Asymptotic stability in $L^1$ of a transport equation

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Abstract. We study the asymptotic behaviour of solutions of a transport equation. We give some sufficient conditions for the complete mixing property of the Markov semigroup generated by this equation.

1. Introduction. In this paper we study the equation

$$\frac{\partial u}{\partial t} + \lambda u = Au + \lambda Pu(t, \cdot),$$

where

$$Au = \sum_{i,j=1}^{d} \frac{\partial^2 (a_{ij}(x)u)}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \frac{\partial (b_i(x)u)}{\partial x_i},$$

$\lambda \geq 0$ and $P : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is a Markov operator. Equation (1) can be interpreted as a diffusion process with jumps (see [12]). Our aim is to give some sufficient conditions for complete mixing (see [7]) of the Markov semigroup generated by (1). Asymptotic properties of similar equations were investigated in [12] and [5]. The complete mixing property of diffusion processes (without jumps) was studied in [11] and [3]. In this paper we generalize these results to equation (1). Our proofs are based on results from [2] (see also [1] and [8]), where some spectral techniques of studying asymptotics of semigroups of linear operators are developed.

The paper is organized as follows. In Section 2 we rewrite (1) as an evolution equation in $L^1$ and give some basic definitions. Section 3 starts with Theorem 2 which is an answer to an open problem posed in [13]. This is the problem of characterisation of asymptotics of the parabolic equation (1) by solutions of a proper elliptic equation. We underline that Theorem 2 in case $\lambda = 0$ is stated in [2]. Theorem 2 allows us to formulate in Theorem 3 some

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sufficient conditions for asymptotic stability of (1) in the one-dimensional case.

2. Preliminaries. We denote by $D$ the set of all nonnegative elements of $L^1(\mathbb{R}^d)$ with norm one. The elements of $D$ will be called densities. A linear operator $P : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is called a Markov operator if $P(D) \subset D$. Every Markov operator is bounded. Equation (1) can be rewritten as an evolution equation

\begin{equation}
  u'(t) = (A - \lambda I + \lambda P)u(t), \quad u(0) = u_0,
\end{equation}

where $P : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is a Markov operator. We assume that $a_{ij} \in C^3_b(\mathbb{R}^d)$ and $b_i \in C^2_b(\mathbb{R}^d)$ for $i, j = 1, \ldots, d$, where $C^k_b(\mathbb{R}^d)$ is the space of $k$ times differentiable bounded functions on $\mathbb{R}^d$ whose derivatives of order $\leq k$ are continuous and bounded. We also assume that $A$ is an elliptic operator, i.e.

\begin{equation}
  \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \mu |\xi|^2
\end{equation}

for some $\mu > 0$ and all $\xi \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$.

The domain of the operator $A$ is given by

\[ D(A) = \{ f \in L^1(\mathbb{R}^d) \cap C^{1+a}(\mathbb{R}^d) : Af \in L^1(\mathbb{R}^d) \}. \]

Here $C^{1+a}(\mathbb{R}^d)$ is the space of all differentiable functions having absolutely continuous derivatives. It is well known that under the above assumptions the operator $A$ generates a continuous semigroup $(T(t))_{t \geq 0}$ of Markov operators on $L^1(\mathbb{R}^d)$.

The semigroup $(T(t))_{t \geq 0}$ is an integral semigroup with strictly positive and continuous kernel, i.e. there exists a continuous function $p : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$ such that

\begin{equation}
  T(t)f(x) = \int_{\mathbb{R}^d} p(t, x, y)f(y)\, dy.
\end{equation}

From the Phillips perturbation theorem equation (3) generates a continuous semigroup $(S(t))_{t \geq 0}$ of Markov operators on $L^1(\mathbb{R}^d)$.

We say that a semigroup $(P(t))_{t \geq 0}$ of Markov operators is completely mixing if for any two densities $f$ and $g$ we have

\begin{equation}
  \|P(t)f - P(t)g\| \to 0 \quad \text{as} \quad t \to \infty,
\end{equation}

where $\| \cdot \|$ denotes the norm in $L^1(\mathbb{R}^d)$.

