On strong tracts of subharmonic functions of infinite lower order

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Abstract. The notion of a strong asymptotic tract for subharmonic functions is defined. Eremenko's value $b(\infty, u)$ for subharmonic functions is introduced and it is used to provide an exact upper estimate of the number of strong tracts of subharmonic functions of infinite lower order. It is also shown that $b(\infty, u) \leq \pi$ for subharmonic functions of infinite lower order.

1. Introduction. Let $u(z)$ be a function subharmonic in the plane and with infinite lower order, which means that

$$\liminf_{r \to \infty} \frac{\log \max\{u(z) : |z| = r\}}{\log r} = \infty.$$

Let us consider the sets $E(n) = \{z \in \mathbb{C} : u(z) \geq n\}$ for $n \in \mathbb{N}$. Let $C(n)$ be a thick component of $E(n)$, which means that $u(z) \not\equiv n$ on $C(n)$ (see [7]). There is $k \in \mathbb{N}$ such that for every $n \geq k$ the function $u(z)$ is unbounded on all thick components $C(n)$. We call a thick component $C(n)$ an n-*tract* of $u(z)$. Let $n_2 > n_1 > k$. Then every n_1-tract contains at least one n_2-tract. If $q(n)$ is the number of different n-tracts $C(n)$, then $q(n_2) \geq q(n_1)$. We call $q = \lim_{n \to \infty} q(n)$ the *number of tracts* of the function $u(z)$.

If $C(n)$ is an n-tract of $u(z)$, then (see [11]) there exists a continuous curve $\Gamma \subset C(n)$ given by the equation $z = z(t)$, where $0 \leq t < \infty$ and $z(t) \to \infty$ as $t \to \infty$, such that

$$\lim_{\substack{z \to \infty \\ z \in \Gamma}} u(z) = \lim_{t \to \infty} u(z(t)) = \infty.$$

THEOREM A ([7]). *Every subharmonic function in the plane has at least one tract.*

DEFINITION. We call an n-tract $C(n)$ of a subharmonic function $u(z)$ a *strong n-tract* if there is a continuous curve $\Gamma \subset C(n)$, $z = z(t)$, $0 \leq t < \infty$,

such that $z(t) \to \infty$ as $t \to \infty$ and

$$\lim_{t \to \infty} \frac{u(z(t))}{\max\{u(z) : |z| = |z(t)|\}} = 1.$$

Let $\mathrm{p}(n)$ be the number of strong n-tracts of the function $u(z)$. We call $p = \lim_{n \to \infty} \mathrm{p}(n)$ the *number of strong tracts* of the function $u(z)$.

In [10] we provided a similar definition of strong tracts for subharmonic functions of finite lower order and estimated, in that case, the number of strong tracts.

In this paper we are going to estimate the number of strong tracts of subharmonic functions of infinite lower order. To state our results, we introduce *Eremenko's value* for a subharmonic function $u(z)$ (see [3]) as

$$b(\infty, u) = \liminf_{r \to \infty} \frac{\max\{u(z) : |z| = r\}}{rT'_-(r, u)},$$

where $T(r, u)$ is the Nevanlinna characteristic of $u(z)$ and $T'_-(r, u)$ is its left derivative (see [7]).

We prove the following results.

THEOREM 1. *For a subharmonic function $u(z)$ of infinite lower order we have*

$$b(\infty, u) \le \pi.$$

This result was obtained for $u(z) = \log |f(z)|$, where $f(z)$ is a meromorphic function, by Bergweiler and Bock in [2]. The estimate given in Theorem 1 is exact (see [6]).

THEOREM 2. *Let $u(z)$ be a subharmonic function with infinite lower order and with p different strong tracts. Then*

$$p \le \left[\frac{\pi}{b(\infty, u)} \right].$$

In the case $u(z) = \log |f(z)|$, where $f(z)$ is an entire function, the theorem was proved by one of the authors ([9]). The estimate of Theorem 2 is exact (see [9]).

