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Some uniqueness results on meromorphic functions sharing three sets

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Abstract. We discuss the uniqueness of meromorphic functions when they share three sets with the notion of weighted sharing and improve two results of Lahiri–Banerjee and Yi–Lin. We also improve a recent result of the present author and thus provide an answer to a question of Gross, in a new direction.

1. Introduction, definitions and results. Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ designates any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty$, outside a possible exceptional set of finite linear measure.

If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities then we say that f and g *share the value a CM* (counting multiplicities). If we do not take the multiplicities into account, f and g are said to *share the value a IM* (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count multiplicities the set $E_f(S)$ is denoted by $\overline{E}_f(S)$.

If $E_f(S) = E_g(S)$ we say that f and g *share the set S CM*. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g *share the set S IM*.

In [3] Gross posed the following question:

> *Can one find two finite sets S_j $(j = 1, 2)$ such that any two nonconstant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?*

Fang and Xu [2] considered the case of meromorphic functions and proved the following result.

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Theorem A ([2]). *Let* $S_1 = \{z : z^3 - z^2 - 1 = 0\}$, $S_2 = \{0\}$ *and* $S_3 = \{\infty\}$. *Suppose that* f *and* g *are two nonconstant meromorphic functions satisfying* $\Theta(\infty; f) > 1/2$ *and* $\Theta(\infty; g) > 1/2$. *If* $E_f(S_j) = E_g(S_j)$ *for* $j = 1, 2, 3$ *then* $f \equiv g$.

Dealing with the question of Gross, Qiu and Fang [12] proved the following theorem.

Theorem B ([12]). *Let* $n \geq 3$ *be a positive integer*, $S_1 = \{z : z^n - z^{n-1} - 1 = 0\}$, $S_2 = \{0\}$, *and let* f *and* g *be two nonconstant meromorphic functions whose poles are of multiplicities at least* 2. *If* $E_f(\{\infty\}) = E_g(\{\infty\})$ *and* $E_f(S_i) = E_g(S_i)$ *for* $i = 1, 2$ *then* $f \equiv g$.

They also gave an example to show that the condition that the poles of f and g are of multiplicities at least 2 cannot be removed in Theorem B.

It should be noted that if two meromorphic functions f and g have no simple pole then clearly $\Theta(\infty; f) \geq 1/2$ and $\Theta(\infty; g) \geq 1/2$.

Lahiri and Banerjee [9] investigated the situation for $\Theta(\infty; f) \leq 1/2$ and $\Theta(\infty, g) \leq 1/2$ in Theorem A and proved the following result.

Theorem C ([9]). *Let* $S_1 = \{z : z^n + az^{n-1} + b = 0\}$, $S_2 = \{0\}$ *and* $S_3 = \{\infty\}$, *where* a, b *are nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no multiple root and* n (≥ 4) *is an integer. If for two nonconstant meromorphic functions* f *and* g, $E_f(S_i) = E_g(S_i)$ *for* $i = 1, 2, 3$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > 0$, *then* $f \equiv g$.

In 2004 Yi and Lin [17] independently proved Theorem C assuming $\Theta(\infty; f) > 0$ instead of $\Theta(\infty; f) + \Theta(\infty; g) > 0$. They remarked that the assumption $E_f(S_2) = E_g(S_2)$ in the above result can be relaxed to $\overline{E}_f(S_2) = \overline{E}_g(S_2)$.

Recently the present author [1] has investigated the relaxation of the nature of sharing the set S_1 in Theorem C using the idea of gradation of sharing of values and sets, known as weighted sharing, introduced in [6, 7]; it consists in measuring how close a shared value is to being shared IM or to being shared CM. We now give the definition.

Definition 1.1 ([6, 7]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g *share the value* a *with weight* k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p with $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2 ([6]). Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and k be a nonnegative integer or ∞. Let

$$E_f(S,k)=\bigcup_{a\in S}E_k(a;f).$$

Clearly $E_f(S)=E_f(S,\infty)$ and $\overline{E}_f(S)=E_f(S,0)$.

Improving the result of Lahiri–Banerjee [9] and Yi–Lin [17] the present author has recently proved the following result.

Theorem D ([1]). *Let* $S_1=\{z:z^n+az^{n-1}+b=0\}$, $S_2=\{0\}$ *and* $S_3=\{\infty\}$, *where* a, b *are nonzero constants such that* $z^n+az^{n-1}+b=0$ *has no multiple root and* $n\,(\geq 4)$ *is an integer. If for two nonconstant meromorphic functions* f *and* g, $E_f(S_1,4)=E_g(S_1,4)$, $E_f(S_2,0)=E_g(S_2,0)$ *and* $E_f(S_3,\infty)=E_g(S_3,\infty)$ *and* $\Theta(\infty;f)+\Theta(\infty;g)>0$, *then* $f\equiv g$.

