is the double layer potential with density f and $\mathcal{U}\nu$ is a single layer potential corresponding to an unknown measure ν on ∂G^+. We get an integral equation $T\nu = \widetilde{\mu}$ (see §4). Under the assumption that the essential spectral radius of the operator $\widetilde{T} - \frac{1}{2}I$ is smaller than $\frac{1}{2}$ necessary and sufficient conditions for the solvability of the problem are stated and the uniqueness of the solution is proved. Moreover, the solution of the equation $T\nu = \widetilde{\mu}$ is expressed in the form of a Neumann series. More precisely, if $\alpha > \alpha_0$, where α_0 is a constant depending on G^+,

a and b (see Theorem 5.3), then

$$\nu = \alpha^{-1} \sum_{n=0}^{\infty} (I - \alpha^{-1} T)^n \widetilde{\mu}.$$

This enables us to use the successive approximation method for obtaining an approximate solution of the equation $T\nu = \widetilde{\mu}$.

2. Formulation of the problem. Suppose that $G^+ \subset \mathbb{R}^m$ $(m > 2)$ is a bounded open set. Set $G^- = \mathbb{R}^m \setminus \operatorname{cl} G^+$, where $\operatorname{cl} G^+$ is the closure of G^+. We suppose that $\partial G^+ = \partial G^-$ where ∂G^+ is the boundary of G^+. If u is a function on $\mathbb{R}^m \setminus \partial G^+$ and $x \in \partial G^+$ we denote by $u_+(x)$ the limit of u at x with respect to G^+ and by $u_-(x)$ the limit of u at x with respect to G^-.

We study generalized solutions of the following transmission problem:

$$\Delta u = 0 \quad \text{in } G^+, \tag{1}$$

$$\Delta u = 0 \quad \text{in } G^-, \tag{2}$$

$$u_+ - a u_- = f \quad \text{on } \partial G^+, \tag{3}$$

$$\frac{\partial u_+}{\partial n^+} - b \frac{\partial u_-}{\partial n^+} = g \quad \text{on } \partial G^+, \tag{4}$$

$$\lim_{|x| \to \infty} u(x) = 0. \tag{5}$$

Here n^+ is the unit outer normal of G^+, and a, b are positive constants.

Denote by $\mathcal{H}_k$ the k-dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesgue measure in $\mathbb{R}^k$. If G^+ has a smooth boundary and u is a classical solution of the above problem then Green's formula yields

$$\int_{G^+} \nabla u \cdot \nabla \varphi \, d\mathcal{H}_m + b \int_{G^-} \nabla u \cdot \nabla \varphi \, d\mathcal{H}_m = \int_{\partial G^+} g \varphi \, d\mathcal{H}_{m-1}$$

for each $\varphi \in \mathcal{D}$ (= the space of all compactly supported infinitely differentiable functions on $\mathbb{R}^m$).

Suppose that $G \subset \mathbb{R}^m$ is an open set with compact boundary. If u is a harmonic function in G such that

$$\int_H |\nabla u| \, d\mathcal{H}_m < \infty \tag{6}$$

for all bounded open subsets H of G, we define the *weak normal derivative* $N^G u$ of u as the distribution

$$\langle N^G u, \varphi \rangle = \int_G \nabla u \cdot \nabla \varphi \, d\mathcal{H}_m \quad \text{for } \varphi \in \mathcal{D}. \tag{7}$$

Suppose that G^+ has a smooth boundary and u is a classical solution of the problem (1)–(5). Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ onto ∂G^+. Then

$N^{G^+}u + bN^{G^-}u = g\mathcal{H}$. This motivates the following weak formulation of the problem:

Let G^+ be a bounded open set with $\mathcal{H}_{m-1}(\partial G^+) < \infty$. Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ onto ∂G^+. If $K \subset \mathbb{R}^m$ is a compact set denote by $\mathcal{C}'(K)$ the Banach space of all finite real Borel measures with support in K, with the total variation norm. Let a, b be positive constants, $f \in \mathcal{C}(\partial G^+)$, $\mu \in \mathcal{C}'(\partial G^+)$. We say that a function u defined in $\mathbb{R}^m \setminus \partial G^+$ is a weak solution of the transmission problem

$$\Delta u = 0 \quad \text{in } G^+, \tag{8}$$

$$\Delta u = 0 \quad \text{in } G^-, \tag{9}$$

$$u_+ - au_- = f \quad \text{on } \partial G^+, \tag{10}$$

$$\frac{\partial u_+}{\partial n^+} - b\,\frac{\partial u_-}{\partial n^+} = \mu, \tag{11}$$

$$\lim_{|x|\to\infty} u(x) = 0 \tag{12}$$

if $u \in \mathcal{C}^\infty(G^+ \cup G^-)$, u is continuously extendible onto $\operatorname{cl} G^+$ and onto $\operatorname{cl} G^-$, there are $N^{G^+}u, N^{G^-}u \in \mathcal{C}'(\partial G^+)$, the relations (8), (9), (10), (12) hold and $N^{G^+}u + bN^{G^-}u = \mu$. If $\mu = g\mathcal{H}$ we can say that u is a weak solution of the problem (1)–(5).

3. Potentials. For $x, y \in \mathbb{R}^m$ and $r > 0$ define $\Omega(x;r) = \{z \in \mathbb{R}^m; |z - x| < r\}$ and

$$h_x(y) = \begin{cases} (m-2)^{-1}(\mathcal{H}_{m-1}(\partial\Omega(0;1)))^{-1}|x-y|^{2-m} & \text{for } x \neq y, \\ \infty & \text{for } x = y. \end{cases}$$

If ν is a finite real Borel measure, write

$$\mathcal{U}\nu(x) = \int_{\mathbb{R}^m} h_x(y)\,d\nu(y)$$

whenever this integral makes sense.

Suppose that G is an open subset of $\mathbb{R}^m$ with compact boundary and $\nu \in \mathcal{C}'(\partial G)$. Then the *single layer potential* $\mathcal{U}\nu$ corresponding to ν is a harmonic function in $\mathbb{R}^m \setminus \partial G$ and $|\nabla \mathcal{U}\nu|$ is integrable in each bounded open subset of G (see [8, Remark on p. 9]).

For $x \in \mathbb{R}^m$ put

$$v^G(x) = \sup\Big\{\int_G \nabla\phi \cdot \nabla h_x\,d\mathcal{H}_m;\ \phi \in \mathcal{D},\ |\phi| \leq 1,\ \operatorname{spt}\phi \subset \mathbb{R}^m \setminus \{x\}\Big\},$$

$$V^G = \sup_{x \in \partial G} v^G(x).$$

It was shown in [8] that $N^G\mathcal{U}\nu \in \mathcal{C}'(\partial G)$ for each $\nu \in \mathcal{C}'(\partial G)$ if and only if $V^G < \infty$. There are more geometrical characterizations of $v^G(x)$ which ensure $V^G < \infty$ for G convex or for G with $\partial G \subset \bigcup_{i=1}^k L_i$, where the L_i are $(m-1)$-dimensional Lyapunov surfaces (i.e. of class $\mathcal{C}^{1+\alpha}$). Denote by

$$\partial_{\mathrm{e}} G = \{x \in \mathbb{R}^m;\ \bar{d}_G(x) > 0,\ \bar{d}_{\mathbb{R}^m \setminus G}(x) > 0\}$$

the *essential boundary* of G where

$$\bar{d}_M(x) = \limsup_{r \to 0_+} \frac{\mathcal{H}_m(M \cap \Omega(x;r))}{\mathcal{H}_m(\Omega(x;r))}$$

is the upper density of M at x. Then

$$v^G(x) = (\mathcal{H}_{m-1}(\partial\Omega(0;1)))^{-1} \int_{\partial\Omega(0;1)} n(\theta, x)\, d\mathcal{H}_{m-1}(\theta), \tag{13}$$

where $n(\theta, x)$ is the number of points of $\partial_{\mathrm{e}} G \cap \{x + t\theta;\ t > 0\}$ (see [2]). This expression is a modification of a similar expression in [8]. As a consequence we see that $V^G \le 1/2$ if G is convex. Since $v^G(x) \le V^G + 1/2$ by [8, Theorem 2.16], we see that if

$$\partial G \subset \bigcup_{i=1}^n \partial G_i$$

and $G_1, \ldots, G_n$ are convex then $V^G \le n$.

