ANNALES
POLONICI MATHEMATICI
92.3 (2007)

Proper holomorphic mappings in the special class of Reinhardt domains

by Łukasz Kosiński (Kraków)

Abstract. A complete characterization of proper holomorphic mappings between domains from the class of all pseudoconvex Reinhardt domains in $\mathbb{C}^2$ with the logarithmic image equal to a strip or a half-plane is given.

1. Statement of results. We adopt the standard notations of complex analysis. Given $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$ and $z = (z_1, z_2) \in \mathbb{C}^2$ we put $|z^\gamma| = |z_1|^{\gamma_1}|z_2|^{\gamma_2}$ whenever it makes sense. The unit disc in $\mathbb{C}$ is denoted by $\mathbb{D}$ and the set of proper holomorphic mappings between domains $D, G \subset \mathbb{C}^n$ is denoted by $\mathrm{Prop}(D, G)$.

In this paper we deal with those pseudoconvex Reinhardt domains in $\mathbb{C}^2$ whose logarithmic image is equal to a strip or a half-plane. Observe that such domains are always algebraically equivalent to domains of the form

$$D_{\alpha,r^-,r^+} := \{z \in \mathbb{C}^2 : r^- < |z^\alpha| < r^+\},$$

where $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{R}^2)_*,\ 0 < r^+ < \infty,\ -\infty < r^- < r^+$.

We say that D_{α,r^-,r^+} is of the *irrational type* if $\alpha_1/\alpha_2 \in \mathbb{R} \setminus \mathbb{Q}$. In the other case it is of the *rational type*.

Recall that if $r^- < 0 < r^+$ and $\alpha \in (\mathbb{R}^2)_*$, then D_{α,r^-,r^+} are so-called *elementary Reinhardt domains*.

Below we shall give a complete description of all proper holomorphic mappings from D_{α,r_1^-,r_1^+} to D_{β,r_2^-,r_2^+} for arbitrary $\alpha, \beta \in (\mathbb{R}^2)_*$ and $0 < r_i^+ < \infty,\ -\infty < r_i^- < r_i^+,\ i = 1, 2$. Similar problems have been studied in the literature. In [Shi1] and [Shi2] the problem of holomorphic equivalence of elementary Reinhardt domains was considered. Those results were partially extended by A. Edigarian and W. Zwonek [Edi-Zwo] who gave a charac-

2000 *Mathematics Subject Classification*: 32H35, 32A07.
Key words and phrases: proper holomorphic mappings, Reinhardt domains, elementary Reinhardt domains.

terization of proper holomorphic mappings between elementary Reinhardt domains of the rational type.

Set $\mathbb{A}(\varrho^-,\varrho^+) := \{z \in \mathbb{C} : \varrho^- < |z| < \varrho^+\}$ for $\varrho^+ > 0$, $\varrho^- < \varrho^+$ and $\mathbb{A}_\varrho := \mathbb{A}(1/\varrho,\varrho)$, $\varrho > 1$. Moreover, put

$$D_{\gamma,r} := \{z \in \mathbb{C}^2 : 1/r < |z_1|\,|z_2|^\gamma < r\}, \quad \gamma \in \mathbb{R}_*,\ r > 1,$$
$$D_\gamma := \{z \in \mathbb{C}^2 : |z_1|\,|z_2|^\gamma < 1\}, \quad \gamma \in \mathbb{R}_*,$$
$$D^*_\gamma := \{z \in \mathbb{C}^2 : 0 < |z_1|\,|z_2|^\gamma < 1\}, \quad \gamma \in \mathbb{R}_*.$$

Note that if γ is rational, i.e. $\gamma = p/q$ for some relatively prime $p, q \in \mathbb{Z}$, $q > 0$, then $D_{\gamma,r}$ is biholomorphically equivalent to $\mathbb{A}_{r^q} \times \mathbb{C}_*$ and D^*_γ is biholomorphically equivalent to $\mathbb{D}_* \times \mathbb{C}$. Indeed, put

$$\psi(z_1,z_2) := (z_1^q z_2^p, z_1^m z_2^n) \quad \text{for } (z_1,z_2) \in \mathbb{C}^2,$$

where $m, n \in \mathbb{Z}$ are such that $pm - qn = 1$. One can check that the mappings $\psi|_{D_{\gamma,r}} : D_{\gamma,r} \to \mathbb{A}_{r^q} \times \mathbb{C}_*$ and $\psi|_{D^*_\gamma} : D^*_\gamma \to \mathbb{D}_* \times \mathbb{C}_*$ are biholomorphic.

Moreover, one may easily prove that D_{α,r^-,r^+} is algebraically equivalent to a domain of one of the following types:

(i) If $r^- > 0$:
  (a) $\mathbb{A}_\varrho \times \mathbb{C}$, $\alpha_1\alpha_2 = 0$,
  (b) $\mathbb{A}_\varrho \times \mathbb{C}_*$, $\alpha_1/\alpha_2 \in \mathbb{Q}_*$,
  (c) $D_{\gamma,\varrho}$, $\gamma = \alpha_2/\alpha_1 \in \mathbb{R} \setminus \mathbb{Q}$.

(ii) If $r^- = 0$:
  (a) $\mathbb{D}_* \times \mathbb{C}$, $\alpha_1\alpha_2 = 0$,
  (b) $\mathbb{D}_* \times \mathbb{C}_*$, $\alpha_1/\alpha_2 \in \mathbb{Q}_*$,
  (c) D^*_γ, $\gamma = \alpha_2/\alpha_1 \in \mathbb{R} \setminus \mathbb{Q}$.

(iii) If $r^- < 0$:
  (a) $\mathbb{D} \times \mathbb{C}$, $\alpha_1\alpha_2 = 0$,
  (b) D_γ, $\gamma = \alpha_2/\alpha_1 \neq 0$.

Our main result is the following:

THEOREM 1.

(a) *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then the set of proper holomorphic mappings from $D_{\alpha,r}$ to $D_{\beta,R}$ is non-empty if and only if*

$$\frac{\log R}{\log r} \in \mathbb{Z} + \beta\mathbb{Z}, \qquad \alpha\frac{\log R}{\log r} \in \mathbb{Z} + \beta\mathbb{Z}. \tag{1}$$

(b) *Let $\alpha, \beta \in \mathbb{R}\setminus\mathbb{Q}$ and let $r, R > 1$ be such that $\frac{\log R}{\log r} = k_1 + l_1\beta$ and $\alpha\frac{\log R}{\log r} = k_2 + l_2\beta$ for some integers k_i, l_i, $i = 1, 2$. Then any proper holomorphic mapping $f : D_{\alpha,r} \to D_{\beta,R}$ is of one of the following forms:*

$$\text{(2)}\qquad \begin{cases} f(z)=(az_1^{k_1}z_2^{k_2}, bz_1^{l_1}z_2^{l_2}),\\ f(z)=(az_1^{-k_1}z_2^{-k_2}, bz_1^{-l_1}z_2^{-l_2}),\end{cases}\qquad z=(z_1,z_2)\in D_{\alpha,r},$$

where $a,b\in\mathbb{C}$ *satisfy* $|a|\,|b|^\beta=1$. *Moreover, any of the mappings given by* (2) *is proper.*

Notice that in Theorem 1(a) we do not demand β to be irrational.