Now considering all the above theorems it is natural to ask the following question which is one of the motivations of the paper.

(i) *What happens in Theorem D if* f *and* g *share the set* S_1 *with weight* 2 *and* 3 *respectively?*

Also note that to deal with the question of Gross none of the previous authors considered any further relaxation of the nature of sharing the set S_3 in Theorem C; they have all confined their investigations to the relaxation of the nature of sharing the sets S_1 and S_2 of Theorem C.

In this paper we concentrate our attention on relaxation of sharing S_3. We now state the following three theorems which are the main results of the paper.

Theorem 1.1. *Let* $S_1=\{z:z^n+az^{n-1}+b=0\}$, $S_2=\{0\}$ *and* $S_3=\{\infty\}$, *where* a, b *are nonzero constants such that* $z^n+az^{n-1}+b=0$ *has no multiple root and* $n\ (\geq 4)$ *is an integer. If for two nonconstant meromorphic functions* f *and* g, $E_f(S_1,3)=E_g(S_1,3)$, $E_f(S_2,0)=E_g(S_2,0)$ *and* $E_f(S_3,\infty)=E_g(S_3,\infty)$ *and* $\Theta(\infty;f)+\Theta(\infty;g)>\max\{0,\frac{20-4n}{7n-11}\}$, *then* $f\equiv g$.

Remark 1.1. If $n\geq 5$ then the assertion of Theorem 1.1 is true for $\Theta(\infty;f)+\Theta(\infty;g)>0$.

Theorem 1.2. *Let* S_1, S_2 *and* S_3 *be defined as in Theorem* 1.1 *and* $n\ (\geq 4)$ *be an integer. If for two nonconstant meromorphic functions* f *and* g, $E_f(S_1,2)=E_g(S_1,2)$, $E_f(S_2,0)=E_g(S_2,0)$ *and* $E_f(S_3,\infty)=E_g(S_3,\infty)$ *and* $\Theta(\infty;f)+\Theta(\infty;g)>\max\{0,\frac{32-4n}{5n-4}\}$, *then* $f\equiv g$.

Remark 1.2. If $n\geq 8$ then Theorem 1.2 is true for $\Theta(\infty;f)+\Theta(\infty;g)>0$.

Theorem 1.3. *Let* S_1, S_2 *and* S_3 *be defined as in Theorem* 1.1 *and* $n\,(\geq 4)$ *be an integer. If for two nonconstant meromorphic functions* f *and* g,

$E_f(S_1,4)=E_g(S_1,4)$, $E_f(S_2,0)=E_g(S_2,0)$ and $E_f(S_3,6)=E_g(S_3,6)$, and $\Theta(\infty;f)+\Theta(\infty;g)>0$, then $f\equiv g$.

REMARK 1.3. Clearly Theorem 1.3 improves Theorem D.

The following example shows that the condition $\Theta(\infty;f)+\Theta(\infty;g)>0$ is not only sharp in Theorem 1.3 but also in Theorems 1.1 and 1.2 when $n\geq 5$ and $n\geq 8$ respectively.

EXAMPLE 1.1. Let

$$g(z)=-a\,\frac{e^{(n-1)z}-1}{e^{nz}-1},\qquad f(z)=e^zg(z)$$

and S_i be as in Theorem 1.1. Then $E_f(S_i,\infty)=E_g(S_i,\infty)$ for $i=1,2,3$ because $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$ and $f\equiv e^zg$. Also $\Theta(\infty;f)+\Theta(\infty;g)=0$ and $f\not\equiv g$.

For the standard definitions and notations of value distribution theory we refer to [4]; we now explain some specific notations used in this paper.

DEFINITION 1.3 ([5]). For $a\in\mathbb{C}\cup\{\infty\}$we denote by $N(r,a;f\mid=1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r,a;f\mid\leq m)$ (resp. $N(r,a;f\mid\geq m)$) the counting function of those a-points of f whose multiplicities are not greater (resp. less) than m where each a-point is counted according to its multiplicity.

$\overline{N}(r,a;f\mid\leq m)$ and $\overline{N}(r,a;f\mid\geq m)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f\mid<m)$, $N(r,a;f\mid>m)$, $\overline{N}(r,a;f\mid<m)$ and $\overline{N}(r,a;f\mid>m)$ are defined analogously.