Let us recall another characterization of $v^G(x)$ using the notion of an exterior normal in Federer's sense. If $z \in \mathbb{R}^m$ and θ is a unit vector such that the symmetric difference of G and the half-space $\{x \in \mathbb{R}^m;\ (x - z) \cdot \theta < 0\}$ has m-dimensional density zero at z then $n^G(z) = \theta$ is termed the *exterior normal of G at z in Federer's sense*. (The symmetric difference of B and C is $(B \setminus C) \cup (C \setminus B)$.) If there is no exterior normal of G at z in this sense, we denote by $n^G(z)$ the zero vector in $\mathbb{R}^m$. (Note that the exterior normal in the classical sense is an exterior normal in Federer's sense.)

If $\mathcal{H}_{m-1}(\partial G) < \infty$ then

$$v^G(x) = \int_{\partial G} |n^G(y) \cdot \nabla h_x(y)|\, d\mathcal{H}_{m-1}(y) \tag{14}$$

for each $x \in \mathbb{R}^m$ (see [8, Lemma 2.15]).

Suppose

$$V^G < \infty.$$

Then $\mathcal{H}_{m-1}(\partial_{\mathrm{e}} G) < \infty$ and (14) holds (see [8, Chapter 2]). For each $x \in \mathbb{R}^m$ the density of G at x,

$$d_G(x) = \lim_{r \to 0_+} \frac{\mathcal{H}_m(G \cap \Omega(x;r))}{\mathcal{H}_m(\Omega(x;r))},$$

exists (see [8, Lemma 2.9]). If $\nu \in \mathcal{C}'(\partial G)$ and M is a Borel set then

$$N^G\mathcal{U}\nu(M) = \int_{M\cap\partial G} d_G(x)\,d\nu(x) + \int_{\partial G}\int_{M\cap\partial G} n^G(y)\cdot\nabla h_x(y)\,d\mathcal{H}_{m-1}(y)\,d\nu(x) \tag{15}$$

(see [8, pp. 73–74]).

Denote by $\mathcal{C}'_{\rm c}(\partial G)$ the set of all $\nu\in\mathcal{C}'(\partial G)$ for which there is a bounded and continuous function $\mathcal{U}_{\rm c}\nu$ in $\mathbb{R}^m$ such that $\mathcal{U}\nu=\mathcal{U}_{\rm c}\nu$ in $\mathbb{R}^m\setminus\partial G$. If $\nu\in\mathcal{C}'(\partial G)$ and there are constants $\alpha>m-2$ and $k>0$ such that $|\nu|(\Omega(x;r))\le kr^\alpha$ for all $x\in\mathbb{R}^m$ and all $r>0$ then $\nu\in\mathcal{C}'_{\rm c}(\partial G)$ (see [8, Lemma 2.18]).

For $f\in\mathcal{C}(\partial G)$ define

$$\mathcal{D}^G f(x) = \frac{1}{\mathcal{H}_{m-1}(\Omega(0;1))}\int_{\partial G}\frac{n^G(y)\cdot(y-x)}{|x-y|^m}\,f(y)\,d\mathcal{H}_{m-1}(y), \tag{16}$$

the double layer potential with density f. Then $\mathcal{D}^G f$ is a harmonic function in G which is continuously extendible onto $\operatorname{cl} G$. If $x\in\partial G$ then

$$\lim_{y\to x,\,y\in G}\mathcal{D}^G f(y) = \mathcal{D}^G f(x) + d_G(x)f(x) \tag{17}$$

(see [8, Theorem 2.19, Lemma 2.15, Proposition 2.8, Lemma 2.9]).

If L is a bounded linear operator on a Banach space X we denote by $\|L\|_{\rm ess}$ the *essential norm* of L, i.e. the distance of L from the space of all compact linear operators on X. The *essential spectral radius* of L is defined by

$$r_{\rm ess}L = \lim_{n\to\infty}\|L^n\|_{\rm ess}^{1/n}.$$

The operator $N^G\mathcal{U}:\nu\mapsto N^G\mathcal{U}\nu$ is a bounded linear operator in $\mathcal{C}'(\partial G)$ (see [8, Theorem 1.13]). We shall need the condition $r_{\rm ess}(N^G\mathcal{U}-\frac12 I)<\frac12$. (Here I denotes the identity operator.) It is well-known that it holds for sets with a smooth boundary (of class $C^{1+\alpha}$) (see [9]) and for convex sets (see [22]). A. Rathsfeld showed in [26], [27] that polyhedral cones in $\mathbb{R}^3$ have this property. (By a *polyhedral cone* in $\mathbb{R}^3$ we mean an open set Ω whose boundary is locally a hypersurface (i.e. every point of $\partial\Omega$ has a neighbourhood in $\partial\Omega$ which is homeomorphic to $\mathbb{R}^2$) and $\partial\Omega$ is formed by a finite number of plane angles. By a *polyhedral open set* with bounded boundary in $\mathbb{R}^3$ we mean an open set Ω whose boundary is locally a hypersurface and $\partial\Omega$ is formed by a finite number of polygons. (Observe that a polyhedral open set may not have Lipschitz boundary.) In [14] it was shown that the condition $r_{\rm ess}(N^G\mathcal{U}-\frac12 I)<\frac12$ has a local character. Hence it holds for $G\subset\mathbb{R}^3$ such that for each $x\in\partial G$ there are $r(x)>0$, a domain D_x which is polyhedral or smooth or convex or a complement of a convex domain and a diffeomorphism $\psi_x\colon\Omega(x;r(x))\to\mathbb{R}^3$ of class $C^{1+\alpha}$, where $\alpha>0$, such that $\psi_x(G\cap\Omega(x;r(x)))=D_x\cap\psi_x(\Omega(x;r(x)))$. V. G. Maz'ya and N. V. Grachev proved this condition for several types of sets with piecewise-smooth boundary in Euclidean space (see [6]).

LEMMA 3.1. *Let $G \subset \mathbb{R}^m$ be an open set with compact boundary. Put $C = \mathbb{R}^m \setminus \operatorname{cl} G$. Suppose $\partial G = \partial C$, $V^G < \infty$ and $r_{\mathrm{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$. Then $\mathcal{H}_{m-1}(\partial G) < \infty$, $V^C < \infty$, $r_{\mathrm{ess}}(N^C\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$,*

$$0 < \inf_{x\in\partial G} d_G(x) \le \sup_{x\in\partial G} d_G(x) < \infty$$

and $G \cup C$ has finitely many components. Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ onto ∂G. If $f \in L^1(\mathcal{H})$ then

$$\widetilde{T}f(x) = d_G(x)f(x) - \int_{\partial G} f(y)n^G(x)\cdot\nabla h_x(y)\,d\mathcal{H}_{m-1}(y)$$

makes sense for almost all $x \in \partial G$, and $N^G\mathcal{U}(f\mathcal{H}) = (\widetilde{T}f)\mathcal{H}$. If G is bounded then

$$\widetilde{T}f(x) = -\int_{\partial G} (f(x)n^G(y)\cdot\nabla h_x(y) + f(y)n^G(x)\cdot\nabla h_x(y))\,d\mathcal{H}_{m-1}(y).$$

Proof. We have $\mathcal{H}_{m-1}(\partial G) < \infty$ and

$$0 < \inf_{x\in\partial G} d_G(x) \le \sup_{x\in\partial G} d_G(x) < \infty,$$

and G has finitely many components by [15, Corollary 1] and [18, Lemma 3]. Since $\mathcal{H}_{m-1}(\partial G) < \infty$ we obtain $\partial_e G = \partial_e C$ and thus $V^C < \infty$ by (13). Since $\mathcal{H}_{m-1}(\partial G) < \infty$ we deduce that $N^C\mathcal{U} = I - N^G\mathcal{U}$. Hence $r_{\mathrm{ess}}(N^C\mathcal{U} - \frac{1}{2}I) = r_{\mathrm{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$. If $f \in L^1(\mathcal{H})$ then $\widetilde{T}f \in L^1(\mathcal{H})$ and $N^G\mathcal{U}(f\mathcal{H}) = (\widetilde{T}f)\mathcal{H}$ by [10]. Suppose that G is bounded and $x \in \partial G$. According to [8, Lemma 2.9 and Proposition 2.8] we have

$$d_G(x) = -\int_{\partial G} n^G(y)\cdot\nabla h_x(y)\,d\mathcal{H}_{m-1}(y).$$

LEMMA 3.2. *Let $G \subset \mathbb{R}^m$ be an open set with compact boundary $\partial G = \partial(\mathbb{R}^m \setminus \operatorname{cl} G)$, $V^G < \infty$ and $r_{\mathrm{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$. If H is a bounded component of G then there is $\nu \in \mathcal{C}'_c(\partial G)$ so that $\mathcal{U}\nu = 1$ in H and $\mathcal{U}\nu = 0$ in $G \setminus H$.*