COROLLARY. *If $u(z)$ is a subharmonic function satisfying $b(\infty, u) > 0$, then the number of different strong tracts of $u(z)$ is finite.*

The condition $b(\infty, u) > 0$ is essential for the Corollary, because for the function $u(z) = \log |e^{e^z}|$, $b(\infty, u) = 0$ and $u(z)$ has infinitely many strong tracts.

2. Auxiliary results. Let $u(z)$ be a subharmonic function. Let

$$m^*(r, \theta, u) = \frac{1}{2\pi} \sup_{|E|=2\theta} \int_E u^+(re^{i\varphi}) \, d\varphi,$$

where $|E|$ is the Lebesgue measure of the set E and $u^+(z) = \max\{u(z), 0\}$, be the function defined by Baernstein (see [1]). According to Baernstein's theorem the function $m^*(r, \theta, u)$ is subharmonic in $K = \{re^{i\theta} : 0 < r < \infty, 0 < \theta < \pi\}$, continuous on $K \cup (\mathbb{R}\setminus\{0\}) \times \{0\}$ and convex in $\log r$ for any fixed $\theta \in [0, \pi]$. Also the function defined above is nondecreasing with respect to r for any fixed θ. Moreover

$$m^*(r, \theta, u) = \frac{1}{\pi} \int\limits_0^\theta \widetilde{u}(re^{i\varphi}) \, d\varphi, \quad \frac{\partial m^*}{\partial \theta}(r, \theta, u) = \frac{1}{\pi} \widetilde{u}(re^{i\theta}), \quad 0 < \theta < \pi,$$

where $\widetilde{u}(re^{i\theta})$ is the circular rearrangement of the function $u^+(re^{i\theta})$ ([5]).

Every subharmonic function $u(z)$ is a pointwise limit of a nonincreasing sequence $\{v^k(z)\}$ of subharmonic functions with continuous partial derivatives of second order ([7]).

LEMMA A ([10]). *The sequence $\{m^*(r, \theta, v^k)\}$ converges to $m^*(r, \theta, u)$ uniformly on the set $\{re^{i\theta} : 1 \leq r \leq R, 0 \leq \theta \leq \pi\}$ for every $R > 1$.*

For a real function $\varphi(r)$, we consider the operator

$$\mathrm{L}\varphi(r) = \liminf_{h \to 0} \frac{\varphi(re^h) + \varphi(re^{-h}) - 2\varphi(r)}{h^2}.$$

If the function $\varphi(r)$ is twice differentiable in r, then

$$\mathrm{L}\varphi(r) = r \frac{d}{dr} r \frac{d}{dr} \varphi(r).$$

Since $m^*(r, \theta, u_j)$ is convex in $\log r$, for all $r > 0$ and $\theta \in [0, \pi]$ we have

$$\mathrm{L}m^*(r, \theta, u_j) \geq 0.$$

LEMMA B ([10, 9]). *Let $u(z)$ be a subharmonic function with continuous partial derivatives of second order. For all $r > 0$ and for almost all $\theta \in (0, \pi)$ we have*

$$\mathrm{L}m^*(r, \theta, u_j) \geq -\frac{1}{\pi} \frac{\partial \widetilde{u}_j(re^{i\varphi})}{\partial \varphi}\bigg|_{\varphi=\theta}.$$

Bergweiler and Bock introduced in [2] the sequences of Pólya peaks for meromorphic functions of infinite lower order. Such sequences can also be constructed for subharmonic functions, so let us recall the basic facts. For any sequences $M_n \to \infty$ and $\varepsilon_n \to 0$ there are sequences $\varrho_n \to \infty$ and $\mu_n \to \infty$ such that, for every r satisfying $\log(r/\varrho_n) \leq M_n/\mu_n$, we have

$$(1) \qquad T(r, u) \leq (1 + \varepsilon_n)\left(\frac{r}{\varrho_n}\right)^{\mu_n} T(\varrho_n, u).$$