DEFINITION 1.4. We denote by $\overline{N}(r,a;f\mid=k)$ the reduced counting function of those a-points of f whose multiplicity is exactly k, where $k\geq 2$ is an integer.

DEFINITION 1.5. Let f and g be two nonconstant meromorphic functions such that f and g share (a,k) where $a\in\mathbb{C}\cup\{\infty\}$. Let z_0 be an a-point of f with multiplicity p and an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the counting function of those a-points of f and g where $p>q$, and by $\overline{N}_E^{(k+1}(r,a;f)$ the counting function of those a-points of f and g where $p=q\geq k+1$; each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,a;g)$ and $\overline{N}_E^{(k+1}(r,a;g)$.

DEFINITION 1.6 ([7]). We set

$$N_2(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2).$$

DEFINITION 1.7 ([6, 7]). Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

Definition 1.8 ([10]). Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;f \mid g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.9 ([10]). Let $a, b_1, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a; f \mid g \neq b_1, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not b_i-points of g for $i = 1, \ldots, q$.

2. Lemmas. Let F and G be two nonconstant meromorphic functions defined as follows:

$$F = \frac{f^{n-1}(f+a)}{-b}, \qquad G = \frac{g^{n-1}(g+a)}{-b}. \tag{2.1}$$

We shall denote by H, Φ and V the following three functions:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

$$\Phi = \frac{F'}{F-1} - \frac{G'}{G-1},$$

$$V = \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}.$$

Lemma 2.1 ([7, Lemma 1]). *Let F, G share $(1,1)$ and $H \not\equiv 0$. Then*

$$N(r,1;F \mid= 1) = N(r,1;G \mid= 1) \le N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.2. *Let S_1, S_2 and S_3 be as in Theorem 1.1 and F, G be given by (2.1). If for two nonconstant meromorphic functions f and g, $E_f(S_1,0) = E_g(S_1,0)$, $E_f(S_2,0) = E_g(S_2,0)$, $E_f(S_3,0) = E_g(S_3,0)$ and $H \not\equiv 0$, then*

$$\begin{aligned} N(r,H) \le{}& \overline{N}_*(r,0,f,g) + \overline{N}(r,0;f+a \mid\ge 2) + \overline{N}(r,0;g+a \mid\ge 2) \\ &+ \overline{N}_*(r,1;F,G) + \overline{N}_*(r,\infty;f,g) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'), \end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$, and $\overline{N}_0(r,0;G')$ is similarly defined.

Proof. Since $E_f(S_1,0) = E_g(S_1,0)$ it follows that F and G share $(1,0)$. We can easily verify that possible poles of H occur at (i) those zeros of f and g whose multiplicities differ from the multiplicities of the corresponding zeros of g and f respectively, (ii) multiple zeros of $f+a$ and $g+a$, (iii) those poles of f and g whose multiplicities differ from the multiplicities of the corresponding poles of g and f respectively, (iv) 1-points of F and G with different multiplicities, (v) zeros of F' which are not the zeros of $F(F-1)$, (v) zeros of G' which are not the zeros of $G(G-1)$. Since H has only simple poles, the lemma follows from the above. ∎

Lemma 2.3 ([13]). *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r, P(f)) = nT(r, f) + O(1)$.*

Lemma 2.4. *Let F and G be given by* (2.1)*. If f, g share $(0,0)$ and 0 is not a Picard exceptional value of f and g, then $\Phi \equiv 0$ implies $F \equiv G$.*

Proof. Suppose $\Phi \equiv 0$. Then by integration we obtain $F - 1 \equiv C(G-1)$. It is clear that if z_0 is a zero of f then it is a zero of g. So from (2.1) it follows that $F(z_0) = 0$ and $G(z_0) = 0$. So $C = 1$ and hence $F \equiv G$. ∎

Lemma 2.5. *Let F and G be given by* (2.1), *$n \geq 3$ an integer and $\Phi \not\equiv 0$. If F, G share $(1, m)$ and f, g share $(0, p)$ and (∞, k), where $0 \leq p < \infty$, then*

$$[(n-1)p + n - 2]\overline{N}(r, 0; f \mid\geq p+1) \leq \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; F, G) + S(r, f) + S(r, g).$$

Proof. If 0 is an e.v.P. (Picard exceptional value) of f and g then the assertion follows immediately.