Let

\[ L^1_0(\mathbb{R}^d) = \{ f \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} f(x)\, dx = 0 \}. \]
Then condition (6) is equivalent to
\[
\lim_{t \to \infty} \| P(t)f \| = 0 \quad \text{for every } f \in L^1_\sigma(\mathbb{R}^d).
\]

If \( B \) is a linear operator then by \( \sigma(B) \), \( P\sigma(B) \) and \( R\sigma(B) \) we denote respectively the spectrum, point spectrum and residual spectrum of \( B \).

The following theorem will be useful in studying the complete mixing property of Markov semigroups.

**Theorem 1.** Let \( (P(t))_{t \geq 0} \) be a continuous semigroup of contractions on a Banach space \((X, \| \cdot \|_X)\). If \( Z \) is the generator of \( (P(t))_{t \geq 0} \) with adjoint \( Z^* \) then denote by \( \overline{N} \) the weak* closure of the linear span of unitary eigenvectors of \( Z^* \), where a unitary eigenvector is an eigenvector corresponding to the eigenvalue \( \mu \in P\sigma(Z^*) \cap i\mathbb{R} \). If the set \( \sigma(Z) \cap i\mathbb{R} \) is countable then for every \( x \in X \),
\[
\lim_{t \to \infty} \| P(t)x \|_X = \sup \{ |\varphi(x)| : \varphi \in \overline{N}, \| \varphi \|_{X^*} \leq 1 \}.
\]

**Remark 1.** Theorem 1 is a special case of a result by Batty, Brzeźniak and Greenfield [2], who consider contractive representations of abstract semigroups in Banach spaces.

Let \( A f = Af - \lambda f + \lambda P f \). Then (3) can be rewritten as
\[
\begin{align*}
\frac{d}{dt}u(t) = A u(t) \\
u(0) = u_0, u_0 \in L^1(\mathbb{R}^d).
\end{align*}
\]
and \( u \) satisfies the initial condition \( u(0) = u_0, u_0 \in L^1(\mathbb{R}^d) \). Let \( A^* \) be the linear operator given by
\[
A^* g(x) = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 g}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x) \frac{\partial g}{\partial x_i}(x),
\]
and denote by \( P^* : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d) \) the adjoint operator of \( P \). Then the adjoint operator of \( A \) is of the form
\[
A^* g = A^* g - \lambda g + \lambda P^* g.
\]

**3. Results.** Theorem 1 allows us to formulate the following result.

**Theorem 2.** The semigroup \( (S(t))_{t \geq 0} \) generated by equation (1) is completely mixing iff the only bounded solutions of
\[
A^* g = 0
\]
are constant functions.

In the case when \( \lambda = 0 \), i.e. \( A = A \), the above theorem can be found in [2] (see Examples 6.1 and 6.4) but it is formulated only for doubly stochastic and one-dimensional equations. For the convenience of the reader we give its proof.
Proof of Theorem 2. Let \((S(t))_{t \geq 0}\) be completely mixing. We denote by \((S^*(t))_{t \geq 0}\) the conjugate semigroup to \((S(t))_{t \geq 0}\), i.e. \(S^*(t) : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)\) and for all \(f \in L^1(\mathbb{R}^d), g \in L^\infty(\mathbb{R}^d)\) and \(t > 0\) we have
\[
\int_{\mathbb{R}^d} (S(t)f)g = \int_{\mathbb{R}^d} f(S^*(t)g).
\]
If \(A^*g = 0\) for some \(g \in L^\infty(\mathbb{R}^d)\) then \(g\) is a fixed point of \((S^*(t))_{t \geq 0}\) and for all \(f \in L^1_0(\mathbb{R}^d)\) we have
\[
\int_{\mathbb{R}^d} fg = \int_{\mathbb{R}^d} (S(t)f)g \to 0 \quad \text{as} \quad t \to \infty.
\]
This implies that \(g = \text{const}\).