Proof. The set G has finitely many components by Lemma 3.1. Let $G_1, \dots, G_n$ be all bounded components of G. The codimension of the range of $N^G\mathcal{U}$ is n by [14, Theorem 1.14]. Since $r_{\mathrm{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$ the operator $N^G\mathcal{U}$ is a Fredholm operator with index 0 by [14, Lemma 1.2]. Therefore the dimension of its kernel is n. If ν is a real measure in this kernel then $\nu \in \mathcal{C}'_c(\partial G)$ and there are constants $c_1, \dots, c_{n+1}$ such that $\mathcal{U}\nu = c_i$ in G_i, $i = 1, \dots, n$, and $\mathcal{U}\nu = c_{n+1}$ in $G \setminus \bigcup G_i$ (see [14, Theorem 1.12] and [16, Lemma 4]). Since $\mathcal{U}\nu(x) \to 0$ as $|x| \to \infty$ we deduce that $c_{n+1} = 0$. If $c_1 = \dots = c_n = 0$ then $\nu = 0$ by [14, Theorem 1.12]. Since the dimension of the kernel of $N^G\mathcal{U}$ is n we have $\nu \in \mathcal{C}'_c(\partial G)$ so that $\mathcal{U}\nu = 1$ in H and $\mathcal{U}\nu = 0$ in $G \setminus H$.

LEMMA 3.3. *Let $G \subset \mathbb{R}^m$ be an open set with compact boundary. Put $C = \mathbb{R}^m \setminus \mathrm{cl}\, G$. Suppose $\partial G = \partial C$, $V^G < \infty$ and $r_{\mathrm{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$. Let $w \in \mathcal{C}(\mathbb{R}^m)$ be a harmonic function in $G \cup C$ such that $w(x) \to 0$ as $|x| \to \infty$. If there is $N^G w \in \mathcal{C}'(\partial G)$ then there is $N^C w \in \mathcal{C}'(\partial G)$.*

Proof. According to [19, Lemma 3] there is $p \geq 1$ such that $w \in L^p(\mathbb{R}^m)$. [19, Theorem 2] implies that $N^G w \in \mathcal{C}'_c(\partial G)$. [18, Theorem 2] and Lemma 3.2 show that there is $\nu \in \mathcal{C}'(\partial G)$ such that $w = \mathcal{S}\nu$ in G. According to [18, Theorem 1] we have $\nu \in \mathcal{C}'_c(\partial G)$. The functions w and $\mathcal{U}_c\nu$ are harmonic in C, continuous in $\mathrm{cl}\, C$, $w = \mathcal{U}_c\nu$ on ∂G, and $w(x) \to 0$ and $\mathcal{U}_c\nu(x) \to 0$ as $|x| \to \infty$. The uniqueness of the Dirichlet problem implies that $w = \mathcal{U}_c\nu$ in C. Lemma 3.1 gives $V^C < \infty$. Hence $N^C w = N^C\mathcal{U}\nu \in \mathcal{C}'(\partial G)$.

PROPOSITION 3.4. *Let $G \subset \mathbb{R}^m$ be an open set with compact boundary. Put $C = \mathbb{R}^m \setminus \mathrm{cl}\, G$. Suppose $\partial G = \partial C$, $V^G < \infty$ and $r_{\mathrm{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$. Let $u \in \mathcal{C}(\mathrm{cl}\, G)$ be a harmonic function in G. If G is unbounded, suppose moreover that $u(x) \to 0$ as $|x| \to \infty$. Then there is $N^G u \in \mathcal{C}'(\partial G)$ if and only if there are $N^G\mathcal{D}^G u \in \mathcal{C}'(\partial G)$ and $N^C\mathcal{D}^G u \in \mathcal{C}'(\partial G)$.*

Proof. We have $r_{\mathrm{ess}}(N^C\mathcal{U} - \frac{1}{2}I) = r_{\mathrm{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$ by Lemma 3.1. According to [17, Theorem 1] the function u can be extended onto $\mathbb{R}^m$ so that $u \in \mathcal{C}(\mathbb{R}^m)$, u is a harmonic function in $G \cup C$ and $u(x) \to 0$ as $|x| \to \infty$. Since there is $N^G u \in \mathcal{C}'(\partial G)$ if and only if there is $N^C u \in \mathcal{C}'(\partial G)$ by Lemma 3.3 and $\mathcal{D}^G u = -\mathcal{D}^C u$ we can suppose that G is bounded.

Put

$$w(x) = \begin{cases} u(x) - \mathcal{D}^G u(x) & \text{for } x \in G, \\ -\mathcal{D}^G u(x) & \text{for } x \in C. \end{cases}$$

The function w is harmonic in $G \cup C$ and $w(x) \to 0$ as $|x| \to \infty$. Let $x \in \partial G$. Since $\mathcal{H}_m(\partial G) = 0$, using the boundary behaviour of the double layer potential we get

$$\lim_{y \to x,\, y \in G} w(y) = (1 - d_G)u(x) - \mathcal{D}^G u(x) = \lim_{y \to x,\, y \in C} w(y).$$

Thus $w \in \mathcal{C}(\mathbb{R}^m)$.

Suppose first that there are $N^G\mathcal{D}^G u \in \mathcal{C}'(\partial G)$ and $N^C\mathcal{D}^G u \in \mathcal{C}'(\partial G)$. Since there is $N^C w \in \mathcal{C}'(\partial G)$ Lemma 3.3 shows that there is $N^G w \in \mathcal{C}'(\partial G)$. Therefore $N^G u = N^G w + N^G\mathcal{D}^G u \in \mathcal{C}'(\partial G)$.

Suppose now that there is $N^G u \in \mathcal{C}'(\partial G)$. Then $u = \mathcal{U}(N^G u) - \mathcal{D}^G u$ in G by [18, Lemma 4]. Since $V^G < \infty$ we have $N^G\mathcal{D}^G u = N^G\mathcal{U}(N^G u) - N^G u \in \mathcal{C}'(\partial G)$. Since $N^G w(x) = N^G u(x) - N^G\mathcal{D}^G u \in \mathcal{C}'(\partial G)$ Lemma 3.3 gives $N^C\mathcal{D}^G u = -N^C w \in \mathcal{C}'(\partial G)$.

NOTATION 3.5. If G is an open subset of $\mathbb{R}^m$, $1 \le p \le \infty$, and $k \in \mathbb{N}$, denote by $W^{k,p}(G)$ the set of all functions $u \in L^p(G)$ for which the partial derivatives up to order k in the sense of distributions are in $L^p(G)$.

LEMMA 3.6. *Let $K \subset \mathbb{R}^m$ be a compact set with $\mathcal{H}_{m-1}(K) < \infty$. Put $F = \{y \in \mathbb{R}^{m-1};\ \{t \in \mathbb{R}^1;\ [t,y] \in K\}$ is infinite$\}$. Then $\mathcal{H}_{m-1}(F) = 0$.*

Proof. Denote by π the projection from $\mathbb{R}^m$ onto $\mathbb{R}^{m-1}$ defined by $\pi(t,y) = y$. If n is a positive integer and j is an integer, set $K_{n,j} = K \cap \{[t,y];\ y \in \mathbb{R}^{m-1},\ t \in [j2^{-n},\ (j+1)2^{-n})\}$ and $F_{n,j} = \pi(K_{n,j})$. As $|\pi(t_1,y_1) - \pi(t_2,y_2)| \le |[t_1,y_1] - [t_2,y_2]|$ we have $\mathcal{H}_{m-1}(F_{n,j}) \le \mathcal{H}_{m-1}(K_{n,j})$. If k, n are positive integers denote by F_n^k the set of $y \in \mathbb{R}^{m-1}$ which are in m sets $F_{n,j}$. Then

$$k\mathcal{H}_{m-1}(F_n^k) \le \sum_j \mathcal{H}_{m-1}(F_{n,j}) \le \sum_j \mathcal{H}_{m-1}(K_{n,j}) = \mathcal{H}_{m-1}(K) < \infty.$$

Since $F \subset \bigcap_{k=1}^\infty \bigcup_{n=1}^\infty F_n^k$ and

$$\mathcal{H}_{m-1}\Big(\bigcap_{k=1}^\infty \bigcup_{n=1}^\infty F_n^k\Big) \le \lim_{k\to\infty} \lim_{n\to\infty} \mathcal{H}_{m-1}(F_n^k) \le \lim_{k\to\infty} \lim_{n\to\infty} \mathcal{H}_{m-1}(K)/k = 0$$

we obtain $\mathcal{H}_{m-1}(F) = 0$.