Using Theorem 1 we will easily obtain analogous results for domains of the forms (ii) and (iii) of the irrational type.

THEOREM 2. *Let* $\alpha,\beta\in\mathbb{R}\setminus\mathbb{Q}$. *The set of proper holomorphic mappings from* D_α^* *to* D_β^* *is non-empty if and only if* $\alpha=(k_2+\beta l_2)/(k_1+\beta l_1)$ *for some* $k_i,l_i\in\mathbb{Z}$, $i=1,2$. *Moreover, in that case, if* $k_1+l_1\beta>0$, *then any proper holomorphic mapping* $f:D_\alpha^*\to D_\beta^*$ *is of the form*

$$\text{(3)}\qquad f(z_1,z_2)=(az_1^{k_1}z_2^{k_2}, bz_1^{l_1}z_2^{l_2}),\quad (z_1,z_2)\in D_\alpha^*,$$

where $a,b\in\mathbb{C}$ *satisfy* $|a|\,|b|^\beta=1$.

THEOREM 3. *Let* $\alpha,\beta\in\mathbb{R}\setminus\mathbb{Q}$. *Then the set* $\mathrm{Prop}(D_\alpha,D_\beta)$ *is non-empty if and only if* $\alpha=(k_2+\beta l_2)/(k_1+\beta l_1)$ *for some* $k_i,l_i\in\mathbb{Z}_{\geq 0}$, $i=1,2$. *Moreover, in that case any proper holomorphic mapping* $f:D_\alpha\to D_\beta$ *is of the form*

$$\text{(4)}\qquad f(z_1,z_2)=(az_1^{k_1}z_2^{k_2}, bz_1^{l_1}z_2^{l_2}),\quad (z_1,z_2)\in D_\alpha,$$

where $a,b\in\mathbb{C}$ *are such that* $|a|\,|b|^\beta=1$.

Next we prove the following

THEOREM 4. *Let* $\alpha,\beta\in(\mathbb{R}^2)_*$, $r_i^+>0$, $r_i^-<r_i^+$, $i=1,2$. *Assume that the sets* D_{α,r_1^-,r_1^+}, D_{β,r_2^-,r_2^+} *are of the same type* (*either rational or irrational*). *If there exists a proper holomorphic mapping between them, then either* $r_1^-r_2^->0$ *or* $r_1^-=r_2^-=0$.

For domains of different types we have the following result:

THEOREM 5. *Let* $\alpha,\beta\in(\mathbb{R}^2)_*$, $r_i^+>0$, $r_i^-<r_i^+$, $i=1,2$. *If the sets* D_{α,r_1^-,r_1^+} *and* D_{β,r_2^-,r_2^+} *are of different types, then there is no proper holomorphic mapping between them.*

Finally, we discuss the rational case. As already mentioned, the set of proper holomorphic mappings between elementary Reinhardt domains of the rational type was described in [Edi-Zwo]. Thus, in order to obtain the desired characterization, it suffices to prove the following three theorems.

THEOREM 6. *Let* $r,R>1$. *If* $R\neq r^m$ *for any natural number* m, *then* $\mathrm{Prop}(\mathbb{A}_r\times\mathbb{C},\mathbb{A}_R\times\mathbb{C})$, $\mathrm{Prop}(\mathbb{A}_r\times\mathbb{C}_*,\mathbb{A}_R\times\mathbb{C})$ *and* $\mathrm{Prop}(\mathbb{A}_r\times\mathbb{C}_*,\mathbb{A}_R\times\mathbb{C}_*)$ *are empty. Moreover, for any* $m\in\mathbb{N}$:

(a) $\mathrm{Prop}(\mathbb{A}_r \times \mathbb{C}, \mathbb{A}_{r^m} \times \mathbb{C})$ *consists of the mappings of the form*

$$\mathbb{A}_r \times \mathbb{C} \ni (z,w) \mapsto (e^{i\theta} z^{\varepsilon m}, a_N(z)w^N + \cdots + a_0(z)) \in \mathbb{A}_{r^m} \times \mathbb{C},$$

where $\theta \in \mathbb{R}$, $N \in \mathbb{N}$, $\varepsilon = \pm 1$ *and* $a_0, \ldots, a_N \in \mathcal{O}(\mathbb{A}_r)$ *are such that* $|a_0(z)| + \cdots + |a_N(z)| > 0$, $z \in \mathbb{A}_r$.

(b) $\mathrm{Prop}(\mathbb{A}_r \times \mathbb{C}_*, \mathbb{A}_{r^m} \times \mathbb{C})$ *consists of the mappings of the form*

$$\mathbb{A}_r \times \mathbb{C}_* \ni (z,w) \mapsto \left(e^{i\theta} z^{\varepsilon m}, \frac{a_N(z)w^N + \cdots + a_0(z)}{w^k} \right) \in \mathbb{A}_{r^m} \times \mathbb{C},$$

where $\theta \in \mathbb{R}$, $k, N \in \mathbb{N}$, $0 < k < N$, $\varepsilon = \pm 1$ *and* $a_i \in \mathcal{O}(\mathbb{A}_r)$, $i = 1, \ldots, N$, *satisfy* $|a_0(z)| + \cdots + |a_{k-1}(z)| > 0$ *and* $|a_{k+1}(z)| + \cdots + |a_N(z)| > 0$ *for* $z \in \mathbb{A}_r$.

(c) $\mathrm{Prop}(\mathbb{A}_r \times \mathbb{C}_*, \mathbb{A}_{r^m} \times \mathbb{C}_*)$ *consists of the mappings of the form*

$$\mathbb{A}_r \times \mathbb{C}_* \ni (z,w) \mapsto (e^{i\theta} z^m, a(z)w^k) \in \mathbb{A}_{r^m} \times \mathbb{C}_*,$$

where $\varepsilon = \pm 1$, $\theta \in \mathbb{R}$, $k \in \mathbb{N}$ *and* $a \in \mathcal{O}(\mathbb{A}_r, \mathbb{C}_*)$.

THEOREM 7. *There are no proper holomorphic mappings from* $\mathbb{A}_r \times \mathbb{C}$ *to* $\mathbb{A}_R \times \mathbb{C}_*$ *for any* $r, R > 1$.

THEOREM 8.