Suppose now that every bounded solution of (12) is constant. From [10] the semigroup \((T(t))_{t \geq 0}\) generated by \(A\) is analytic, which implies that \((S(t))_{t \geq 0}\) generated by \(A\) is also analytic. For such semigroups we always have \(\sigma(A) \cap i\mathbb{R} \subset \{0\}\) (see [4]), so the assumptions of Theorem 1 are satisfied.

Since \(P\sigma(A^*) \subset R\sigma(A) \subset \sigma(A)\), the only unitary eigenvectors of \(A^*\) correspond to the eigenvalue \(\mu = 0\), and are constant by our assumption. From (8) we get
\[
\lim_{t \to \infty} \|S(t)f\| = \left| \int_{\mathbb{R}^d} f(x) \, dx \right| = 0 \quad \text{for every} \quad f \in L^1_0(\mathbb{R}^d)
\]
and the semigroup \((S(t))_{t \geq 0}\) is completely mixing.

In the rest of this paper we investigate the one-dimensional case of equation (1). If \(d = 1\) then the formulas for \(A\) and \(A^*\) are as follows:

\begin{align*}
(13) \quad Af &= \frac{d^2}{dx^2}(a(x)f) - \frac{d}{dx}(b(x)f), \\
(14) \quad A^*g &= a(x) \frac{d^2g}{dx^2} + b(x) \frac{dg}{dx},
\end{align*}

where \(a \in C^3_b(\mathbb{R}), b \in C^2_b(\mathbb{R})\) and \(\inf_{x \in \mathbb{R}} a(x) > 0\).

A mapping \(\mathcal{P} : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]\), where \(\mathcal{B}(\mathbb{R})\) is the family of all Borel measurable subsets of \(\mathbb{R}\), is called a transition probability function if \(\mathcal{P}(x, \cdot)\) is a probability measure for all \(x \in \mathbb{R}\) and \(\mathcal{P}(\cdot, B)\) is a Borel measurable function for every \(B \in \mathcal{B}(\mathbb{R})\). From now on we assume that the operator \(P\) in (3) is induced by a transition probability function \(\mathcal{P}\), i.e. for every \(f \in D\) the measure
\[
\mu_f(B) = \int_{\mathbb{R}} f(x)\mathcal{P}(x, B) \, dx
\]
is absolutely continuous with respect to the Lebesgue measure and

\[(15)\] \[Pf = \frac{d\mu_f}{dx}.\]

The formula for the dual operator is as follows:

\[(16)\] \[P^*g(x) = \int_R g(y) \mathcal{P}(x, dy).\]

The following result gives a sufficient condition for complete mixing of the semigroup generated by equation (1).

**Theorem 3.** Let \(P\) be a Markov operator induced by a transition probability function \(\mathcal{P}\) satisfying the following condition.

\[(T)\] There exist nonnegative constants \(M\) and \(L\) such that

\[\mathcal{P}(x, [-M, x]) = 1 \quad \text{for } x \geq L, \quad \mathcal{P}(x, [x, M]) = 1 \quad \text{for } x \leq -L.\]

If

\[(17)\] \[\int_0^{+\infty} \exp\left(-\int_0^x \frac{b(r)}{a(r)} dr\right) dx = \int_{-\infty}^0 \exp\left(-\int_0^x \frac{b(r)}{a(r)} dr\right) dx = +\infty\]

then the semigroup \((S(t))_{t \geq 0}\) generated by (1) is completely mixing.

We need the following two lemmas.