LEMMA 3.7. *Let $G \subset \mathbb{R}^m$ be an open set with compact boundary. Put $C = \mathbb{R}^m \setminus \operatorname{cl} G$. Suppose $\partial G = \partial C$, $V^G < \infty$ and $r_{\mathrm{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$. Let $\nu \in \mathcal{C}'_{\mathrm{c}}(\partial G)$ and $\varphi \in \mathcal{D}$ be such that $\varphi = 1$ on a neighbourhood of ∂G. Then $\varphi\mathcal{U}_{\mathrm{c}}\nu \in W^{1,2}(\mathbb{R}^m) \cap \mathcal{C}(\mathbb{R}^m)$ and $|\nabla\mathcal{U}_{\mathrm{c}}\nu| \in L^2(\mathbb{R}^m)$.*

Proof. For $z \in \mathbb{R}^{m-1}$ write $V_z = \{t \in \mathbb{R}^1;\ [t,z] \in \mathbb{R}^m \setminus \partial G\}$ and $v_z(t) = \varphi(t,z)\mathcal{U}_{\mathrm{c}}\nu(t,z)$ for $t \in \mathbb{R}^1$. Since $\mathcal{H}_{m-1}(\partial G) < \infty$ by Lemma 3.1, we see from Lemma 3.6 that there is $F \subset \mathbb{R}^{m-1}$ with $\mathcal{H}_{m-1}(F) = 0$ such that $\mathbb{R}^1 \setminus V_z$ is finite for each $z \in \mathbb{R}^{m-1} \setminus F$. We infer from [8, Lemma 5.8] that $|\nabla\mathcal{U}_{\mathrm{c}}\nu| \in L^2(\mathbb{R}^m \setminus \partial G)$. Since $\mathcal{U}_{\mathrm{c}}\nu \in \mathcal{C}(\mathbb{R}^m) \cap C^\infty(\mathbb{R}^m \setminus \partial G)$ and $\varphi \in \mathcal{D}$ we deduce that $|\nabla(\varphi\mathcal{U}_{\mathrm{c}}\nu)| \in L^2(\mathbb{R}^m \setminus \partial G) \cap L^1(\mathbb{R}^m \setminus \partial G)$. Using Fubini's theorem we conclude that there is $\widetilde{F} \subset \mathbb{R}^{m-1}$ with $\mathcal{H}_{m-1}(\widetilde{F}) = 0$ such that $v'_z \in L^1(V_z)$ for each $z \in \mathbb{R}^{m-1} \setminus \widetilde{F}$. If $z \in \mathbb{R}^{m-1} \setminus (F \cup \widetilde{F})$ then $v_z \in \mathcal{C}(R^1) \cap \mathcal{C}^\infty(V_z)$, $v'_z \in L^1(V_z)$ and $\mathbb{R}^1 \setminus V_z$ is a finite set, which forces that v_z is an absolutely continuous function in $\mathbb{R}^1$. Similarly, $\varphi\mathcal{U}_{\mathrm{c}}\nu$ is absolutely continuous on almost all lines parallel to the coordinate axes. Since its partial derivatives belong to $L^2(\mathbb{R}^m)$, using [29, Theorem 2.1.4] we find that $\varphi\mathcal{U}_{\mathrm{c}}\nu \in W^{1,2}(\mathbb{R}^m)$.

PROPOSITION 3.8. *Let $G \subset \mathbb{R}^m$ be an open set with compact boundary. Put $C = \mathbb{R}^m \setminus \operatorname{cl} G$. Suppose $\partial G = \partial C$, $V^G < \infty$ and $r_{\mathrm{ess}}(N^G\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$. Denote by $\mathcal{C}_\nabla(\partial G)$ the set of all $f \in \mathcal{C}(\partial G)$ for which there are $N^G\mathcal{D}^G f \in \mathcal{C}'(\partial G)$ and $N^C\mathcal{D}^G f \in \mathcal{C}'(\partial G)$. If $f \in \mathcal{C}(\partial G)$ then $f \in \mathcal{C}_\nabla(\partial G)$ if and only if f can be extended to an $f \in \mathcal{C}(\mathbb{R}^m) \cap W^{1,2}(\mathbb{R}^m)$ such that Δf in* *the sense of distributions is a real measure. Thus* $\mathcal{C}^2(\mathbb{R}^m) \subset \mathcal{C}_\nabla(\partial G)$ *and* $\mathcal{C}(\mathbb{R}^m) \cap W^{1,2}(\mathbb{R}^m) \cap W^{2,1}(\mathbb{R}^m) \subset \mathcal{C}_\nabla(\partial G)$.

Proof. Since $V^C < \infty$ and $r_{\text{ess}}(N^C\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$ by Lemma 3.1, and $\mathcal{D}^C u = -\mathcal{D}^G u$, we can suppose that G is bounded.

Suppose first that $f \in \mathcal{C}(\mathbb{R}^m) \cap W^{1,2}(\mathbb{R}^m)$ and Δf in the sense of distributions is a real measure. Fix $\varphi \in \mathcal{D}$ with $\varphi = 1$ on a neighbourhood of $\operatorname{cl} G$. Put $u = f\varphi$. Since $\partial_j u = f\partial_j\varphi + \varphi\partial_j f$ and $\Delta u = f\Delta\varphi + 2\nabla\varphi\cdot\nabla f + \varphi\Delta f$ we can suppose that the support of f is compact. Set $\mu = \Delta f$. Then $f = \mathcal{U}\mu$ on $\mathbb{R}^m \setminus F$ where $F = \{x \in \mathbb{R}^m; \mathcal{U}|\mu|(x) = \infty\}$. Since F is a polar set, it has zero Newton capacity (see [12, Chapters I, III and VI]; cf. also [4]). Since $\mathcal{U}\mu \in W^{1,2}(\mathbb{R}^m)$ the real measure μ has finite energy (see [12, Chapter VI]).

Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition. The nonnegative measures μ^+, μ^- have finite energy (see [12, Chapters I and VI]). The potentials $\mathcal{U}\mu_+$, $\mathcal{U}\mu_-$ are lower semicontinuous in $\mathbb{R}^m$ (see [12, Theorem 1.3]) and finite on $\mathbb{R}^m \setminus F$. Since $\mathcal{U}\mu^+ - \mathcal{U}\mu^-$ is finite and continuous in $\mathbb{R}^m \setminus F$ the functions $\mathcal{U}\mu^+$, $\mathcal{U}\mu^-$ are continuous in $\mathbb{R}^m \setminus F$.

Denote by $\widetilde{\mu}_+$, $\widetilde{\mu}_-$ the balayages of the measures μ_+, μ_- relative to $\operatorname{cl} G$ (see [7, Chapter 11, §3]). Then $\widetilde{\mu}_+$, $\widetilde{\mu}_-$ are nonnegative measures from $\mathcal{C}'(\operatorname{cl} G)$ with finite energy such that $\mathcal{U}\widetilde{\mu}_+ \le \mathcal{U}\mu_+$, $\mathcal{U}\widetilde{\mu}_- \le \mathcal{U}\mu_-$ and $\mathcal{U}\widetilde{\mu}_+ = \mathcal{U}\mu_+$, $\mathcal{U}\widetilde{\mu}_- = \mathcal{U}\mu_-$ in G (see [7, Theorem 11.16] and [1, Theorem VIII.3]). The functions $\mathcal{U}\widetilde{\mu}_+$, $\mathcal{U}\widetilde{\mu}_-$ are continuous with respect to the fine topology (see [7, Chapter 10]). This topology is stronger than the ordinary topology.