Next suppose 0 is not an e.v.P. of f and g. Let z_0 be a zero of f with multiplicity q and a zero of g with multiplicity r. From (2.1) we know that z_0 is a zero of F with multiplicity $(n-1)q$ and a zero of G with multiplicity $(n-1)r$. We note that F and G have no zero of multiplicity t where $(n-1)p < t < (n-1)(p+1)$. So from the definition of Φ it is clear that z_0 is a zero of Φ with multiplicity at least $(n-1)(p+1) - 1$. So we have

$$\begin{aligned}
&[(n-1)p + n - 2]\overline{N}(r, 0; f \mid\geq p+1) \\
&\quad = [(n-1)p + n - 2]\overline{N}(r, 0; g \mid\geq p+1) \\
&\quad = [(n-1)p + n - 2]\overline{N}(r, 0; F \mid\geq (n-1)(p+1)) \\
&\quad \leq N(r, 0; \Phi) \leq N(r, \infty; \Phi) + S(r, f) + S(r, g) \\
&\quad \leq \overline{N}_*(r, \infty; F, G) + \overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g). \ \blacksquare
\end{aligned}$$

Lemma 2.6. *Let F and G be given by* (2.1), *and suppose f, g share $(\infty, 0)$ and ∞ is not an Picard exceptional value of f and g. Then $V \equiv 0$ implies $F \equiv G$.*

Proof. Suppose $V \equiv 0$. Then by integration we obtain

$$1 - \frac{1}{F} \equiv A\left(1 - \frac{1}{G}\right).$$

It is clear that if z_0 is a pole of f then it is a pole of g. Hence from the definition of F and G we have $1/F(z_0) = 0$ and $1/G(z_0) = 0$. So $A = 1$ and hence $F \equiv G$. ∎

LEMMA 2.7. *Let F, G be given by* (2.1) *and $V \not\equiv 0$. If f, g share $(0,0)$ and (∞, k), where $0 \le k < \infty$, and F, G share $(1,m)$, then the poles of F and G are the zeros of V and*

$$\begin{aligned}(nk+n-1)\overline{N}(r,\infty;f \mid\ge k+1) &= (nk+n-1)\overline{N}(r,\infty;g \mid\ge k+1)\\ &\le \overline{N}_*(r,0;f,g) + \overline{N}(r,0;f+a)\\ &\quad + \overline{N}(r,0;g+a) + \overline{N}_L(r,1;F)\\ &\quad + \overline{N}_L(r,1;G) + S(r,f) + S(r,g).\end{aligned}$$

Proof. If ∞ is an e.v.P. of f and g then the assertion follows immediately.

Next suppose ∞ is not an e.v.P. of f and g. Since f, g share (∞,k), it follows that F, G share (∞, nk) and so a pole of F with multiplicity $p\,(\ge nk+1)$ is a pole of G with multiplicity $r\,(\ge nk+1)$ and vice versa. We note that F and G have no pole of multiplicity q where $nk < q < nk+n$. So using Lemma 2.3 and noting that f, g share $(0,0)$ and F, G share $(1,m)$ we get, from the definition of V,

$$\begin{aligned}&(nk+n-1)\overline{N}(r,\infty;f \mid\ge k+1)\\ &\qquad = (nk+n-1)\overline{N}(r,\infty;g \mid\ge k+1)\\ &\qquad = (nk+n-1)\overline{N}(r,\infty;F \mid\ge nk+n)\\ &\qquad \le N(r,0;V) \le N(r,\infty;V) + S(r,f) + S(r,g)\\ &\qquad \le \overline{N}_*(r,0;f,g) + \overline{N}(r,0;f+a) + \overline{N}(r,0;g+a)\\ &\qquad\quad + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g). \ \blacksquare\end{aligned}$$

LEMMA 2.8 ([1, Lemma 3]). *Let f and g be two nonconstant meromorphic functions sharing $(1,m)$, where $2 \le m < \infty$. Then*

$$\begin{aligned}&\overline{N}(r,1;f \mid= 2) + 2\overline{N}(r,1;f \mid= 3) + \cdots + (m-1)\overline{N}(r,1;f \mid= m)\\ &\quad + m\overline{N}_L(r,1;f) + (m+1)\overline{N}_L(r,1;g) + m\overline{N}_E^{(m+1}(r,1;f)\\ &\qquad\qquad \le N(r,1;g) - \overline{N}(r,1;g).\end{aligned}$$