**Lemma 1.** Let \(\alpha < x_0 < \beta\). Assume that a function \(u : [\alpha, \beta] \to [0, 1]\) has absolutely continuous derivative on \([x_0, \beta]\) and satisfies

\[a(x)u''(x) + b(x)u'(x) \geq 0 \quad \text{a.e. in } [x_0, \beta].\]

Also assume that \(u'(x_0) > 0\). Then there exists a function \(v : [\alpha, \beta] \to [0, 1]\) with absolutely continuous derivative and constants \(\alpha < x_1 < x_0 < x_2 < \beta\) such that

\[v(x) = u(x_0) \quad \text{for } x \in [\alpha, x_1],\]
\[v(x) = u(x) \quad \text{for } x \in [x_2, \beta],\]
\[a(x)v''(x) + b(x)v'(x) \geq 0 \quad \text{a.e. in } [\alpha, \beta].\]

Moreover, \(v'(x) \geq 0, v''(x) \geq 0\) and \(v'''(x) \geq 0\) for \(x \in (x_1, x_2)\).

**Proof.** Fix positive constants \(c, d\) satisfying

\[\frac{1}{3} < \frac{d}{c + d} < \frac{1}{2}.\]

Since

\[\frac{u(x_0 + dh) - u(x_0)}{(c + d)h} \to \frac{d}{c + d} u'(x_0) \quad \text{as } h \to 0\]
and $u'$ is continuous, there exists $\delta > 0$ such that for $0 < h < \delta$ we have
\[(18) \quad \frac{u'(x_0 + dh)}{3} < \frac{u(x_0 + dh) - u(x_0)}{(c + d)h} < \frac{u'(x_0 + dh)}{2}.
\]

There exist positive constants $\mu, B$ such that
\[(19) \quad \mu \leq a(x), \quad |b(x)| \leq B \quad \text{for} \quad x \in \mathbb{R}.
\]

We take $h > 0$ such that $h < \min \left( \delta, \frac{\mu}{2Bd}, \frac{\mu}{2Bc} \right)$ and put $x_1 = x_0 - ch, x_2 = x_0 + dh$. There exists a polynomial $f(x) = \gamma_3 x^3 + \gamma_2 x^2 + \gamma_1 x + \gamma_0$ such that $f(x_1) = u(x_0), f'(x_1) = 0, f(x_2) = u(x_2)$ and $f''(x_2) = u''(x_2)$. Such an $f$ satisfies
\[
\begin{align*}
  f(x_1) &= u(x_0), \\
  f'(x_1) &= 0, \\
  f''(x_1) &= \frac{6}{(x_2 - x_1)^2} \left[ u(x_2) - u(x_0) - \frac{x_2 - x_1}{3} u'(x_2) \right], \\
  f'''(x_1) &= \frac{12}{(x_2 - x_1)^3} \left[ \frac{x_2 - x_1}{2} u'(x_2) - (u(x_2) - u(x_0)) \right].
\end{align*}
\]

From (18) and (20) we have
\[(21) \quad f''(x_1) > 0, \quad f'''(x_1) > 0.
\]

Consequently,
\[(22) \quad f'(x) > 0, \quad f''(x) > 0 \quad \text{for} \quad x \in (x_1, x_2).
\]

By the definition of $x_1$ and $x_2$ and from (21) it follows that for $x \in (x_1, x_2)$ we have
\[
(x - x_1) f''(x_1) \leq \frac{\mu}{B} f''(x_1),
\]
\[
(x - x_1)^2 \frac{f'''(x_1)}{2} \leq \frac{\mu}{B} f'''(x_1)(x - x_1).
\]

Hence
\[
\frac{\mu}{B} \left[ f''(x_1) + f'''(x_1)(x - x_1) \right] \geq f''(x_1)(x - x_1) + \frac{f'''(x_1)}{2} (x - x_1)^2 \quad \text{for} \quad x \in (x_1, x_2).
\]

Since $f'(x_1) = 0$ it follows that
\[
\mu f''(x) \geq B f'(x) \quad \text{for} \quad x \in (x_1, x_2).
\]

From (19) and (22) we have
\[
a(x) f''(x) + b(x) f'(x) \geq 0 \quad \text{for} \quad x \in (x_1, x_2).
\]
If we define

\[ v(x) = \begin{cases} 
  u(x_0) & \text{for } x \in [\alpha, x_1], \\
  f(x) & \text{for } x \in (x_1, x_2), \\
  u(x) & \text{for } x \in [x_2, \beta],
\end{cases} \]

then \( v \) has all the required properties.