Fix $x \in \partial G \setminus F$. Since $d_G(x) > 0$ every fine neighbourhood of x intersects G (see [7, Corollary 10.5] and [12, Theorem 5.11]) and the fine topology is stronger than the ordinary topology, and $\mathcal{U}\mu_+$, $\mathcal{U}\mu_-$ are continuous, we deduce that $\mathcal{U}\widetilde{\mu}_+(x) = \mathcal{U}\mu_+(x)$, $\mathcal{U}\widetilde{\mu}_-(x) = \mathcal{U}\mu_-(x)$. Thus $\mathcal{U}(\widetilde{\mu}_+ - \widetilde{\mu}_-) = \mathcal{U}\mu = f$ on $G \cup (\partial G \setminus F)$. Since $\operatorname{spt}\widetilde{\mu}_+ \subset \operatorname{cl} G$ and $\operatorname{spt}\widetilde{\mu}_- \subset \operatorname{cl} G$ the functions $\mathcal{U}\widetilde{\mu}_+$, $\mathcal{U}\widetilde{\mu}_-$ are harmonic in $\mathbb{R}^m \setminus \operatorname{cl} G$.

Fix $R > 0$ so that $\operatorname{cl} G \subset \Omega(0;R)$. Denote by $\widehat{\mu}_+$, $\widehat{\mu}_-$ the balayages of the measures $\widetilde{\mu}_+$, $\widetilde{\mu}_-$ relative to $\operatorname{cl}\Omega(0;R) \setminus G$. Then $\widehat{\mu}_+$, $\widehat{\mu}_-$ are nonnegative measures in $\mathcal{C}'(\operatorname{cl}\Omega(0;R) \setminus G)$ with finite energy such that $\mathcal{U}\widehat{\mu}_+ \le \mathcal{U}\widetilde{\mu}_+$, $\mathcal{U}\widehat{\mu}_- \le \mathcal{U}\widetilde{\mu}_-$ and $\mathcal{U}\widehat{\mu}_+ = \mathcal{U}\widetilde{\mu}_+$ and $\mathcal{U}\widehat{\mu}_- = \mathcal{U}\widetilde{\mu}_-$ in $\Omega(0;R) \setminus \operatorname{cl} G$. In the same way as above we prove that $\mathcal{U}\widehat{\mu}_+ = \mathcal{U}\widetilde{\mu}_+$ and $\mathcal{U}\widehat{\mu}_- = \mathcal{U}\widetilde{\mu}_-$ on $(\operatorname{cl}\Omega(0;R) \setminus \operatorname{cl} G) \cup (\partial G \setminus F)$. Thus $\mathcal{U}(\widehat{\mu}_+ - \widehat{\mu}_-) = f$ on $\partial G \setminus F$. Since $\mathcal{U}\widehat{\mu}_+$, $\mathcal{U}\widetilde{\mu}_+$ are harmonic functions in $\mathbb{R}^m \setminus \operatorname{cl}\Omega(0;R)$, continuous in $\mathbb{R}^m \setminus \Omega(0;R)$, tending to 0 at infinity, and $\mathcal{U}\widehat{\mu}_+ = \mathcal{U}\widetilde{\mu}_+$ on $\partial\Omega(0;R)$, the uniqueness of the solution of the Dirichlet problem implies that $\mathcal{U}\widehat{\mu}_+ = \mathcal{U}\widetilde{\mu}_+$ in $\mathbb{R}^m \setminus \Omega(0;R)$. Since $\mathcal{U}\widehat{\mu}_+ = \mathcal{U}\widetilde{\mu}_+$ in $\mathbb{R}^m \setminus \operatorname{cl} G$ the potential $\mathcal{U}\widehat{\mu}_+$ is harmonic in $\mathbb{R}^m \setminus \partial G$. Therefore $\widehat{\mu}_+ \in \mathcal{C}'(\partial G)$. Similarly, $\widehat{\mu}_- \in \mathcal{C}'(\partial G)$.

Denote by u the classical solution of the Dirichlet problem for the Laplace equation in G with the boundary condition f. Then $u = \mathcal{U}(\widehat{\mu}_+ - \widehat{\mu}_-)$ on

$\partial G \setminus F$. Thus $u = \mathcal{U}(\widehat{\mu}_+ - \widehat{\mu}_-)$ in G by [18, Lemma 1]. Since $V^G < \infty$ we have $N^G u = N^G \mathcal{U}(\widehat{\mu}_+ - \widehat{\mu}_-) \in \mathcal{C}'(\partial G)$. Proposition 3.4 shows that there are $N^G \mathcal{D}^G u \in \mathcal{C}'(\partial G)$ and $N^C \mathcal{D}^G u \in \mathcal{C}'(\partial G)$. Since $u = f$ on ∂G we obtain $f \in \mathcal{C}_\nabla(\partial G)$.

Suppose conversely that $f \in \mathcal{C}_\nabla(\partial G)$. Denote by u the classical solution of the Dirichlet problem in G with the boundary condition f. Then there is $N^G u \in \mathcal{C}'(\partial G)$ by Lemma 3.4. According to [18, Theorem 2] and Lemma 3.2, there is $\nu \in \mathcal{C}'(\partial G)$ such that $u = \mathcal{U}\nu$ in G. Fix $\varphi \in \mathcal{D}$ such that $\varphi = 1$ on a neighbourhood of ∂G. Then $\varphi \mathcal{U}_{\mathrm{c}}\nu$ is an extension of f such that $\varphi \mathcal{U}_{\mathrm{c}}\nu \in \mathcal{C}(\mathbb{R}^m) \cap W^{1,2}(\mathbb{R}^m)$ (see Lemma 3.7). According to [8, Remark 5.7], we have $\Delta \mathcal{U}\nu = \nu$ in the sense of distributions. Since $\mathcal{U}_{\mathrm{c}}\nu$ and $\mathcal{U}\nu$ differ on a set of zero Lebesgue measure we obtain $\Delta \mathcal{U}_{\mathrm{c}}\nu = \nu$. Since $\varphi \in \mathcal{D}$, $\mathcal{U}_{\mathrm{c}}\nu \in W^{1,2}(\mathbb{R}^m)$ and $\Delta(\varphi \mathcal{U}_{\mathrm{c}}\nu) = (\Delta\varphi)\mathcal{U}_{\mathrm{c}}\nu + 2\nabla\varphi \cdot \nabla \mathcal{U}_{\mathrm{c}}\nu + \varphi \Delta \mathcal{U}_{\mathrm{c}}\nu$ we deduce that $\Delta(\varphi \mathcal{U}_{\mathrm{c}}\nu)$ in the sense of distributions is a real measure.

REMARK 3.9. Let $G \subset \mathbb{R}^m$ be an open set with compact boundary. Put $C = \mathbb{R}^m \setminus \mathrm{cl}\, G$. Suppose that $\partial G = \partial C$ is locally the graph of a Lipschitz function, $V^G < \infty$, $r_{\mathrm{ess}}(N^G \mathcal{U} - \frac{1}{2} I) < \frac{1}{2}$ and $f \in \mathcal{C}(\partial G) \cap W^{1,p}(\partial G)$ with $1 < p < \infty$. The nontangential maximal functions of $|\nabla \mathcal{D}^G f|$ with respect to G and C are from $L^p(\partial G)$ by [5, p. 149]. According to [3, Lemma 2.10] the nontangential limits of $\nabla \mathcal{D}^G f$ with respect to G and with respect to C exist on ∂G. Since $\mathcal{D}^G f$ is an L^p-solution of some Neumann problem in G and in C we conclude by [20, Lemma 4.1] that there are $N^G \mathcal{D}^G f, N^C \mathcal{D}^G f \in \mathcal{C}'(\partial G)$. Thus $f \in \mathcal{C}_\nabla(\partial G)$.