(a) $\mathrm{Prop}(\mathbb{D}_* \times \mathbb{C}, \mathbb{D}_* \times \mathbb{C})$ *consists of the mappings of the form*

$$\mathbb{D}_* \times \mathbb{C} \ni (z,w) \mapsto (e^{i\theta} z^m, a_N(z)w^N + \cdots + a_0(z)) \in \mathbb{D}_* \times \mathbb{C},$$

where $\theta \in \mathbb{R}$, $N \in \mathbb{N}$, $m \in \mathbb{N}$ *and* $a_0, \ldots, a_N \in \mathcal{O}(\mathbb{D}_*)$ *are such that* $|a_0(z)| + \cdots + |a_N(z)| > 0$, $z \in \mathbb{D}_*$.

(b) $\mathrm{Prop}(\mathbb{D}_* \times \mathbb{C}_*, \mathbb{D}_* \times \mathbb{C})$ *consists of the mappings of the form*

$$\mathbb{D}_* \times \mathbb{C}_* \ni (z,w) \mapsto \left(e^{i\theta} z^m, \frac{a_N(z)w^N + \cdots + a_0(z)}{w^k} \right) \in \mathbb{D}_* \times \mathbb{C},$$

where $\theta \in \mathbb{R}$, $m \in \mathbb{N}$, $k, N \in \mathbb{N}$, $0 < k < N$ *and* $a_i \in \mathcal{O}(\mathbb{D}_*)$, $i = 1, \ldots, N$ *satisfy* $|a_0(z)| + \cdots + |a_{k-1}(z)| > 0$ *and* $|a_{k+1}(z)| + \cdots + |a_N(z)| > 0$ *for* $z \in \mathbb{D}_*$.

(c) $\mathrm{Prop}(\mathbb{D}_* \times \mathbb{C}_*, \mathbb{D}_* \times \mathbb{C}_*)$ *consists of the mappings of the form*

$$\mathbb{D}_* \times \mathbb{C}_* \ni (z,w) \mapsto (e^{i\theta} z^m, a(z)w^k) \in \mathbb{D}_* \times \mathbb{C}_*,$$

where $\theta \in \mathbb{R}$, $k \in \mathbb{N}$ *and* $a \in \mathcal{O}(\mathbb{D}_*, \mathbb{C}_*)$.

(d) $\mathrm{Prop}(\mathbb{D}_* \times \mathbb{C}, \mathbb{D}_* \times \mathbb{C}_*) = \emptyset$.

2. Proofs. The following result is probably known. However, we could not find it in the literature, so we present a proof.

LEMMA 9. *Let* $D \subset \mathbb{C}^n$ *be a domain,* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *and let* $f, g : D \to \mathbb{C}$ *be holomorphic mappings with* $|f(z)| = |g(z)|^\alpha$, $z \in D$. *Then either* $f = g = 0$ *on* D *or there exists a holomorphic branch of the logarithm of* g, *i.e.*

a mapping $\psi \in \mathcal{O}(D)$ such that $e^{\psi} = g$ on D. In particular, there exists a $\theta \in \mathbb{R}$ such that $f = e^{i\theta + \alpha\psi}$ on D.

Proof. Comparing multiplicities of the roots of the functions f and g composed with affine mappings we may reduce our considerations to the case when $f, g : D \to \mathbb{C}_*$. Moreover, we may assume that $g(x') \in \mathbb{R}_{>0}$ for some $x' \in D$.

Obviously, there exists an $\eta \in \mathbb{R}$ such that the set $G_\eta := \{z \in D : e^{i\eta} f(z) \in g(z)^\alpha\}$ is non-empty. Considering, if necessary, a mapping $e^{i\eta} f$ instead of f we may assume that $\eta = 0$.

It is easy to see that G_0 is an open-and-closed subset of D, and so $G_0 = D$. Thus, there exists a holomorphic branch of g^α (also denoted by g^α) such that $g^\alpha(x') \in \mathbb{R}_{>0}$. It follows that there exist g^t for any $t \in Q := \{k + l\alpha : k, l \in \mathbb{Z}\}$. Fix a sequence $(t_m)_{m=1}^\infty \subset Q$ converging to 0. In virtue of Montel's theorem, it is clear that $g^{t_m} \to 1$ locally uniformly.

Put $\psi_m := (g^{t_m} - 1)/t_m$. Then $\lim_{m\to\infty} \psi_m(x') = \log g(x')$ and the sequence $(\psi'_m)_{m=1}^\infty = (g^{t_m - 1} g')_{m=1}^\infty$ is locally uniformly convergent with limit $(1/g)g'$. Thus $(\psi_m)_{m=1}^\infty$ converges locally uniformly on D. Denote its limit by ψ. By the Weierstrass theorem, ψ is holomorphic on D and $\psi' = \lim_{m\to\infty} \psi'_m = (1/g)g'$.

Let $\widetilde{D} \subset D$ be any simply connected neighborhood of x'. Let $\widetilde{\psi}$ be a holomorphic mapping on $\widetilde{D}$ such that $g|_{\widetilde{D}} = e^{\widetilde{\psi}}$ and $\widetilde{\psi}(x') = \log g(x')$. It is easy to see that $\widetilde{\psi} = \psi$ on $\widetilde{D}$, and so, by the identity principle, $g = e^\psi$ on D. ■

LEMMA 10. *Let $0 < r_i^+$, $-\infty < r_i^- < r_i^+$, $i = 1, 2$, $\alpha, \beta \in \mathbb{R}$. Let $(\lambda_n)_{n=1}^\infty \subset \mathbb{A}(r_1^-, r_1^+)$. Let $\phi \in \operatorname{Prop}(D_{(1,\alpha), r_1^-, r_1^+}, D_{(1,\beta), r_2^-, r_2^+})$. Put*

$$v(\lambda) := |\phi_1(\lambda, 1)|\, |\phi_2(\lambda, 1)|^\beta, \qquad \lambda \in \mathbb{A}(r_1^-, r_1^+).$$

If the sequence $(\lambda_n)_{n=1}^\infty$ has no accumulation points in $\mathbb{A}(r_1^-, r_1^+)$, then $(v(\lambda_n))_{n=1}^\infty$ has no accumulation points in $\mathbb{A}(r_2^-, r_2^+)$.

Proof. Assume that $v(\lambda_n) \to q$. It suffices to show that $q \in \partial\mathbb{A}(r_2^-, r_2^+)$.

Otherwise $q \in \mathbb{A}(r_2^-, r_2^+)$. Note that for any $\lambda \in \mathbb{A}(r_1^-, r_1^+)$ the function

$$u_\lambda : \mathbb{C} \ni z \mapsto |\phi_1(\lambda e^{-\alpha z}, e^z)|\, |\phi_2(\lambda e^{-\alpha z}, e^z)|^\beta \tag{5}$$

is bounded and subharmonic, so u_λ is constant.