We put $P_n = \varrho_n e^{-M_n/\mu_n}$ and $Q_n = \varrho_n e^{M_n/\mu_n}$. Then inequality (1) holds for all $r \in [P_n, Q_n]$. We assume that $M_n > 1$. Consider the sets

$$A_n = \left\{ r \in [\varrho_n, Q_n] : T(r, u) \leq \frac{1}{\sqrt{M_n}} \left(\frac{r}{\varrho_n} \right)^{\mu_n} T(\varrho_n, u) \right\},$$

$$B_n = \left\{ r \in [P_n, \varrho_n] : T(r, u) \leq \frac{1}{\sqrt{M_n}} \left(\frac{r}{\varrho_n} \right)^{\mu_n} T(\varrho_n, u) \right\}.$$

We put

$$R_n = \begin{cases} \min A_n & \text{if } A_n \neq \emptyset, \\ Q_n & \text{if } A_n = \emptyset, \end{cases} \qquad t_n = \begin{cases} \max B_n & \text{if } B_n \neq \emptyset, \\ P_n & \text{if } B_n = \emptyset, \end{cases}$$

$$T_n = e^{-2/\mu_n} R_n.$$

Then $t_n < \varrho_n < T_n < R_n$. We may achieve that $M_n \to \infty$ as slowly as we please, so we have the following lemma.

LEMMA C ([2]). *For a subharmonic function of infinite lower order and the sequences defined above,*

$$\frac{T(R_n, u)}{R_n^{\mu_n}} + \frac{T(T_n, u)}{T_n^{\mu_n}} + \frac{T(t_n, u)}{t_n^{\mu_n}} = o\left(\mu_n \int\limits_{t_n}^{T_n} \frac{T(r, u)}{r^{\mu_n+1}} \, dr \right), \qquad n \to \infty.$$

We need one more lemma.

LEMMA D ([8]). *Let a function $f(x)$ be nondecreasing on an interval $[a, b]$ and let $\varphi(x)$ be a nonnegative function having a bounded derivative on $[a, b]$. Then*

$$\int\limits_a^b f'(x)\varphi(x) \, dx \leq f(x)\varphi(x)\big|_a^b - \int\limits_a^b \varphi'(x)f(x) \, dx.$$

Now let $\{v^k(z)\}$ be a nonincreasing sequence of subharmonic functions with continuous partial derivatives of second order, pointwise converging to $u(z)$. For every k and for every n we put (see [4])

$$(2) \qquad \sigma_{k,n}(r) = \int\limits_0^{\pi/2\mu_n} m^*(r, \theta, v^k) \cos \mu_n \theta \, d\theta.$$

LEMMA 1. *For every $\varepsilon > 0$ and sequences $\{\mu_n\}$ and $\{T_n\}$ defined above, there exist $N(\varepsilon) \in \mathbb{N}$ and $K(\varepsilon, n) \in \mathbb{N}$ such that*

$$\frac{(\sigma_{k,n})'_-(T_n)}{T_n^{\mu_n-1}} + \frac{\mu_n \sigma_{k,n}(T_n)}{T_n^{\mu_n}} < \varepsilon \mu_n \int\limits_{t_n}^{T_n} \frac{T(r, u)}{r^{\mu_n+1}} \, dr$$

for $n > N(\varepsilon)$ and $k > K(\varepsilon, n)$.