LEMMA 2.9. *Let F, G be given by* (2.1) *and suppose they share $(1,m)$, where $2 \le m < \infty$. If f, g share $(0,p)$ and (∞,k), and $H \not\equiv 0$, then*

$$\begin{aligned}T(r,F) \le\ & \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}_*(r,0;f,g) + N_2(r,0;f+a)\\ & + N_2(r,0;g+a) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\\ & + \overline{N}_*(r,\infty;f,g) - m(r,1;G) - \overline{N}(r,1;F \mid= 3)\\ & - \cdots - (m-2)\overline{N}(r,1;F \mid= m) - (m-2)\ \overline{N}_L(r,1;F)\\ & - (m-1)\overline{N}_L(r,1;G) - (m-1)\overline{N}_E^{(m+1}(r,1;F)\\ & + S(r,F) + S(r,G).\end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned}(2.2)\quad T(r,F)+T(r,G) \le\ & \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)\\ &+\overline{N}(r,\infty;G)+\overline{N}(r,1;F)+\overline{N}(r,1;G)\\ &-N_0(r,0;F')-N_0(r,0;G')+S(r,F)+S(r,G).\end{aligned}$$

In view of Definition 1.7, using Lemmas 2.1, 2.2 and 2.8 we see that

$$\begin{aligned}(2.3)\quad & \overline{N}(r,1;F)+\overline{N}(r,1;G)\\ &\le N(r,1;F\mid=1)+\overline{N}(r,1;F\mid=2)+\overline{N}(r,1;F\mid=3)\\ &\quad+\cdots+\overline{N}(r,1;F\mid=m)+\overline{N}_E^{(m+1}(r,1;F)\\ &\quad+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}(r,1;G)\\ &\le \overline{N}_*(r,0;f,g)+\overline{N}(r,0;f+a\mid\ge 2)\\ &\quad+\overline{N}(r,0;g+a\mid\ge 2)+\overline{N}_*(r,\infty;f,g)+\overline{N}_L(r,1;F)\\ &\quad+\overline{N}_L(r,1;G)+\overline{N}(r,1;F\mid=2)+\cdots\\ &\quad+\overline{N}(r,1;F\mid=m)+\overline{N}_E^{(m+1}(r,1;F)\\ &\quad+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+T(r,G)-m(r,1;G)\\ &\quad+O(1)-\overline{N}(r,1;F\mid=2)-2\overline{N}(r,1;F\mid=3)\\ &\quad-(m-1)\overline{N}(r,1;F\mid=m)-\cdots-m\overline{N}_E^{(m+1}(r,1;F)\\ &\quad-m\overline{N}_L(r,1;F)-(m+1)\overline{N}_L(r,1;G)+\overline{N}_0(r,0;F')\\ &\quad+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\ &\le \overline{N}_*(r,0;f,g)+\overline{N}(r,0;f+a\mid\ge 2)+\overline{N}(r,0;g+a\mid\ge 2)\\ &\quad+\overline{N}_*(r,\infty;f,g)+T(r,G)-m(r,1;G)-\overline{N}(r,1;F\mid=3)\\ &\quad-2\overline{N}(r,1;F\mid=4)-\cdots-(m-2)\overline{N}(r,1;F\mid=m)\\ &\quad-(m-2)\overline{N}_L(r,1;F)-(m-1)\overline{N}_L(r,1;G)\\ &\quad-(m-1)\overline{N}_E^{(m+1}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')\\ &\quad+S(r,F)+S(r,G).\end{aligned}$$

From (2.2) and (2.3) in view of Definition 1.6 the lemma follows. ■

LEMMA 2.10 ([9, Lemma 3]). *Let f, g be two nonconstant meromorphic functions sharing $(0,\infty)$ and (∞,∞), and $\Theta(\infty;f)+\Theta(\infty;g)>0$. Then $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$ implies $f\equiv g$, where n (≥ 2) is an integer and a is a nonzero finite constant.*

Lemma 2.11 ([8, Lemma 5]). *If two nonconstant meromorphic functions f, g share $(\infty, 0)$, then for $n \geq 2$,*

$$f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2,$$

where a, b are finite nonzero constants.

Lemma 2.12 ([16, Lemma 6]). *If $H \equiv 0$, then F, G share $(1, \infty)$. If further F, G share $(\infty, 0)$, then they share (∞, ∞).*

Lemma 2.13 ([11]). *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f \mid< k) + k\overline{N}(r, 0; f \mid\geq k) + S(r, f).$$

Lemma 2.14. *Let F, G be given by* (2.1) *and suppose F, G share $(1, m)$, $0 \leq m < \infty$ and $\omega_1, \ldots, \omega_n$ are the distinct roots of the equation $z^n + az^{n-1} + b = 0$ and $n \geq 3$. Then*

$$\overline{N}_L(r, 1; F) \leq \frac{1}{m+1}\,[\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - N_{\otimes}(r, 0; f')] + S(r, f),$$

where $N_{\otimes}(r, 0; f') = N(r, 0; f' \mid f \neq 0, \omega_1, \ldots, \omega_n)$.