PROPOSITION 3.10. *Let $G \subset \mathbb{R}^m$ be an open set with compact boundary. Put $C = \mathbb{R}^m \setminus \mathrm{cl}\, G$. Suppose $\partial G = \partial C$, $V^G < \infty$ and $r_{\mathrm{ess}}(N^G \mathcal{U} - \frac{1}{2} I) < \frac{1}{2}$. If $f \in \mathcal{C}_\nabla(\partial G)$ then $N^G \mathcal{D}^G f = N^C \mathcal{D}^C f = -N^C \mathcal{D}^G f$.*

Proof. We can suppose that G is bounded. If $x \in G$ then

$$\mathcal{D}^G f(x) = \mathcal{U}(N^G \mathcal{D}^G f)(x) + \mathcal{D}^G(\mathcal{D}^G f + d_G f)(x)$$

(see [18, Lemma 4] and (17)). Define $C(r) = C \cap \Omega(0; r)$. Put $f = 0$ on $\mathbb{R}^m \setminus \partial G$. If $x \in C$ then

$$\begin{aligned}\mathcal{D}^C f(x) &= \lim_{r \to \infty} [\mathcal{U}(N^{C(r)} \mathcal{D}^C f)(x) + \mathcal{D}^{C(r)}(\mathcal{D}^C f + d_C f)(x)] \\ &= \mathcal{U}(N^C \mathcal{D}^C f)(x) + \mathcal{D}^C(\mathcal{D}^C f + d_C f)(x).\end{aligned}$$

Since

$$\mathcal{U}(N^G \mathcal{D}^G f)(x) = \mathcal{D}^G f(x) - \mathcal{D}^G(\mathcal{D}^G f + d_G f)(x)$$

in G we have $N^G \mathcal{D}^G f \in \mathcal{C}'_{\mathrm{c}}(\partial G)$ (see [18, Theorem 1]) and

$$\begin{aligned}\mathcal{U}(N^G \mathcal{D}^G f)(x) = \mathcal{D}^G f(x) + d_G(x) f(x) - \mathcal{D}^G(\mathcal{D}^G f + d_G f)(x) \\ - d_G(x)(\mathcal{D}^G f + d_G f)\end{aligned}$$

on ∂G. Since $d_G = 1/2$ a.e. on ∂G (see [16, Lemma 2]),

$$\mathcal{U}(N^G \mathcal{D}^G f) = \frac{1}{4} f - \mathcal{D}^G(\mathcal{D}^G f)$$

a.e. on ∂G. Similarly, $N^C \mathcal{D}^C f \in \mathcal{C}'_{\rm c}(\partial G)$ and

$$\mathcal{U}(N^C \mathcal{D}^C f) = \frac{1}{4} f - \mathcal{D}^C(\mathcal{D}^C f)$$

a.e. on ∂G. Since $\mathcal{U}(N^C \mathcal{D}^C f) = \mathcal{U}(N^G \mathcal{D}^G f)$ a.e. on ∂G and $N^C \mathcal{D}^C f \in \mathcal{C}'_{\rm c}(\partial G)$ and $N^G \mathcal{D}^G f \in \mathcal{C}'_{\rm c}(\partial G)$, we deduce that $N^G \mathcal{D}^G f - N^C \mathcal{D}^C f \in \mathcal{C}'_{\rm c}(\partial G)$ and $\mathcal{U}_{\rm c}(N^G \mathcal{D}^G f - N^C \mathcal{D}^C f) = 0$ on ∂G. Since $\mathcal{U}_{\rm c}(N^G \mathcal{D}^G f - N^C \mathcal{D}^C f)$ is a continuous function in $\mathbb{R}^m$, harmonic in $\mathbb{R}^m \setminus \partial G$ and vanishing at infinity the maximum principle implies that $\mathcal{U}_{\rm c}(N^G \mathcal{D}^G f - N^C \mathcal{D}^C f) = 0$ in $\mathbb{R}^m$. Therefore

$$\begin{aligned} &N^G \mathcal{D}^G f - N^C \mathcal{D}^C f \\ &\qquad = N^G \mathcal{U}(N^G \mathcal{D}^G f - N^C \mathcal{D}^C f) + N^C \mathcal{U}(N^G \mathcal{D}^G f - N^C \mathcal{D}^C f) = 0. \end{aligned}$$

4. Reduction of the problem

PROPOSITION 4.1. *Let $V^{G^+} < \infty$, $r_{\rm ess}(N^{G^+}\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$, u be a weak solution of the transmission problem* (8)–(12) *with $a = 1$, and $f \equiv 0$. Then there is $\nu \in \mathcal{C}'_{\rm c}(\partial G^+)$ such that $u = \mathcal{U}\nu$.*

Proof. According to [19, Lemma 3], there is $p \geq 1$ such that $u \in L^p(\mathbb{R}^m)$. [18, Theorem 2] and Lemma 3.2 yield a $\nu \in \mathcal{C}'(\partial G)$ such that $u = \mathcal{U}\nu$ in G^-. [18, Theorem 1] shows that $\nu \in \mathcal{C}'_{\rm c}(\partial G)$. The function $\mathcal{U}\nu$ is a solution of the Dirichlet problem for the Laplace equation in G^+ with the boundary condition $\mathcal{U}\nu = u_- = u_+$. The uniqueness of the solution of the Dirichlet problem implies that $u = \mathcal{U}\nu$ in G^+.

We look for a solution of the problem (8)–(12) in the form of the sum of a single layer potential and a double layer potential. For this we need

$$V^{G^+} < \infty, \qquad r_{\rm ess}(N^{G^+}\mathcal{U} - \tfrac{1}{2}I) < \tfrac{1}{2}. \tag{18}$$

In the remainder of the paper we suppose that these conditions are satisfied.

If there is a weak solution of the problem (8)–(12) then $u_+, u_- \in \mathcal{C}_\nabla(\partial G^+)$ by Proposition 3.4. Hence $f \in \mathcal{C}_\nabla(\partial G^+)$. So, we can suppose that $f \in \mathcal{C}_\nabla(\partial G^+)$. We look for a solution in the form

$$u = \begin{cases} \mathcal{D}^{G^+} f + v & \text{in } G^+, \\ \mathcal{D}^{G^+} f/a + v/a & \text{in } G^-. \end{cases}$$

Then u is a weak solution of (8)–(12) if and only if v is a weak solution of the problem

$$\Delta v = 0 \quad \text{in } G^+, \tag{19}$$

$$\Delta v = 0 \quad \text{in } G^-, \tag{20}$$

$$v_+ - v_- = 0 \quad \text{on } \partial G^+, \tag{21}$$

$$\frac{\partial v_+}{\partial n^+} - c\,\frac{\partial v_-}{\partial n^+} = \widetilde{\mu}, \tag{22}$$

$$\lim_{|x|\to\infty} v(x) = 0, \tag{23}$$

where $c = b/a$ and

$$\widetilde{\mu} = \mu - N^{G^+}\mathcal{D}^{G^+} f - cN^{G^-}\mathcal{D}^{G^+} f = \mu - (1-c)N^{G^+}\mathcal{D}^{G^+} f \tag{24}$$

(see §3). According to Proposition 4.1 we can look for a solution of this problem in the form $v = \mathcal{U}\nu$ where $\nu \in \mathcal{C}'_{\rm c}(\partial G^+)$. The problem (19)–(23) reduces to the equation

$$T\nu = \widetilde{\mu} \tag{25}$$

where

$$T\nu = N^{G^+}\mathcal{U}\nu + cN^{G^-}\mathcal{U}\nu = c\nu + (1-c)N^{G^+}\mathcal{U}\nu. \tag{26}$$

Now, T is a bounded linear operator in $\mathcal{C}'(\partial G)$ such that $T(\mathcal{C}'_{\rm c}(\partial G^+)) \subset \mathcal{C}'_{\rm c}(\partial G^+)$. Since $T(\mathcal{C}'_{\rm c}(\partial G^+)) \subset \mathcal{C}'_{\rm c}(\partial G^+)$ a necessary condition for the solvability of the problem (19)–(23) is $\widetilde{\mu} \in \mathcal{C}'_{\rm c}(\partial G^+)$.

5. Solution of the problem

NOTATION 5.1. Let X be a real Banach space. Denote by $\operatorname{compl} X$ the complexification of X, i.e. $\operatorname{compl} X = \{x + iy;\, x \in X,\, y \in X\}$. If A is a linear operator on X, we extend A onto $\operatorname{compl} X$ by $A(x + iy) = Ax + iAy$. Denote by $\sigma(A)$ the spectrum of A and by $r(A)$ the spectral radius of A.

LEMMA 5.2. *Let $V^{G^+} < \infty$, $r_{\rm ess}(N^{G^+}\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$, and c be a positive constant. Let $\nu \in \operatorname{compl}\mathcal{C}'_{\rm c}(\partial G^+)$ and $\beta \in \mathbb{C}$ be such that $T\nu = \beta\nu$. If $|\nu|(\partial G^+) > 0$ then β is real and* $\min(1,c) \le \beta \le \max(1,c)$.