Proof. Let $\varepsilon > 0$ be given. Let $\{T_n\}$, $\{R_n\}$ and $\{\mu_n\}$ be the sequences defined above. By Lemma C there exists $N \in \mathbb{N}$ such that for $n > N$,

$$(3) \qquad \frac{T(R_n, u)}{R_n^{\mu_n}} + \frac{T(T_n, u)}{T_n^{\mu_n}} < \frac{\varepsilon}{8} \, \mu_n \int\limits_{t_n}^{T_n} \frac{T(r, u)}{r^{\mu_n + 1}} \, dr.$$

Let us choose $N(\varepsilon) \geq N$ so that

$$\varepsilon < T(T_n, u).$$

Applying Lemma A, for $k > K(\varepsilon, n)$, we get

$$m^*(r, \theta, v^k) \leq m^*(r, \theta, u) + \varepsilon \leq T(r, u) + \varepsilon$$

on the set $\{re^{i\theta} : 1 \leq r \leq R_n, \, 0 \leq \theta \leq \pi\}$. Hence

$$\sigma_{k,n}(r) = \int\limits_0^{\pi/2\mu_n} m^*(r, \theta, v^k) \cos \mu_n \theta \, d\theta \leq \int\limits_0^{\pi/2\mu_n} (T(r, u) + \varepsilon) \cos \mu_n \theta \, d\theta$$

$$= \frac{1}{\mu_n} (T(r, u) + \varepsilon).$$

Therefore

$$(4) \qquad \frac{\mu_n \sigma_{k,n}(T_n)}{T_n^{\mu_n}} \leq \frac{T(T_n, u) + \varepsilon}{T_n^{\mu_n}} \leq \frac{2T(T_n, u)}{T_n^{\mu_n}}.$$

Now, because $m^*(r, \theta, v^k)$ is a nondecreasing function of r,

$$(\sigma_{k,n})'_-(r) = \int\limits_0^{\pi/2\mu_n} \frac{\partial m^*(r, \theta, v^k)}{\partial r} \cos \mu_n \theta \, d\theta.$$

Since $m^*(r, \theta, v^k)$ is convex in $\log r$ if θ is fixed, $r\partial m^*(r, \theta, v^k)/\partial r$ is nondecreasing and from Lemma D for $x > r$, we get

$$m^*(x, \theta, v^k) \geq m^*(x, \theta, v^k) - m^*(r, \theta, v^k) \geq \int\limits_r^x \frac{\partial m^*(t, \theta, v^k)}{\partial t} \, dt$$

$$= \int\limits_r^x \frac{t \frac{\partial}{\partial t} m^*(t, \theta, v^k)}{t} \, dt = \int\limits_r^x t \frac{\partial m^*(t, \theta, v^k)}{\partial t} \, d(\log t)$$

$$\geq r \frac{\partial m^*(r, \theta, v^k)}{\partial r} \log \frac{x}{r}.$$

Hence

$$\frac{\partial m^*(r, \theta, v^k)}{\partial r} \leq \frac{1}{r \log \frac{x}{r}} m^*(x, \theta, v^k) \leq \frac{1}{r \log \frac{x}{r}} (T(x, u) + \varepsilon),$$

so, using $T_n = e^{-2/\mu_n} R_n$, we have

$$\left. \frac{\partial m^*(r, \theta, v^k)}{\partial r} \right|_{r=T_n} \leq \frac{1}{T_n \log \frac{R_n}{T_n}} (T(R_n, u) + \varepsilon) \leq \frac{\mu_n}{T_n} T(R_n, u).$$

Thus

$$(5) \qquad \frac{(\sigma_{k,n})'_-(T_n)}{T_n^{\mu_n-1}} = \frac{1}{T_n^{\mu_n-1}} \int\limits_0^{\pi/2\mu_n} \frac{\partial m^*(r,\theta,v^k)}{\partial r}\bigg|_{r=T_n} \cos \mu_n\theta \, d\theta$$

$$\leq \frac{1}{T_n^{\mu_n}} T(R_n,u) = \frac{e^2 T(R_n,u)}{R_n^{\mu_n}} < 8\, \frac{T(R_n,u)}{R_n^{\mu_n}}.$$

Lemma 1 follows from (3), (4) and (5).