Since ϕ is proper, the mapping $\mathbb{C} \ni z \mapsto \phi_2(\lambda_n e^{-\alpha z}, e^z) \in \mathbb{C}$ is non-constant for any $n \in \mathbb{N}$. Picard's theorem implies that there is a sequence $(z_n)_{n=0}^\infty \subset \mathbb{C}$ such that $|\phi_2(\lambda_n e^{-\alpha z_n}, e^{z_n})|^\beta = 1$. Obviously $u_\lambda(z) = u_\lambda(1) = v(\lambda)$ for all $z \in \mathbb{C}$ and $v(\lambda_n) \to q$, so $|\phi_1(\lambda_n e^{-\alpha z_n}, e^{z_n})| \to q$. In particular, the set $\{\phi(\lambda_n e^{-\alpha z_n}, e^{z_n}) : n \in \mathbb{N}\}$ is relatively compact in $D_{(1,\beta), r_2^-, r_2^+}$; however, $((\lambda_n e^{-\alpha z_n}, e^{z_n}))_{n=1}^\infty$ has no accumulation points in $D_{(1,\alpha), r_1^-, r_1^+}$, a contradiction. ■

COROLLARY 11. *Let $\phi=(\phi_1,\phi_2):D_{\alpha,r}\to D_{\beta,R}$ be a proper holomorphic mapping and let $\alpha,\beta\in\mathbb{R}_{>0}$, $r,R>1$. Put $v(\lambda):=|\phi_1(\lambda,1)|\,|\phi_2(\lambda,1)|^\beta$, $\lambda\in\mathbb{A}_r$. Then either*

$$\lim_{|\lambda|\to 1/r} v(\lambda)=1/R,\ \lim_{|\lambda|\to r} v(\lambda)=R \quad \textit{or} \quad \lim_{|\lambda|\to 1/r} v(\lambda)=R,\ \lim_{|\lambda|\to r} v(\lambda)=1/R.$$

LEMMA 12. *Let $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, $\beta\in\mathbb{R}$, $-\infty<r_i^-<r_i^+<\infty$, $0<r_i^+$, $i=1,2$, and let $\phi:D_{(1,\alpha),r_1^-,r_1^+}\to D_{(1,\beta),r_2^-,r_2^+}$ be a holomorphic mapping. Then for any $\lambda\in\mathbb{A}(r_1^-,r_1^+)$,*

$$\phi(\{(z_1,z_2)\in\mathbb{C}^2:|z_1|\,|z_2|^\alpha=|\lambda|\}) \subset \{(w_1,w_2)\in\mathbb{C}^2:|w_1|\,|w_2|^\beta=|\phi_1(\lambda,1)|\,|\phi_2(\lambda,1)|^\beta\}.$$

Proof. Note that for any $\lambda\in\mathbb{A}(r_1^-,r_1^+)$ the function

$$u:\mathbb{C}\ni z\mapsto|\phi_1(\lambda e^{\alpha z},e^{-z})|\,|\phi_2(\lambda e^{\alpha z},e^{-z})|^\beta$$

is subharmonic and bounded. Hence u is constant.

In virtue of Kronecker's theorem, the set $\{(|\lambda|e^{\alpha z},e^{-z}):z\in\mathbb{C}\}$ is dense in $\{(z_1,z_2)\in\mathbb{C}^2:|z_1|\,|z_2|^\alpha=|\lambda|\}$. Thus, there is $t\in\mathbb{R}$ such that

$$\phi(\{(z_1,z_2)\in\mathbb{C}^2:|z_1|\,|z_2|^\alpha=|\lambda|\})\subset\{(w_1,w_2)\in\mathbb{C}^2:|w_1|\,|w_2|^\beta=t\}.$$

It is easy to see that $t=|\phi_1(\lambda,1)|\,|\phi_2(\lambda,1)|^\beta$. ∎

Proof of Theorem 1(a). Let $\phi:D_{\alpha,r}\to D_{\beta,R}$ be a proper holomorphic mapping. Put $v(\lambda):=|\phi_1(\lambda,1)|\,|\phi_2(\lambda,1)|^\beta$, $\lambda\in\mathbb{A}_r$. Obviously, $\log v$ is a harmonic function. Applying Corollary 11 and Hadamard's theorem we infer that

$$v(\lambda)=|\lambda|^{\frac{\log R}{\log r}},\ \lambda\in\mathbb{A}_r \quad\text{or}\quad v(\lambda)=|\lambda|^{-\frac{\log R}{\log r}},\ \lambda\in\mathbb{A}_r.$$

From this and Lemma 12 we easily conclude that there is $\varepsilon=\pm1$ such that

$$|\phi_1(z)|\,|\phi_2(z)|^\beta=|z_1|^{\varepsilon\frac{\log R}{\log r}}|z_2|^{\varepsilon\alpha\frac{\log R}{\log r}},\qquad z\in D_{\alpha,r}. \tag{6}$$

Let $z_2=1$, $z_1=z\in\mathbb{A}_r$, $\psi_i(z):=\phi_i(z,1)$, $i=1,2$. Then

$$\log(\psi_1(z)\overline{\psi}_1(z))+\beta\log(\psi_2(z)\overline{\psi}_2(z))=\varepsilon\frac{\log R}{\log r}\log(z\overline{z}).$$

Differentiating with respect to z we get

$$\frac{\psi_1'(z)}{\psi_1(z)}+\beta\frac{\psi_2'(z)}{\psi_2(z)}=\varepsilon\frac{\log R}{\log r}\cdot\frac{1}{z},\qquad z\in\mathbb{A}_r. \tag{7}$$

It follows that

$$\operatorname{Ind}(\psi_1\circ\gamma;0)+\beta\operatorname{Ind}(\psi_2\circ\gamma;0)=\varepsilon\frac{\log R}{\log r},$$

where γ is the unit circle. Hence $\frac{\log R}{\log r}\in\mathbb{Z}+\beta\mathbb{Z}$. The same argument with respect to the second variable shows that $\alpha\frac{\log R}{\log r}\in\mathbb{Z}+\beta\mathbb{Z}$.

To prove the converse, assume that

$$\frac{\log R}{\log r} = k_1 + l_1\beta, \qquad \alpha\,\frac{\log R}{\log r} = k_2 + l_2\beta,$$

where $k_i, l_i \in \mathbb{Z}$, $i = 1, 2$. Define $\phi_1(z) := z_1^{k_1} z_2^{k_2}$, $\phi_2(z) := z_1^{l_1} z_2^{l_2}$ for $z = (z_1, z_2) \in \mathbb{C}^2$, and $\phi := (\phi_1, \phi_2)$. Observe that $\phi|_{D_{\alpha,r}} \in \mathrm{Prop}(D_{\alpha,r}, D_{\beta,R})$. Indeed, it is easy to check that

$$|\phi_1(z)|\,|\phi_2(z)|^{\beta} = |z_1|^{\log R/\log r}|z_2|^{\alpha\log R/\log r}, \qquad (z_1, z_2) \in D_{\alpha,r}, \tag{8}$$

so $\phi|_{D_{\alpha,r}} \in \mathcal{O}(D_{\alpha,r}, D_{\beta,R})$. Since $k_1 l_2 \neq k_2 l_1$, ϕ is a proper holomorphic mapping from $(\mathbb{C}_*)^2$ into itself (see [Zwo, Theorem 2.1]). Now we immediately conclude from (8) that $\phi|_{D_{\alpha,r}}$ is a proper holomorphic mapping from $D_{\alpha,r}$ to $D_{\beta,R}$. ■