**Lemma 2.** For fixed \( x_0 \in \mathbb{R} \) and \( \eta > 0 \) let \( u : (x_0 - \eta, \infty) \to [0, 1) \) be a function with absolutely continuous derivative such that \( \sup_{x \in (x_0, \infty)} u(x) = 1 \) and \( u'(x_0) > 0 \). Define

\[ U = \{ x \in (x_0, \infty) : u(y) \leq u(x) \text{ for } y \in (x_0, x) \}. \]

If \( a(x)u''(x) + b(x)u'(x) \geq 0 \) a.e. in \( U \), then there exists a nondecreasing function \( v : (x_0, \infty) \to [0, 1) \) such that \( \sup_{x \in (x_0, \infty)} v(x) = 1 \), \( v' \) is absolutely continuous, and

\[ a(x)v''(x) + b(x)v'(x) \geq 0 \quad \text{a.e. in } (x_0, \infty). \]

**Proof.** For \( x > x_0 \) define

\[ c(x) = \sup \{ y \in U : y \leq x \}, \quad d(x) = \inf \{ y \in U : x \leq y \}. \]

Set \( K = \{ x \geq x_0 : c(x) < x < d(x) \} \). Then \( K = \bigcup_{n=1}^{\infty} (c_n, d_n) \), where the intervals in the union are pairwise disjoint. We have \( u(x) \leq u(c_n) = u(d_n) \) for \( x \in (c_n, d_n), n \in \mathbb{N} \). Define \( \tilde{v}_1 : (x_0, \infty) \to [0, 1) \) by

\[ \tilde{v}_1(x) = \begin{cases} 
  u(d_1) & \text{for } x \in (c_1, d_1), \\
  u(x) & \text{for } x \in (x_0, \infty) \setminus (c_1, d_1)
\end{cases} \]

If \( u'(d_1) = 0 \) then set \( v_1 = \tilde{v}_1 \). If \( u'(d_1) > 0 \) then there exists \( \varepsilon_1 > 0 \) such that \( u'(x) > 0 \) for all \( x \in [d_1, d_1 + \varepsilon_1] \). According to Lemma 1 we modify \( \tilde{v}_1 \) on \( [c_1, d_1 + \varepsilon_1] \), obtaining a function \( v : (x_0, \infty) \to [0, 1) \) such that \( v_1 \) has absolutely continuous derivative,

\[ v_1(x) = u(x) \quad \text{for } x \in (x_0, \infty) \setminus (c_1, d_1 + \varepsilon_1), \]

\[ a(x)v''_1(x) + b(x)v'_1(x) \geq 0 \quad \text{a.e. in } (c_1, d_1 + \varepsilon_1). \]

By induction we define a sequence of functions \( v_n : (x_0, \infty) \to [0, 1) \) with absolutely continuous derivatives such that

\[ v_n(x) = u(x) \quad \text{for } x \in (x_0, \infty) \setminus \bigcup_{k=1}^{n} (c_k, d_k + \varepsilon_k), \]

\[ v_m(x) = v_n(x) \quad \text{for } x \in \bigcup_{k=1}^{n} (c_k, d_k + \varepsilon_k), \quad m \geq n, \]

\[ a(x)v''_n(x) + b(x)v'_n(x) \geq 0 \quad \text{for } x \in \bigcup_{k=1}^{n} (c_k, d_k + \varepsilon_k), \]