3. Proof of Theorem 1.   Applying the definition of $\sigma_{k,n}(r)$ and Lemma B, we get

$$\mathrm{L}\sigma_{k,n}(r) \geq -\frac{1}{\pi} \int\limits_0^{\pi/2\mu_n} \frac{\partial \widetilde{v^k}(re^{i\theta})}{\partial \theta} \cos \mu_n\theta \, d\theta.$$

Integrating by parts we obtain

$$(1) \quad \mathrm{L}\sigma_{k,n}(r) \geq -\frac{1}{\pi}\, \widetilde{v^k}(re^{i\theta}) \cos \mu_n\theta \bigg|_0^{\pi/2\mu_n} - \frac{\mu_n}{\pi} \int\limits_0^{\pi/2\mu_n} \widetilde{v^k}(re^{i\theta}) \sin \mu_n\theta \, d\theta$$

$$= \frac{1}{\pi}\, \widetilde{v^k}(r) - \mu_n m^*\left(r, \frac{\pi}{2\mu_n}, v^k\right) + \mu_n^2 \int\limits_0^{\pi/2\mu_n} m^*(r,\theta,v^k) \cos \mu_n\theta \, d\theta$$

$$= \frac{1}{\pi}\, \widetilde{v^k}(r) - \mu_n m^*\left(r, \frac{\pi}{2\mu_n}, v^k\right) + \mu_n^2 \sigma_{k,n}(r)$$

$$\equiv h_{k,n}(r) + \mu_n^2 \sigma_{k,n}(r).$$

Since $\mathrm{L}m^*(r,\theta,v^k) \geq 0$, from Fatou's lemma we get

$$\mathrm{L}\sigma_{k,n}(r) \geq \int\limits_0^{\pi/2\mu_n} \mathrm{L}m^*(r,\theta,v^k) \cos \mu_n\theta \, d\theta \geq 0$$

and $\sigma_{k,n}(r)$ is convex in $\log r$. It follows that $r(\sigma_{k,n})'_-(r)$ is an increasing function on $(0,\infty)$. Thus, for almost all $r > 0$ we have

$$\mathrm{L}\sigma_{k,n}(r) = r\, \frac{d}{dr} r(\sigma_{k,n})'_-(r).$$

We now divide the inequality (1) by r^{μ_n+1} and integrate it over the interval $[t_n, T_n]$ to obtain

$$\int\limits_{t_n}^{T_n} \frac{\frac{d}{dr} r(\sigma_{k,n})'_-(r)}{r^{\mu_n}} \, dr \geq \int\limits_{t_n}^{T_n} \frac{h_{k,n}(r)}{r^{\mu_n+1}} \, dr + \mu_n^2 \int\limits_{t_n}^{T_n} \frac{\sigma_{k,n}(r)}{r^{\mu_n+1}} \, dr,$$

where the numbers t_n and T_n are defined just before Lemma C. Since

the function $r(\sigma_{k,n})'_-(r)$ is increasing, by Lemma A we have

$$\int_{t_n}^{T_n} \frac{\frac{d}{dr}r(\sigma_{k,n})'_-(r)}{r^{\mu_n}} dr \le \left(\frac{r(\sigma_{k,n})'_-(r)}{r^{\mu_n}} + \mu_n \frac{\sigma_{k,n}(r)}{r^{\mu_n}} \right)\Bigg|_{t_n}^{T_n} + \mu_n^2 \int_{t_n}^{T_n} \frac{\sigma_{k,n}(r)}{r^{\mu_n+1}} dr.$$

Hence

$$\int_{t_n}^{T_n} \frac{h_{k,n}(r)}{r^{\mu_n+1}} dr \le \left(\frac{(\sigma_{k,n})'_-(r)}{r^{\mu_n-1}} + \mu_n \frac{\sigma_{k,n}(r)}{r^{\mu_n}} \right)\Bigg|_{t_n}^{T_n}.$$

Let $\varepsilon > 0$ be given. By Lemma 1, for $n > N(\varepsilon)$ and $k > K(\varepsilon, n)$, we have

$$\int_{t_n}^{T_n} \frac{h_{k,n}(r)}{r^{\mu_n+1}} dr < \varepsilon\mu_n \int_{t_n}^{T_n} \frac{T(r,u)}{r^{\mu_n+1}} dr.$$