Proof. Using Lemmas 2.3 and 2.13 we see that

$$\begin{aligned}\overline{N}_L(r, 1; F) &\leq \overline{N}(r, 1; F \mid\geq m+2)\\ &\leq \frac{1}{m+1}\,(N(r, 1; F) - \overline{N}(r, 1; F))\\ &\leq \frac{1}{m+1}\,\sum_{j=1}^{n}(N(r, \omega_j; f) - \overline{N}(r, \omega_j; f))\\ &\leq \frac{1}{m+1}\,(N(r, 0; f' \mid f \neq 0) - N_{\otimes}(r, 0; f'))\\ &\leq \frac{1}{m+1}\,[\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - N_{\otimes}(r, 0; f')] + S(r, f). \ \blacksquare\end{aligned}$$

Lemma 2.15. *Under the assumptions of Lemma* 2.14,

$$\overline{N}_*(r, 1; F, G) \leq \frac{1}{m}\,[\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) - N_{\otimes}(r, 0; f')] + S(r, f).$$

Proof. Since

$$\overline{N}_*(r, 1; F, G) \leq \overline{N}(r, 1; F \mid\geq m+1) \leq \frac{1}{m}\,(N(r, 1; F) - \overline{N}(r, 1; F)),$$

the proof can be carried out along the lines of the proof of Lemma 2.14. ∎

LEMMA 2.16 ([14]). *Let F, G be two nonconstant meromorphic functions sharing $(1,\infty)$ and (∞,∞). If*

$$N_2(r,0;F)+N_2(r,0;F)+2\overline{N}(r,\infty;F)<\lambda T_1(r)+S_1(r),$$

where $\lambda<1$ and $T_1(r)=\max\{T(r,F),T(r,G)\}$ and $S_1(r)=o(T_1(r))$, $r\to\infty$, outside a possible exceptional set of finite linear measure, then $F\equiv G$ or $FG\equiv 1$.

LEMMA 2.17. *Let F, G be given by (2.1) and $n\geq 4$, and suppose that F, G share $(1,m)$, where $2\leq m<\infty$. If f, g share $(0,0)$ and (∞,k), and $\Theta(\infty;f)+\Theta(\infty;g)>0$ and $H\equiv 0$, then $f\equiv g$.*

Proof. Since $H\equiv 0$, Lemma 2.12 shows that F, G share $(1,\infty)$ and (∞,∞). If possible, suppose $F\not\equiv G$. Then from Lemmas 2.4 and 2.5 we have

$$\overline{N}(r,0;f)=\overline{N}(r,0;g)=S(r).$$

Again from Lemmas 2.6 and 2.7 we obtain

$$\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\leq\frac{4}{n-1}T(r)+S(r).$$

Therefore

$$\begin{aligned}(2.4)\qquad N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,\infty;F)&\\ \leq 2\overline{N}(r,0;f)+2\overline{N}(r,0;g)+N_2(r,0;f+a)&\\ +N_2(r,0;g+a)+2\overline{N}(r,\infty;f)&\\ \leq N_2(r,0;f+a)+N_2(r,0;g+a)&\\ +\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r).&\end{aligned}$$

Using Lemma 2.3 we obtain

$$(2.5)\qquad T_1(r)=n\ \max\{T(r,f),T(r,g)\}+O(1)=nT(r)+O(1).$$

So again using Lemma 2.3 we get from (2.4) and (2.5),

$$N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,\infty;F)\leq\frac{2+\frac{4}{n-1}}{n}T_1(r)+S_1(r).$$

Since $\frac{2+\frac{4}{n-1}}{n}<1$ for $n\geq 4$, Lemma 2.16 yields $FG\equiv 1$, which is impossible by Lemma 2.11. Hence $F\equiv G$, i.e. $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$. This together with the assumption that f and g share $(0,0)$ implies that f and g share $(0,\infty)$. Now the lemma follows from Lemma 2.10. ∎

3. Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1). Then F and G share $(1,3)$ and (∞,∞). We consider the following cases.