Lemma 9 and (6) lead to the following

COROLLARY 13. *Let $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ and $\phi \in \mathrm{Prop}(D_{\alpha,r}, D_{\beta,R})$. Assume that $\frac{\log R}{\log r} = k_1 + l_1\beta$ and $\alpha\frac{\log R}{\log r} = k_2 + l_2\beta$ for some $k_i, l_i \in \mathbb{Z}$, $i = 1, 2$. Then there are $\theta \in \mathbb{R}$, $\psi \in \mathcal{O}(D_{\alpha,r})$ and $\varepsilon \in \{1, -1\}$ such that*

$$\phi(z) = (z_1^{\varepsilon k_1} z_2^{\varepsilon k_2} e^{i\theta} e^{-\beta\psi(z)}, z_1^{\varepsilon l_1} z_2^{\varepsilon l_2} e^{\psi(z)}), \quad z \in D_{\alpha,r}.$$

REMARK 14. We may always assume that ε in Corollary 13 is equal to 1 (replacing if necessary ϕ by $\phi \circ h$, where $h \in \mathrm{Aut}(D_{\alpha,r})$, $h(z_1, z_2) := (z_1^{-1}, z_2^{-1})$).

To prove Theorem 1(b) we need the following notation. Put $X_{\alpha,r} := \{z \in \mathbb{C}^2 : -\log r < \mathrm{Re}\, z_1 + \alpha\,\mathrm{Re}\, z_2 < \log r\}$ and $\Pi(z_1, z_2) := (e^{z_1}, e^{z_2})$ for $(z_1, z_2) \in \mathbb{C}^2$. It is clear that $(X_{\alpha,r}, \Pi)$ is the universal covering of $D_{\alpha,r}$.

LEMMA 15. *Let $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$, $r, R > 1$ and assume that $\frac{\log R}{\log r} = k_1 + l_1\beta$, $\alpha\frac{\log R}{\log r} = k_2 + l_2\beta$, where $k_i, l_i \in \mathbb{Z}$, $i = 1, 2$. Let $f : D_{\alpha,r} \to D_{\beta,R}$ be a proper holomorphic mapping. Then every continuous lifting of the mapping $f \circ \Pi : X_{\alpha,r} \to D_{\beta,R}$ is proper and holomorphic.*

Proof. By Corollary 13 and Remark 14 we may assume that $f(z) = (z_1^{k_1} z_2^{k_2} e^{-\beta\psi(z)+i\theta}, z_1^{l_1} z_2^{l_2} e^{\psi(z)})$ $(z \in D_{\alpha,r})$, where $\theta \in \mathbb{R}$ and $\psi \in \mathcal{O}(D_{\alpha,r})$.

Let $\widetilde{f}$ be any continuous lifting of $f \circ \Pi : X_{\alpha,r} \to D_{\beta,R}$, that is, $\widetilde{f} : X_{\alpha,r} \to X_{\beta,R}$ and $f \circ \Pi = \Pi \circ \widetilde{f}$. It is obvious that $\widetilde{f}$ is holomorphic. Then by the identity principle

$$\begin{cases} \widetilde{f}_1(z) = k_1 z_1 + k_2 z_2 - \beta\psi(e^{z_1}, e^{z_2}) + i\theta + 2\mu_1\pi i, \\ \widetilde{f}_2(z) = l_1 z_1 + l_2 z_2 + \psi(e^{z_1}, e^{z_2}) + 2\mu_2\pi i, \end{cases} \qquad z \in X_{\alpha,r}, \tag{9}$$

for some $\mu_i \in \mathbb{Z}$, $i = 1, 2$.

Suppose that $\widetilde{f}$ is not proper, i.e. there is a sequence $(z^m)_{m=1}^{\infty} \subset X_{\alpha,r}$, $z^m = (z_1^m, z_2^m)$, $m \in \mathbb{N}$, without any accumulation points in $X_{\alpha,r}$ such that $(\widetilde{f}(z^m))_{m=1}^{\infty}$ is convergent in $X_{\beta,R}$. Put $y_0 := \lim_{m\to\infty} \widetilde{f}(z^m) \in X_{\beta,R}$.

Obviously, $f(\Pi(z_1^m, z_2^m)) = \Pi(\widetilde{f}(z_1^m, z_2^m)) \to \Pi(y_0)$. Since f is proper, the set $\{\Pi(z^m) : m \geq 1\}$ is relatively compact in $D_{\alpha,r}$. Thus we may assume that $(\Pi(z^m))_{m=1}^{\infty}$ is convergent, say to $w_0 \in D_{\alpha,r}$. From (9) we deduce that $(k_1 z_1^m + k_2 z_2^m)_{m=1}^{\infty}$ and $(l_1 z_1^m + l_2 z_2^m)_{m=1}^{\infty}$ are convergent in $\mathbb{C}^2$. Thus $(z^m)_{m=1}^{\infty}$ is also convergent.

Put $z_0 := \lim_{m\to\infty} z^m$. Now it suffices to observe that $\Pi(z_0) = w_0 \in D_{\alpha,r}$, so $z_0 \in X_{\alpha,r}$; a contradiction. ∎

Now we are able to give a description of the set of proper holomorphic mappings between the domains $D_{\alpha,r}$ and $D_{\beta,R}$ of the irrational type.

Proof of Theorem 1(b). Let $f \in \operatorname{Prop}(D_{\alpha,r}, D_{\beta,R})$. In virtue of Corollary 13 and Remark 14 we may assume that

$$f(z) = (z_1^{k_1} z_2^{k_2} e^{-\beta\psi(z)+i\theta}, z_1^{l_1} z_2^{l_2} e^{\psi(z)}), \quad z = (z_1, z_2) \in D_{\alpha,r},$$

for some $\theta \in \mathbb{R}$ and $\psi \in \mathcal{O}(D_{\alpha,r})$. Our aim is to show that ψ is constant.

To simplify notation, for $\gamma \in \mathbb{R}$ put

$$\Lambda_\gamma : \mathbb{C}^2 \ni (z_1, z_2) \mapsto (z_1 + \gamma z_2, z_2) \in \mathbb{C}^2.$$

It is clear that $\Lambda_\gamma(X_{\gamma,\varrho}) = S_\varrho \times \mathbb{C}$, $\varrho > 1$, where $S_\varrho := \{z \in \mathbb{C} : -\log r < \operatorname{Re} z < \log r\}$. Moreover, Λ_γ is biholomorphic with inverse $\Lambda_\gamma^{-1} = \Lambda_{-\gamma}$.