Proof. Denote by $\overline{\mathcal{U}\nu}$ the complex conjugate of $\mathcal{U}\nu$. As $\nu \in \operatorname{compl}\mathcal{C}'_{\rm c}(\partial G^+)$, using [8, Lemma 5.8] and [16, Lemma 7] we get

$$\begin{aligned}\beta \int_{G^+\cup G^-} |\nabla\mathcal{U}\nu|^2\, d\mathcal{H}_m &= \beta \int_{\partial G^+} \overline{\mathcal{U}_{\rm c}\nu}\, d\nu = \int_{\partial G^+} \overline{\mathcal{U}_{\rm c}\nu}\, dT\nu \\ &= \int_{\partial G^+} \overline{\mathcal{U}_{\rm c}\nu}\, d(N^{G^+}\mathcal{U}\nu + cN^{G^-}\mathcal{U}\nu) \\ &= \int_{G^+} |\nabla\mathcal{U}\nu|^2\, d\mathcal{H}_m + c\int_{G^-} |\nabla\mathcal{U}\nu|^2\, d\mathcal{H}_m.\end{aligned}$$

If $\nabla\mathcal{U}\nu = 0$ in $\mathbb{R}^m \setminus \partial G^+$ then $\mathcal{U}\nu$ is constant on each component of $\mathbb{R}^m \setminus \partial G^+$. Since $\mathcal{U}\nu(x) \to 0$ as $|x| \to \infty$ we deduce that $\mathcal{U}\nu \equiv 0$ on the unbounded component of $\mathbb{R}^m \setminus \partial G^+$. Since $\mathcal{U}_c\nu \in \mathcal{C}(\mathbb{R}^m)$ is constant on each component of $\mathbb{R}^m \setminus \partial G^+$ we infer that $\mathcal{U}_c\nu \equiv 0$ in $\mathbb{R}^m$. Hence $0 = N^{G^+}\mathcal{U}\nu + N^{G^-}\mathcal{U}\nu = \nu$, which is a contradiction. Therefore

$$0 < \int_{G^+\cup G^-} |\nabla \mathcal{U}\nu|^2 \, d\mathcal{H}_m < \infty.$$

Since

$$\beta = \Big[\int_{G^+} |\nabla \mathcal{U}\nu|^2 \, d\mathcal{H}_m + c \int_{G^-} |\nabla \mathcal{U}\nu|^2 \, d\mathcal{H}_m \Big] \Big[\int_{G^+\cup G^-} |\nabla \mathcal{U}\nu|^2 \, d\mathcal{H}_m \Big]^{-1},$$

we get $\min(1, c) \le \beta \le \max(1, c)$.

THEOREM 5.3. *Let $V^{G^+} < \infty$, $r_{\mathrm{ess}}(N^{G^+}\mathcal{U} - \frac{1}{2}I) < \frac{1}{2}$, and c be a positive constant. Define by*

$$\|\nu\|_{\mathcal{C}'_c(\partial G^+)} = \|\nu\|_{\mathcal{C}'(\partial G^+)} + \|\mathcal{U}_c\nu\|_{\mathcal{C}(\partial G^+)}$$

the norm on $\mathcal{C}'_c(\partial G^+)$. Then $\mathcal{C}'_c(\partial G^+)$ is a Banach space. The operator T is a bounded continuously invertible linear operator in $\mathcal{C}'(\partial G^+)$ and in $\mathcal{C}'_c(\partial G^+)$. Fix $\alpha > \max(1, c)/2$. Then there are constants $q \in (0, 1)$ and $M > 1$ such that

$$\big\|(I - \alpha^{-1}T)^n\big\| \le Mq^n \tag{27}$$

for each nonnegative integer n and

$$T^{-1} = \alpha^{-1} \sum_{n=0}^{\infty} (I - \alpha^{-1}T)^n \tag{28}$$

in $\mathcal{C}'(\partial G^+)$ and in $\mathcal{C}'_c(\partial G^+)$.

Proof. $\mathcal{C}'_c(\partial G^+)$ is a Banach space and T is a bounded linear operator in $\mathcal{C}'_c(\partial G^+)$ by [15, Corollary 2]. We have

$$r_{\mathrm{ess}}(T - ((1 + c)/2)I) = |1 - c| r_{\mathrm{ess}}(N^{G^+}\mathcal{U} - \tfrac{1}{2}I).$$

If $\beta \in \sigma(T - ((1 + c)/2)I)$ is a complex number such that

$$|\beta| > |1 - c| r_{\mathrm{ess}}(N^{G^+}\mathcal{U} - \tfrac{1}{2}I)$$

then β is an eigenvalue of the operator $T - ((1 + c)/2)I$ by [14, Lemma 1.2]. According to [15, Lemmas 5 and 10] there is a nontrivial $\nu \in \operatorname{compl} \mathcal{C}'_c(\partial G^+)$ such that $[T - ((1 + c)/2)I]\nu = \beta\nu$. Since $T\nu = [\beta + (1 + c)/2]\nu$ Lemma 5.2 gives $\min(1, c) \le \beta + (1 + c)/2 \le \max(1, c)$ and

$$\begin{aligned} \sigma(T - ((1 + c)/2)I) \subset{} & \{\beta \in \mathbb{C}; |\beta| \le |1 - c| r_{\mathrm{ess}}(N^{G^+}\mathcal{U} - \tfrac{1}{2}I)\} \\ & \cup [\min(1, c) - (1 + c)/2, \max(1, c) - (1 + c)/2] \\ \subset{} & \{0\} \cup \{\beta;\ |\beta| < |c - 1|/2\}. \end{aligned}$$

The spectral mapping theorem (see [28, Theorem 9.5]) gives $\sigma(T-\alpha I)\subset\{(1+c)/2-\alpha\}\cup\{\beta;\,|\beta+(c+1)/2-\alpha|<|c-1|/2\}\subset\{\beta\in\mathbb{C};|\beta|<\alpha\}$ in $\mathcal{C}'(\partial G^+)$. According to [15, Lemmas 5 and 8], we have $\sigma(T-\alpha I)\subset\{\beta\in\mathbb{C};\,|\beta|<\alpha\}$ in $\mathcal{C}'_{\mathrm{c}}(\partial G^+)$. Since $r(\alpha^{-1}T-I)<1$ in $\mathcal{C}'(\partial G^+)$ and in $\mathcal{C}'_{\mathrm{c}}(\partial G^+)$, there are constants $q\in(0,1)$ and $M>1$ such that (27) holds. Since $T=\alpha[(\alpha^{-1}T-I)+I]$, an easy calculation yields (28).

Theorem 5.4. *Let $V^{G^+}<\infty$, $r_{\mathrm{ess}}(N^{G^+}\mathcal{U}-\frac{1}{2}I)<\frac{1}{2}$, and a,b be positive constants. Let $\mu\in\mathcal{C}'(\partial G^+)$ and $f\in\mathcal{C}(\partial G^+)$. Then there is a weak solution of the problem* (8)–(12) *if and only if $\mu\in\mathcal{C}'_{\mathrm{c}}(\partial G^+)$ and $f\in\mathcal{C}_\nabla(\partial G^+)$. In that case, let $\widetilde{\mu}$ be given by* (24), *$c=b/a$ and T^{-1} be given by Theorem* 5.3. *Then $u=\mathcal{D}f+\mathcal{U}T^{-1}\widetilde{\mu}$ is a unique weak solution of the problem* (8)–(12). *This solution satisfies $|\nabla u|\in L^2(G^+\cup G^-)$.*

Proof. Suppose first that there is a weak solution u of (8)–(12). It was shown in §4 that $f\in\mathcal{C}_\nabla(\partial G^+)$. According to [19, Lemma 3] there is $p\geq 1$ such that $u\in L^p(\mathbb{R}^m)$. [19, Theorem 2] shows that $N^{G^+}u, N^{G^-}u\in\mathcal{C}'_{\mathrm{c}}(\partial G^+)$. Thus $\mu\in\mathcal{C}'_{\mathrm{c}}(\partial G^+)$.