Case 1: $H \not\equiv 0$. Then $F \not\equiv G$. Suppose 0 is not an e.v.P. of f and g. Then by Lemma 2.4 we get $\Phi \not\equiv 0$. Noting that f and g sharing $(0,0)$ implies $\overline{N}_*(r,0;f,g) \le \overline{N}(r,0;f) = \overline{N}(r,0;g)$, from Lemmas 2.3, 2.5, 2.9 and 2.14 we obtain, for $\varepsilon > 0$,

$$\begin{aligned}
(3.1)\qquad nT(r,f) &\le 3\overline{N}(r,0;f) + N_2(r,0;f+a) + N_2(r,0;g+a) \\
&\quad + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) - \overline{N}_L(r,1;F) \\
&\quad - 2\overline{N}_L(r,1;G) + S(r,f) + S(r,g) \\
&\le 3\overline{N}(r,0;f) + T(r,f) + T(r,g) + \overline{N}(r,\infty;f) \\
&\quad + \overline{N}(r,\infty;g) - \overline{N}_L(r,1;F) - 2\overline{N}_L(r,1;G) \\
&\quad + S(r,f) + S(r,g) \\
&\le \frac{3}{n-2}\,[\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G)] + 2T(r) \\
&\quad + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) - \overline{N}_L(r,1;F) \\
&\quad - 2\overline{N}_L(r,1;G) + S(r,f) + S(r,g) \\
&\le \frac{5-n}{4(n-2)}\,[\overline{N}(r,0;f) + \overline{N}(r,\infty;f)] + 2T(r) \\
&\quad + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g) \\
&\le \frac{5-n}{4(n-2)}\,T(r,f) + 2T(r) + \left(2 + \frac{5-n}{4(n-2)}\right)\overline{N}(r,\infty;f) \\
&\quad + S(r) \\
&\le \left[4 + \frac{5-n}{2(n-2)} - \frac{7n-11}{8(n-2)}\,\{\Theta(\infty;f) + \Theta(\infty;g) - 2\varepsilon\}\right]T(r) \\
&\quad + S(r).
\end{aligned}$$

If 0 is an e.v.P. of f and g then (3.1) holds automatically.

In the same way we can obtain

$$\begin{aligned}
(3.2)\qquad nT(r,g) &\le \left[4 + \frac{5-n}{2(n-2)} - \frac{7n-11}{8(n-2)}\,\{\Theta(\infty;f) + \Theta(\infty;g) - 2\varepsilon\}\right]T(r) \\
&\quad + S(r).
\end{aligned}$$

Combining (3.1) and (3.2) we see that

$$\left[n - 4 - \frac{5-n}{2(n-2)} + \frac{7n-11}{8(n-2)}\,\{\Theta(\infty;f) + \Theta(\infty;g) - 2\varepsilon\}\right]T(r) \le S(r).$$

Since $\Theta(\infty;f) + \Theta(\infty;g) > \max\{0, \frac{20-4n}{7n-11}\}$, there exists a $\delta > 0$ such that

$$\Theta(\infty;f) + \Theta(\infty;g) = \max\left\{0, \frac{20-4n}{7n-11}\right\} + \delta.$$

If we choose $0 < \varepsilon < \delta/2$ then (3.2) leads to a contradiction.

CASE 2: $H \equiv 0$. Then the assertion follows from Lemma 2.17. ■

Proof of Theorem 1.2. Let F, G be given by (2.1). Then F and G share $(1,2)$ and (∞,∞). We consider the following cases.

CASE 1: $H \not\equiv 0$. Then $F \not\equiv G$. Suppose 0 is not an e.v.P. of f and g. Then by Lemma 2.4 we get $\Phi \not\equiv 0$. So from Lemmas 2.3, 2.5, 2.9 and 2.14 we obtain, for $\varepsilon > 0$,

$$\begin{aligned}(3.3)\quad nT(r,f) &\le 3\overline{N}(r,0;f) + N_2(r,0;f+a) + N_2(r,0;g+a)\\ &\quad + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) - \overline{N}_L(r,1;G)\\ &\quad + S(r,f) + S(r,g)\\ &\le \frac{3}{n-2}\,\overline{N}_L(r,1;F) + \frac{5-n}{n-2}\,\overline{N}_L(r,1;G) + 2T(r)\\ &\quad + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g)\\ &\le \left(2 + \frac{8-n}{3(n-2)}\right)T(r) + \left(2 + \frac{8-n}{3(n-2)}\right)\overline{N}(r,\infty;f) + S(r)\\ &\le \left[4 + \frac{16-2n}{3(n-2)} - \frac{5n-4}{6(n-2)}\,\{\Theta(\infty;f) + \Theta(\infty;g) - 2\varepsilon\}\right]T(r)\\ &\quad + S(r).\end{aligned}$$

If 0 is an e.v.P. of f and g then (3.3) holds automatically.