Note that the mapping $\widetilde{f} : X_{\alpha,r} \to X_{\beta,R}$ given by

$$\widetilde{f}(z) = (k_1 z_1 + k_2 z_2 - \beta\psi(e^{z_1}, e^{z_2}) + i\theta, l_1 z_1 + l_2 z_2 + \psi(e^{z_1}, e^{z_2}))$$

is a lifting of $f \circ \Pi$. Thus Lemma 15 implies that $\widetilde{f}$ is proper and holomorphic.

Put $H := (H_1, H_2) := \Lambda_\beta \circ \widetilde{f} \circ \Lambda_\alpha^{-1} : S_r \times \mathbb{C} \to S_R \times \mathbb{C}$. Obviously, H is proper and holomorphic.

Applying the relations $\frac{\log R}{\log r} = k_1 + l_1\beta$, $\alpha\frac{\log R}{\log r} = k_2 + l_2\beta$ we see that

$$H(z) = (z_1(k_1 + \beta l_1) + i\theta, l_1 z_1 + z_2(l_2 - l_1\alpha) + \psi(e^{z_1 - \alpha z_2}, e^{z_2})), \quad z \in S_r \times \mathbb{C}. \tag{10}$$

Hence for any $z_1 \in S_r$ the mapping $\mathbb{C} \ni z \mapsto H_2(z_1, z) \in \mathbb{C}$ is proper and holomorphic. Consequently, due to the form of proper holomorphic self-mappings of $\mathbb{C}$, there is a polynomial $p = p_{z_1} \in \mathcal{P}(\mathbb{C})$ such that $H_2(z_1, z) = p(z)$. Therefore, the polynomial $q(z) := q_{z_1}(z) := p(z) - l_1 z_1 - z(l_2 - l_1\alpha)$ satisfies the equation

$$\psi(e^{z_1} e^{-\alpha z}, e^z) = q(z), \quad z \in \mathbb{C}. \tag{11}$$

Notice that $\{(e^{z_1} e^{-\alpha 2\pi i m}, e^{2\pi i m}) : m \in \mathbb{N}\}$ is a relatively compact subset of $D_{\alpha,r}$ and the sequence $\{q(2\pi i m)\}_{m=1}^{\infty}$ is bounded. Thus the polynomial q is constant.

Put $c(z_1) := \psi(e^{z_1-\alpha z_2}, e^{z_2})$, $z_1 \in S_r$. Fix any $1 < \varrho < R$ and take a constant $M = M(\varrho) > 0$ such that $|c(x)| < M$ for every $x \in [-\log\varrho, \log\varrho]$.

Let $\lambda \in \varrho\mathbb{D} \setminus (1/\varrho)\mathbb{D}$ be arbitrary. Note that for any $z_2 \in \mathbb{C}$ we have $|\psi(|\lambda|e^{-\alpha z_2}, e^{z_2})| = |c(\log|\lambda|)| < M$. Applying Kronecker's theorem we infer that the set $\{(|\lambda|e^{-\alpha z}, e^{z}) : z \in \mathbb{C}\}$ is dense in $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|\,|z_2|^\alpha = |\lambda|\}$. Consequently, $\psi|_{D_{\alpha,\varrho}}$ is bounded.

Now it suffices to repeat the proof of Lemma 2.7.1 of [Jar-Pfl1] in order to show that every bounded holomorphic mapping on $D_{\alpha,\varrho}$ (in particular, ψ) is constant.

On the other hand, we have already mentioned in the proof of Theorem 1(a) that any mapping given by (2) is proper. ∎

Proof of Theorems 2 and 3. We prove both theorem simultaneously. Let $f : D_\alpha \to D_\beta$ (respectively, $f : D_\alpha^* \to D_\beta^*$) be a proper holomorphic function. We aim at reducing the situation to that of Theorem 1. Take any $r > 1$.

From Lemma 12 we see that for any $t \in [0,1)$ (resp. $t \in (0,1)$) there is an $s(t) \in [0,1)$ (resp. $s(t) \in (0,1)$) such that

$$f(\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|\,|z_2|^\alpha = t\}) \subset \{(w_1, w_2) \in \mathbb{C}^2 : |w_1|\,|w_2|^\beta = s(t)\}.$$

Note that $s(|\lambda|) = |f_1(\lambda,1)|\,|f_2(\lambda,1)|^\beta$ and the function v given by $v : \mathbb{D} \ni \lambda \mapsto s(|\lambda|) \in [0,1]$ (resp. $v : \mathbb{D}_* \ni \lambda \mapsto s(|\lambda|) \in [0,1]$) is radial and subharmonic on $\mathbb{D}$ (in the second case we may remove the singularity at 0). The maximum principle applied to v implies that s is increasing.

In particular, $f|_{D_{(1,\alpha),1/r^2,1}} : D_{(1,\alpha),1/r^2,1} \to D_{(1,\beta),1/R^2,1}$ is proper for some $R > 1$. For $\varrho > 1$ put $\widetilde{\Lambda}_\varrho : \mathbb{C}^2 \ni (z_1, z_2) \mapsto (\varrho z_1, z_2) \in \mathbb{C}^2$ and define $\psi := \widetilde{\Lambda}_R \circ f \circ \widetilde{\Lambda}_r^{-1}|_{D_{\alpha,r}}$. Note that $\psi \in \mathrm{Prop}(D_{\alpha,r}, D_{\beta,R})$. Applying Theorem 1 we find that $\frac{\log R}{\log r} = k_1 + l_1\beta$, $\alpha\frac{\log R}{\log r} = k_2 + l_2\beta$ and $\psi(z_1, z_2) = (az_1^{\varepsilon k_1} z_2^{\varepsilon k_2}, bz_1^{\varepsilon l_1} z_2^{\varepsilon l_2})$ for some $k_i, l_i \in \mathbb{Z}$, $i = 1, 2$, $\varepsilon = \pm 1$ and $a, b \in \mathbb{C}$ satisfying $|a|\,|b|^\beta = 1$. Oviously $\alpha = (k_2 + l_2\beta)/(k_1 + l_1\beta)$ and by the identity principle we obtain

$$f(z_1, z_2) = (ar^{\varepsilon l_1\beta} z_1^{\varepsilon k_1} z_2^{\varepsilon k_2}, br^{-\varepsilon l_1} z_1^{\varepsilon l_1} z_2^{\varepsilon l_2}), \qquad (z_1, z_2) \in D_\alpha \text{ (resp. } (z_1, z_2) \in D_\alpha^*). \tag{12}$$

If $f : D_\alpha^* \to D_\beta^*$, then it suffices to notice that $|f_1(z)|\,|f_2(z)|^\beta = (|z_1|\,|z_2|^\alpha)^{\varepsilon k_1 + \varepsilon l_1\beta}$, $z = (z_1, z_2) \in D_\alpha^*$, hence $\varepsilon(k_1 + l_1\beta) > 0$.