Now, if $\{v^k(z)\}$ is a nonincreasing sequence tending to $u(z)$, then $\{\widetilde{v^k}(z)\}$ is a nonincreasing sequence tending to $\widetilde{u}(z)$. By Lemma A, $m^*(r, \pi/2\mu_n, v^k)$ converges to $m^*(r, \pi/2\mu_n, u)$. Hence, using the monotone convergence theorem, we get

$$(2) \qquad \int_{t_n}^{T_n} \frac{h_n(r)}{r^{\mu_n+1}} dr < \varepsilon\mu_n \int_{t_n}^{T_n} \frac{T(r,u)}{r^{\mu_n+1}} dr,$$

where

$$(3) \qquad h_n(r) = \frac{1}{\pi} \widetilde{u}(r) - \mu_n m^*\left(r, \frac{\pi}{2\mu_n}, u\right) \ge \frac{1}{\pi} \widetilde{u}(r) - \mu_n T(r,u).$$

Applying Lemma C, we have

$$(4) \qquad \mu_n \int_{t_n}^{T_n} \frac{T(r,u)}{r^{\mu_n+1}} dr = -\frac{T(r,u)}{r^{\mu_n}}\Bigg|_{t_n}^{T_n} + \int_{t_n}^{T_n} \frac{rT'_-(r,u)}{r^{\mu_n+1}} dr$$

$$\le o\left(\mu_n \int_{t_n}^{T_n} \frac{T(r,u)}{r^{\mu_n+1}} dr \right) + \int_{t_n}^{T_n} \frac{rT'_-(r,u)}{r^{\mu_n+1}} dr$$

$$\le \frac{1}{1-\varepsilon} \int_{t_n}^{T_n} \frac{rT'_-(r,u)}{r^{\mu_n+1}} dr.$$

Now, using (2), (3) and (4) we get

$$\int_{t_n}^{T_n} \frac{\widetilde{u}(r)}{r^{\mu_n+1}} dr < \pi \frac{1+\varepsilon}{1-\varepsilon} \int_{t_n}^{T_n} \frac{rT'_-(r,u)}{r^{\mu_n+1}} dr.$$

Hence there is a sequence $r_n \in [t_n, T_n]$ such that

$$\widetilde{u}(r_n) < \pi \frac{1+\varepsilon}{1-\varepsilon} r_n T'_-(r_n, u)$$

for large enough n's. Since $\widetilde{u}(r_n) \geq \max\{u(z) : |z| = r_n\}$, using the definition of $b(\infty, u)$, we get

$$\widetilde{u}(r_n) > (1 - \varepsilon)b(\infty, u)r_n T'_-(r_n, u).$$

Therefore

$$(1 - \varepsilon)b(\infty, u) < \pi \frac{1 + \varepsilon}{1 - \varepsilon},$$

or

$$b(\infty, u) < \pi \frac{1 + \varepsilon}{(1 - \varepsilon)^2}.$$

Since $\varepsilon > 0$ is arbitrary we obtain Theorem 1.

4. Proof of Theorem 2. Let $u(z)$ be a subharmonic function and let p be the number of strong tracts of $u(z)$. Let $q \leq p$ be a natural number. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\mathrm{E}(n)$ has at least q connected components. Consider the functions (see [7])

$$u_j(z) = \begin{cases} u(z) & \text{if } z \in C_j(n_0), \\ n_0 & \text{if } z \notin C_j(n_0), \end{cases}$$

for $j = 1, \ldots, q$. The functions $u_j(z)$ are subharmonic in $\mathbb{C}$. Also, for every $j \in \{1, \ldots, q\}$, $u_j(z)$ is the pointwise limit of a nonincreasing sequence $\{v_j^k(z)\}$ of subharmonic functions with continuous partial derivatives of second order.