Let now $\mu\in\mathcal{C}'_{\mathrm{c}}(\partial G^+)$ and $f\in\mathcal{C}_\nabla(\partial G^+)$. Then $u=\mathcal{D}f+\mathcal{U}T^{-1}\widetilde{\mu}$ is a weak solution of (8)–(12) by §4 and Theorem 5.3. We now show that $|\nabla u|\in L^2(G^+\cup G^-)$. It was proved in §4 that $\widetilde{\mu}\in\mathcal{C}'_{\mathrm{c}}(\partial G^+)$. Since $T^{-1}\widetilde{\mu}\in\mathcal{C}'_{\mathrm{c}}(\partial G^+)$ by Theorem 5.3, we have $|\nabla\mathcal{U}T^{-1}\widetilde{\mu}|\in L^2(G^+\cup G^-)$ by [8, Lemma 5.8]. According to [19, Lemma 3] there is $p\geq 1$ such that $\mathcal{D}f\in L^p(\mathbb{R}^m)$. [19, Theorem 2] shows that $N^{G^+}\mathcal{D}f\in\mathcal{C}'_{\mathrm{c}}(\partial G^+)$ and $N^{G^-}\mathcal{D}f\in\mathcal{C}'_{\mathrm{c}}(\partial G^+)$. [18, Theorem 2] and Lemma 3.2 imply that there are $\nu^+,\nu^-\in\mathcal{C}'(\partial G^+)$ such that $\mathcal{D}f=\mathcal{S}\nu^+$ in G^+ and $\mathcal{D}f=\mathcal{S}\nu^-$ in G^-. According to [18, Theorem 1] we have $\nu^+,\nu^-\in\mathcal{C}'_{\mathrm{c}}(\partial G^+)$. Thus $|\nabla\mathcal{D}f|\in L^2(G^+\cup G^-)$ by [8, Lemma 5.8]. This gives $|\nabla u|\in L^2(G^+\cup G^-)$. Now we show the uniqueness of a solution of the problem (8)–(12). Let u be a solution of (8)–(12) with $f\equiv 0$ and $\mu\equiv 0$. Then there is $\nu\in\mathcal{C}'_{\mathrm{c}}(\partial G^+)$ such that $u=\mathcal{U}\nu$ by Theorem 4.1. Theorem 5.3 and §4 imply that $\nu\equiv 0$ and therefore $u\equiv 0$.

Remark 5.5. Let $V^{G^+}<\infty$, $r_{\mathrm{ess}}(N^{G^+}\mathcal{U}-\frac{1}{2}I)<\frac{1}{2}$, $\mu\in\mathcal{C}'_{\mathrm{c}}(\partial G^+)$, $f\in\mathcal{C}_\nabla(\partial G^+)$, and a, b be positive constants. Let $\widetilde{\mu}$ be given by (24) and $c=b/a$. If $\nu\in\mathcal{C}'_{\mathrm{c}}(\partial G^+)$ is a solution of the equation $T\nu=\widetilde{\mu}$ then $\mathcal{D}f+\mathcal{U}\nu$ is a weak solution of the problem (8)–(12). Fix $\alpha>\max(1,c)/2$. We can rewrite the equation $T\nu=\widetilde{\mu}$ as $\nu=(I-\alpha^{-1}T)\nu+\alpha^{-1}\widetilde{\mu}$. Fix $\nu_0\in\mathcal{C}_{\mathrm{c}}(\partial G^+)$. Put

$$\nu_{n+1}=(I-\alpha^{-1}T)\nu_n+\alpha^{-1}\widetilde{\mu}$$

for nonnegative integers n. Let $M\in[1,\infty)$ and $q\in(0,1)$ be the constants from Theorem 5.3. Then

$$\begin{aligned}\|\nu_{n+1}-\nu_n\|&=\|(I-\alpha^{-1}T)(\nu_n-\nu_{n+1})\|=\|(I-\alpha^{-1}T)^n(\nu_1-\nu_0)\|\\&\leq Mq^n\|\nu_1-\nu_0\|\end{aligned}$$

in $\mathcal{C}'(\partial G^+)$ and in $\mathcal{C}'_{\rm c}(\partial G^+)$. Since $\{\nu_n\}$ is a Cauchy sequence it has a limit ν in $\mathcal{C}'(\partial G^+)$ (and in $\mathcal{C}'_{\rm c}(\partial G^+)$) and $T\nu = \widetilde{\mu}$. Moreover,

$$\|\nu - \nu_n\| \le \frac{M\|\nu_1 - \nu_0\|}{1-q}\, q^n.$$

References

[1] M. Brelot, *On Topologies and Boundaries in Potential Theory*, Springer, Berlin, 1971.

[2] M. Chlebík, *Tricomi potentials*, Thesis, Math. Inst., Czechoslovak Acad. Sci., Praha, 1988 (in Slovak).

[3] B. E. J. Dahlberg and C. Kenig, *Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains*, Ann. of Math. 125 (1987), 437–465.

[4] J. Deny, *Les potentiels d'énergie finie*, Acta Math. 82 (1950), 107–183.

[5] L. Escauriaza and M. Mitrea, *Transmission problems and spectral theory for singular integral operators on Lipschitz domains*, J. Funct. Anal. 216 (2004), 141–171.

[6] N. V. Grachev and V. G. Maz'ya, *On the Fredholm radius for operators of the double layer potential type on piecewise smooth boundaries*, Vestnik Leningrad. Univ. 19 (1986), no. 4, 60–64 (in Russian).

[7] L. L. Helms, *Introduction to Potential Theory*, Pure Appl. Math. 22, Wiley, New York, 1969.

[8] J. Král, *Integral Operators in Potential Theory*, Lecture Notes in Math. 823, Springer, Berlin, 1980.

[9] —, *The Fredholm method in potential theory*, Trans. Amer. Math. Soc. 125 (1966), 511–547.

[10] —, *Problème de Neumann faible avec condition frontière dans L^1*, in: Séminaire de Théorie du Potentiel (Université Paris VI) No. 9, Lecture Notes in Math. 1393, Springer, 1989, 145–160.

[11] J. Král and I. Netuka, *Contractivity of C. Neumann's operator in potential theory*, J. Math. Anal. Appl. 61 (1977), 607–619.

[12] N. L. Landkof, *Fundamentals of Modern Potential Theory*, Izdat. Nauka, Moscow, 1966 (in Russian).

[13] V. G. Maz'ya, *Boundary integral equations*, in: Analysis IV, Encyclopaedia Math. Sci. 27, Springer, New York, 1991, 127–222.

[14] D. Medková, *The third boundary value problem in potential theory for domains with a piecewise smooth boundary*, Czechoslovak Math. J. 47 (1997), 651–679.

[15] —, *Solution of the Robin problem for the Laplace equation*, Appl. Math. 43 (1998), 133–155.

[16] —, *Solution of the Neumann problem for the Laplace equation*, Czechoslovak Math. J. 48 (1998), 768–784.

[17] —, *Solution of the Dirichlet problem for the Laplace equation*, Appl. Math. 44 (1999), 143–168.

[18] —, *Continuous extendibility of solutions of the Neumann problem for the Laplace equation*, Czechoslovak Math. J. 53 (2003), 377–395.

[19] —, *Continuous extendibility of solutions of the third problem for the Laplace equation*, ibid. 53 (2003), 669–689.

[20] —, *The third problem for the Laplace equation with a boundary condition from L_p*, Integral Equations Operator Theory 54 (2006), 235–258.

[21] D. Medková, *The Neumann problem for the Laplace equation on general domains*, Czechoslovak Math. J. 57 (2007), to appear.

[22] I. Netuka, *Fredholm radius of a potential theoretic operator for convex sets*, Čas. Pěst. Mat. 100 (1975), 374–383.

[23] —, *An operator connected with the third boundary value problem in potential theory*, Czechoslovak Math. J. 22 (1972), 462–489.

[24] —, *The third boundary value problem in potential theory*, ibid. 22 (1972), 554–580.

[25] —, *Generalized Robin problem in potential theory*, ibid. 22 (1972), 312–324.

[26] A. Rathsfeld, *The invertibility of the double layer potential in the space of continuous functions defined on a polyhedron. The panel method*, Appl. Anal. 45 (1992), 1–4, 135–177.

[27] —, *The invertibility of the double layer potential operator in the space of continuous functions defined over a polyhedron. The panel method. Erratum*, ibid. 56 (1995), 109–115.

[28] M. Schechter, *Principles of Functional Analysis*, Amer. Math. Soc., Providence, RI, 2002.

[29] W. P. Ziemer, *Weakly Differentiable Functions*, Springer, New York, 1989.

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