When $f : D_\alpha \to D_\beta$, we conclude that $\varepsilon k_i, \varepsilon l_i \ge 0$, $i = 1, 2$.

Hence we easily get the required formulas.

On the other hand, one can check that any of the mappings given in Theorem 3 is proper (since α is irrational, $k_1l_2 - k_2l_1 \neq 0$). ∎

LEMMA 16. *Let $r^+ > 0$, $r^- < r^+$, $t \in \mathbb{R}$. Suppose that the function $v : \mathbb{A}(r^-, r^+) \to [-\infty, t)$ is subharmonic, radial (i.e. $v(|\lambda|) = v(\lambda)$, $\lambda \in$*

$\mathbb{A}(r^-,r^+))$ *and harmonic on the set* $\{z\in\mathbb{A}(r^-,r^+): v(z)\neq-\infty\}$. *Then there exist* $a,b\in\mathbb{R}$ *such that*

$$v(\lambda)=a\log|\lambda|+b,\qquad \lambda\in\mathbb{A}(r^-,r^+).$$

Proof. It suffices to observe that since v is radial, $\mathbb{A}(r^-,r^+)\setminus\{0\}\subset\{z\in\mathbb{A}(r^-,r^+): v(z)\neq-\infty\}$ (and next one may proceed in a standard way, i.e. solve an easy differential equation). ■

Proof of Theorem 4. First, consider the case when D_{α,r_1^-,r_1^+} and D_{β,r_2^-,r_2^+} are of the irrational type. Then we may assume that $\alpha=(1,\alpha_1)$ for some $\alpha_1\in\mathbb{R}\setminus\mathbb{Q}$. Let

$$v:\mathbb{A}(r_1^-,r_1^+)\ni\lambda\mapsto\log|\psi_1(\lambda,1)|^{\beta_1}|\psi_2(\lambda,1)|^{\beta_2}\in\mathbb{R}.$$

By Lemma 12 we see that $\psi(\{(z_1,z_2)\in\mathbb{C}^2:|z_1|\,|z_2|^\alpha=|\lambda|\}\subset\{(w_1,w_2)\in\mathbb{C}^2:|w_1|^{\beta_1}|w_2|^{\beta_2}=e^{v(\lambda)}\}$. Therefore, v is radial. Observe moreover that v is subharmonic on $\mathbb{A}(r_1^-,r_1^+)$ and harmonic on $\{\lambda\in\mathbb{A}(r_1^-,r_1^+): v(\lambda)>-\infty\}$. Since ψ is surjective, we conclude that

$$v(\mathbb{A}(r_1^-,r_1^+))=\begin{cases}(\log r_2^-,\log r_2^+) & \text{if } r_2^-\geq 0,\\ [-\infty,\log r_2^+) & \text{if } r_2^-<0\end{cases}\tag{13}$$

(we put $\log 0:=-\infty$). However, by Lemma 16, $v(\lambda)=a\log|\lambda|+b$, $\lambda\in\mathbb{A}(r_1^-,r_1^+)$, for some $a,b\in\mathbb{R}$, which easily finishes the proof in this case.

Now suppose that D_{α,r_1^-,r_1^+} and D_{β,r_2^-,r_2^+} are of the rational type; we may assume that $\beta=(p,q)\in\mathbb{Z}^2$ and $\alpha=(1,\alpha_1)$ for some $\alpha_1\in\mathbb{Q}$. Applying Lemma 10 one can see that the mapping

$$\mathbb{A}(r_1^-,r_1^+)\ni\lambda\mapsto\psi_1(\lambda,1)^p\psi_2(\lambda,1)^q\in\mathbb{A}(r_2^-,r_2^+)$$

is proper. Hence this case follows directly from the form of the set of proper holomorphic mappings from $\mathbb{A}(r_1^-,r_1^+)$ to $\mathbb{A}(r_2^-,r_2^+)$. ■

Proof of Theorem 5. Assume that D_{α,r_1^-,r_1^+} is of the rational type and D_{β,r_2^-,r_2^+} is of the irrational type; without loss of generality $\alpha=(1,p/q)$ for some $p,q\in\mathbb{Z}$ and $\beta=(1,\beta_2)$ for some $\beta_2\in\mathbb{R}\setminus\mathbb{Q}$.

Suppose that $\psi\in\operatorname{Prop}(D_{\alpha,r_1^-,r_1^+},D_{\beta,r_2^-,r_2^+})$. Note that for any $\lambda\in\mathbb{A}(r_1^-,r_1^+)$ the mapping

$$u_\lambda:\mathbb{C}_*\ni z\mapsto|\psi_1(\lambda z^p,z^{-q})|\,|\psi_2(\lambda z^p,z^{-q})|^{\beta_2}\tag{14}$$

is constant. Fix λ_0 and $c\neq 0$ such that $u_{\lambda_0}\equiv c$. One can see that $\mathbb{C}_*\ni z\mapsto\psi_i(\lambda_0 z^p,z^{-q})\in\mathbb{C}_*$ is a proper holomorphic self-mapping of $\mathbb{C}_*$, $i=1,2$. Therefore, there are $a_i\in\mathbb{C}_*$ and $\mu_i\in\mathbb{Z}_*$, $i=1,2$, such that $\psi_i(\lambda_0 z^p,z^{-q})=a_i z^{\mu_i}$ for $z\in\mathbb{C}_*$, $i=1,2$. Applying (14) it is clear that $|a_1|\,|a_2|^{\beta_2}|z|^{\mu_1+\mu_2\beta_2}=c$ for $z\in\mathbb{C}_*$. In particular, $\beta_2\in\mathbb{Q}$, a contradiction.

Now, suppose that there exists $\psi \in \operatorname{Prop}(D_{\beta,r_2^-,r_2^+}, D_{\alpha,r_1^-,r_1^+})$. Put $u(\lambda) := |\psi_1(\lambda,1)|\,|\psi_2(\lambda,1)|^{\beta_2}$ for $\lambda \in \mathbb{A}(r_2^-, r_2^+)$.