In the same manner we can obtain

$$\begin{aligned}(3.4)\quad nT(r,g) &\le \left[4 + \frac{16-2n}{3(n-2)} - \frac{5n-4}{6(n-2)}\,\{\Theta(\infty;f) + \Theta(\infty;g) - 2\varepsilon\}\right]T(r)\\ &\quad + S(r).\end{aligned}$$

Combining (3.3) and (3.4) we see that

$$\left[n - 4 - \frac{16-2n}{3(n-2)} + \frac{5n-4}{6(n-2)}\,\{\Theta(\infty;f) + \Theta(\infty;g) - 2\varepsilon\}\right]T(r) \le S(r),$$

which leads to a contradiction for arbitrary $\varepsilon > 0$.

CASE 2: $H \equiv 0$. Then the assertion follows from Lemma 2.17. ■

Proof of Theorem 1.3. Let F, G be given by (2.1). Then F and G share $(1,4)$ and $(\infty,6n)$. We consider the following cases.

CASE 1: $H \not\equiv 0$. Then $F \not\equiv G$. Suppose 0, ∞ are not Picard exceptional values of f and g. Then by Lemmas 2.4 and 2.6 we get $\Phi \not\equiv 0$ and $V \not\equiv 0$. Noting that f, g sharing $(0,0)$ and $(\infty,6)$ implies $\overline{N}_*(r,0;f,g) \le \overline{N}(r,0;f) = \overline{N}(r,0;g)$ and $\overline{N}_*(r,\infty;f,g) \le \overline{N}(r,\infty;f\,|\ge 7) = \overline{N}(r,\infty;g\,|\ge 7)$, from Lemmas 2.3, 2.5 and 2.9 we obtain

$$\begin{aligned}
(3.5) \quad & nT(r,f)+nT(r,g) \\
&\le 6\overline{N}(r,0;f)+2T(r,f)+2T(r,g)+4\overline{N}(r,\infty;f) \\
&\quad + 2\overline{N}(r,\infty;f\mid\ge 7)-5\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g) \\
&\le 2T(r,f)+2T(r,g)+\left\{2+\frac{6}{n-2}\right\}\overline{N}(r,\infty;f\mid\ge 7) \\
&\quad + \frac{6}{n-2}\,\overline{N}_*(r,1;F,G)+4\overline{N}(r,\infty;f) \\
&\quad - 5\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}$$

So using respectively Lemma 2.7 for $k=6$ and $k=0$, Lemma 2.5 for $p=0$ and Lemma 2.15 we deduce from (3.5) that

$$\begin{aligned}
(3.6) \quad & nT(r,f)+nT(r,g) \\
&\le \left(2+\frac{3(n+1)}{(n-2)(7n-1)}\right)\{T(r,f)+T(r,g)\} \\
&\quad + \left(\frac{6}{n-2}+\frac{2(n+1)}{(n-2)(7n-1)}\right)\overline{N}_*(r,1;F,G) \\
&\quad + \frac{4}{n-1}\,[T(r,f)+T(r,g)+\overline{N}(r,0;f)+\overline{N}_*(r,1;F,G)] \\
&\quad - 5\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g) \\
&\le \left(2+\frac{4}{n-1}+\frac{3(n+1)}{(n-2)(7n-1)}\right)\{T(r,f)+T(r,g)\} \\
&\quad + \left(\frac{6}{n-2}+\frac{4}{n-1}+\frac{2(n+1)}{(n-2)(7n-1)}\right)\overline{N}_*(r,1;F,G) \\
&\quad + \frac{4}{(n-1)(n-2)}\,[\overline{N}(r,\infty;f\mid\ge 7)+\overline{N}_*(r,1;F,G)]-5\overline{N}_*(r,1;F,G) \\
&\quad + S(r,f)+S(r,g) \\
&\le \left(2+\frac{4n-6}{(n-1)(n-2)}+\frac{3(n+1)}{(n-2)(7n-1)}\right)\{T(r,f)+T(r,g)\} \\
&\quad + \left(\frac{10}{n-2}+\frac{2(n+1)}{(n-2)(7n-1)}-5\right)\overline{N}_*(r,1;F,G) \\
&\quad + S(r,f)+S(r,g) \\
&\le \left(2+\frac{(4n-6)}{(n-1)(n-2)}+\frac{7(n+1)}{2(n-2)(7n-1)}\right)\{T(r,f)+T(r,g)\} \\
&\quad + S(r,f)+S(r,g).
\end{aligned}$$

From (3.6) we get a contradiction for $n\ge 4$.

If 0, ∞ are e.v.P. of f and g then (3.6) holds automatically.

Case 2: $H \equiv 0$. Now the assertion follows from Lemma 2.17. ∎

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