Now, we define the function

$$m_0^*(r, \theta, u) = \sum_{j=1}^{q} m^*(r, \theta, u_j).$$

The function $m_0^*(r, \theta, u)$, being the sum of $m^*(r, \theta, u_j)$ for $j \in \{1, \ldots, q\}$, is subharmonic in $K = \{re^{i\theta} : 0 < r < \infty, 0 < \theta < \pi\}$, continuous on $K \cup (\mathbb{R} \setminus \{0\}) \times \{0\}$ and convex in $\log r$ for any fixed $\theta \in [0, \pi]$. It is also nondecreasing with respect to r for any fixed θ.

For every k and for every n we put

$$\sigma_{k,n}(r) = \int_0^{\pi/2\mu_n} (m_k)_0^*(r, \theta) \cos \mu_n \theta \, d\theta,$$

where $(m_k)_0^*(r, \theta) = \sum_{j=1}^{q} m^*(r, \theta, v_j^k)$.

It is easy to see that Lemma 1 also holds for $\sigma_{k,n}(r)$ defined above.

Now, using the definition of $\sigma_{k,n}(r)$ and Lemma B, we have

$$\mathrm{L}\sigma_{k,n}(r) \geq -\frac{1}{\pi} \sum_{j=1}^{q} \int_0^{\pi/2\mu_n} \frac{\partial \widetilde{v}_j^k(re^{i\theta})}{\partial \theta} \cos \mu_n \theta \, d\theta.$$

Proceeding analogously to the proof of Theorem 1 we get

$$h_{k,n}(r) = \frac{1}{\pi} \sum_{j=1}^{q} \widetilde{v}_j^k(r) - \mu_n(m_k)_0^* \left(r, \frac{\pi}{2\mu_n}\right)$$

and

$$h_n(r) = \frac{1}{\pi} \sum_{j=1}^{q} \widetilde{u}_j(r) - \mu_n m_0^* \left(r, \frac{\pi}{2\mu_n}, u\right).$$

Then

$$h_n(r) \geq \frac{1}{\pi} \sum_{j=1}^{q} \widetilde{u}_j(r) - \mu_n(T(r,u) + qn_0) \geq \frac{1}{\pi} \sum_{j=1}^{q} \widetilde{u}_j(r) - \mu_n(1+\varepsilon)T(r,u).$$

Finally, we get

$$\int_{t_n}^{T_n} \frac{\sum_{j=1}^{q} \widetilde{u}_j(r)}{r^{\mu_n+1}} \, dr < \pi \frac{1+2\varepsilon}{1-\varepsilon} \int_{t_n}^{T_n} \frac{rT'_-(r,u)}{r^{\mu_n+1}} \, dr.$$

Hence there is a sequence $r_n \in [t_n, T_n]$ such that

$$\sum_{j=1}^{q} \widetilde{u}_j(r_n) < \pi \frac{1+2\varepsilon}{1-\varepsilon} r_n T'_-(r_n, u)$$

for large enough n's. Since our tracts are strong,

$$\widetilde{u}_j(r_n) = \max_{|z|=r_n} u_j(z) \geq \max_{\substack{|z|=r_n \\ z \in \Gamma_j}} u_j(z) = \max_{\substack{|z|=r_n \\ z \in \Gamma_j}} u(z) > (1-\varepsilon) \max_{|z|=r_n} u(z).$$

From the definition of $b(\infty, u)$ we get

$$\widetilde{u}_j(r_n) > (1-\varepsilon)^2 b(\infty, u) r_n T'_-(r_n, u).$$

Hence

$$q(1-\varepsilon)^2 b(\infty, u) < \pi \frac{1+2\varepsilon}{1-\varepsilon},$$

or

$$q < \frac{\pi}{b(\infty, u)} \cdot \frac{1+2\varepsilon}{(1-\varepsilon)^3}.$$

Since $\varepsilon > 0$ is arbitrary and q is an arbitrary natural number such that $q \leq p$, we have proved Theorem 2.

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