Applying Lemmas 10 and 12 we find that u satisfies the assumptions of Lemma 16. Thus, there are $a, b \in \mathbb{R}$ such that $\log u(\lambda) = a \log|\lambda| + b$ for $\lambda \in \mathbb{A}(r_2^-, r_2^+)$. In particular, u is either strictly increasing or strictly decreasing. Take any ϱ_2^-, ϱ_2^+ such that $\varrho_2^- > \max\{0, r_2^-\}$, $\varrho_2^+ < r_2^+$, $\varrho_2^- < \varrho_2^+$. Put $\varrho_1^- := \min\{u(\varrho_2^-), u(\varrho_2^+)\}$, $\varrho_1^+ := \max\{u(\varrho_2^-), u(\varrho_2^+)\}$. Then

$$\psi|_{D_{\beta,\varrho_2^-,\varrho_2^+}} : D_{\beta,\varrho_2^-,\varrho_2^+} \to D_{(1,\alpha),\varrho_1^-,\varrho_1^+}$$

is obviously a proper holomorphic mapping. In virtue of Theorem 1(a) there are $k_i, l_i \in \mathbb{Z}$, $i = 1, 2$, such that $\beta = (k_1 + l_1\alpha)/(k_2 + l_2\alpha)$. In particular, $\beta \in \mathbb{Q}$, a contradiction. ∎

LEMMA 17. *Let $A, B \subset \mathbb{C}^n$ be domains and assume that B is bounded.*

(a) *A mapping $f : A \times \mathbb{C}_* \to B \times \mathbb{C}$ is proper and holomorphic if and only if there are $m \in \operatorname{Prop}(A,B)$, $k \in \mathbb{N}$, $0 < k < N$, $N \in \mathbb{N}$, $a_i \in \mathcal{O}(A)$, $i = 1, \ldots, N$, with $|a_0(z)| + \cdots + |a_{k-1}(z)| > 0$ and $|a_{k+1}(z)| + \cdots + |a_N(z)| > 0$ for $z \in A$, satisfying*

$$f(z,w) = \left(m(z), \frac{a_N(z)w^N + \cdots + a_0(z)}{w^k}\right), \qquad (z,w) \in A \times \mathbb{C}_*.$$

(b) *A mapping $f : A \times \mathbb{C} \to B \times \mathbb{C}$ is proper and holomorphic if and only if there are $a_0, \ldots, a_N \in \mathcal{O}(A)$, $N \in \mathbb{N}$, with $|a_0(z)| + \cdots + |a_N(z)| > 0$ for $z \in A$, and there is a proper holomorphic mapping $m : A \to B$ such that*

$$f(z,w) = (m(z), a_N(z)w^N + \cdots + a_0(z)), \quad (z,w) \in A \times \mathbb{C}.$$

(c) *A mapping $f : A \times \mathbb{C}_* \to B \times \mathbb{C}$ is proper and holomorphic if and only if there are $m \in \operatorname{Prop}(A,B)$, $a \in \mathcal{O}(A, \mathbb{C}_*)$ and $k \in \mathbb{N}$ such that*

$$f(z,w) = (m(z), a(z)w^k), \quad (z,w) \in A \times \mathbb{C}_*.$$

(d) *There is no proper holomorphic mapping from $A \times \mathbb{C}$ to $B \times \mathbb{C}_*$.*

Proof. First of all, notice that for any $z \in A$ the mapping $w \mapsto f_1(z,w) \in \mathbb{C}^n$ is bounded on $\mathbb{C}$ (or $\mathbb{C}_*$), so it is constant.

(a) Observe that $\mathbb{C}_* \ni w \mapsto f_2(z,w) \in \mathbb{C}$ is proper for any $z \in A$. Thus, for any $z \in A$ there is a polynomial $p(z,\cdot)$, $p(z,0) \neq 0$, and a natural $k(z)$ such that

$$\phi_2(z,w) = \frac{p(z,w)}{w^{k(z)}}, \qquad (z,w) \in A \times \mathbb{C}_*. \tag{15}$$

One can see that there is a k such that $k = k(z)$ for $z \in A$ (use Rouche's theorem). Consequently, $p \in \mathcal{O}(A \times \mathbb{C}_*)$.

Fix any domain $A' \subset\subset A$ and put

$$A_\mu := \left\{ z \in \overline{A'} : \frac{\partial^\mu p}{\partial w^\mu}(z,w) = 0 \text{ for any } w \in \mathbb{C} \right\}.$$

The above considerations imply that $\bigcup_{\mu=1}^{\infty} A_\mu = \overline{A'}$. Applying Baire's theorem we find that there exists $N \in \mathbb{N}$ such that A_N has non-empty interior. By the identity principle, $A_N = A$.

Thus, there are holomorphic mappings $a_0, \ldots, a_N : A \to \mathbb{C}$ such that $p(z,w) = a_N(z)w^N + \cdots + a_1(z)w + a_0(z)$ for $(z,w) \in A \times \mathbb{C}$, i.e.

$$(16)\qquad f_2(z,w) = \frac{a_N(z)w^N + \cdots + a_1(z)w + a_0(z)}{w^k}, \qquad (z,w) \in A \times \mathbb{C}.$$

By properness of $f_2(z,\cdot)$ we conclude that $0 < k < N$, and $|a_N(z)| + \ldots + |a_{k+1}(z)| > 0$ and $|a_{k-1}(z)| + \cdots + |a_0(z)| > 0$ for any $z \in A$.

Put $m(z) := f_1(z,1)$, $z \in A$. We claim that m is proper.

Indeed, take any sequence $(z_n)_{n=1}^{\infty}$ and assume that it has no accumulation points in A. We may assume that $a_0(z_n) \neq 0$ for any $n \in \mathbb{N}$ (if necessary replace a_0 with a_1 etc.). Then there exists a sequence $(w_n)_{n=1}^{\infty} \subset \mathbb{C}_*$ such that $a_N(z_n)w_n^N + \cdots + a_1(z_n)w_n + a_0(z_n) = 0$ for any $n \in \mathbb{N}$. Since $f(z_n,w_n) = (m(z_n),0)$, it is obvious that $(m(z_n))_{n=1}^{\infty}$ has no accumulation points in B.

Conversely, one can check that every mapping f defined in this way is proper.

(b) It is easy to see that $\mathbb{C} \ni w \mapsto f_2(z,w) \in \mathbb{C}$ is a proper holomorphic mapping for any $z \in A$. From the form of such mappings we conclude that for every $z \in A$ the mapping $f_2(z,\cdot)$ is a complex polynomial. Now we proceed exactly as in the proof of (a).

(c) We proceed similarly to the proofs of (a) and (b).

(d) Suppose that $f : A \times \mathbb{C} \to B \times \mathbb{C}_*$ is a proper holomorphic function. Fix $z \in A$. Then $\mathbb{C} \ni w \mapsto f_2(z,w) \in \mathbb{C}_*$ is proper.

Take $\psi \in \mathcal{O}(\mathbb{C})$ such that $f_2(1,\cdot) = \exp \circ \psi$. Observe that ψ is a proper holomorphic self-mapping of the complex plane, hence ψ is a polynomial. From this we easily get a contradiction. ∎

Proof of Theorems 6, 7 and 8. These are direct consequences of Lemma 17. ∎

Finally, I take this opportunity to express my deep gratitude to Professor Włodzimierz Zwonek for introducing me to the subject and numerous remarks.

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Instytut Matematyki
Uniwersytet Jagielloński
Reymonta 4
30-059 Kraków, Poland
E-mail: lukasz.kosinski@im.uj.edu.pl

Received 27.7.2007
and in final form 3.8.2007 (1804)