where \( \varepsilon_k \geq 0 \) and \( n \in \mathbb{N} \).
The sequence \((v_n)_{n \in \mathbb{N}}\) is pointwise convergent to a nondecreasing function \(f : (x_0, \infty) \to [0, 1]\). There exists \(C \subset (x_0, \infty)\) with Lebesgue measure zero such that \(f\) is differentiable on \((x_0, \infty) \setminus C\). Set \(D = \{c_n : n \in \mathbb{N}\}\), where \(\overline{K}\) stands for the closure of \(K\). Define \(g : (x_0, \infty) \to \mathbb{R}\) by
\[
g(x) = \begin{cases} f'(x) & \text{for } x \notin D, \\ u'(x) & \text{for } x \in D. \end{cases}
\]
Since \(C \subset D\), it follows that \(g\) is continuous. Moreover, the sequence \((v_n)_{n \in \mathbb{N}}\) is pointwise convergent to \(g\). Since the variation is an additive function of the interval, the definition of \(v_n\) implies that
\[
\int_{c_n}^{d_n + \varepsilon_n} g'(x) \, dx = g(d_n + \varepsilon_n) - g(c_n).
\]
Moreover, outside \(\bigcup_{n=1}^{\infty} (c_n, d_n + \varepsilon_n)\) we have \(g'(x) = u''(x)\) for a.e. \(x\), \(u'\) is absolutely continuous and \(g(c_n) = u'(c_n)\), \(g(d_n + \varepsilon_n) = u'(d_n + \varepsilon_n)\). This implies that \(\int_{x}^{y} g'(t) \, dt = g(y) - g(x)\) for every \(x < y\), which means that \(g\) is absolutely continuous.

Finally, define \(v(x) = u(x_0) + \int_{x_0}^{x} g(t) \, dt\) for \(x \in (x_0, \infty)\). The function \(v\) has all the required properties.

**Proof of Theorem 3.** By Theorem 2 it suffices to show that the only bounded solutions of
\[
A^* g = A^* g - \lambda g + \lambda P^* g = 0
\]
are constants.

Let \(u \in L^\infty(\mathbb{R})\) be a nonconstant function. We show that \(u\) cannot satisfy (23). If \(g \in L^\infty(\mathbb{R}^d)\) satisfies (23) then \(g + c\), where \(c \in \mathbb{R}\), also satisfies (23). Hence, by the linearity of (23), we can assume that \(u \geq 0\) and \(\|u\|_\infty = 1\).

Let us consider two cases.

**Case 1:** There exists \(x_0 \in \mathbb{R}\) such that \(u(x_0) \geq u(x)\) for \(x \in \mathbb{R}\). A function \(g\) satisfies (23) if and only if
\[
\lambda R(\lambda, A)^* P^* g = g,
\]
where \(R(\lambda, A)^*\) stands for the dual of the resolvent operator \(R(\lambda, A)\). Since
\[
R(\lambda, A)f = \int_{0}^{\infty} e^{-\lambda t} T(t) f \, dt,
\]
from (5) it follows that $R(\lambda, A)$ is an integral operator, i.e.

\begin{equation}
R(\lambda, A)f(x) = \int K(x, y)f(y)\, dy,
\end{equation}

where

\begin{equation}
K(x, y) = \int_{0}^{\infty} e^{-\lambda t} p(t, x, y)\, dt > 0
\end{equation}

is a continuous and strictly positive kernel. From the fact that $(T(t))_{t \geq 0}$ is a Markov semigroup it follows that

\begin{equation}
\lambda \int_{\mathbb{R}} K(x, y)\, dx = 1 \quad \text{for } y \in \mathbb{R}.
\end{equation}

For the dual operator $R(\lambda, A)^*$ we have

\begin{equation}
R(\lambda, A)^*g(x) = \int_{\mathbb{R}} K(y, x)g(y)\, dy.
\end{equation}

If $P^*u \equiv \text{const}$ then since $A^*$ is a differential operator it follows that $\lambda R(\lambda, A)^*P^*u \equiv \text{const}$ so equation (24) is not satisfied. If $P^*u \not\equiv \text{const}$ then $P^*u(x) \leq u(x_0)$ for $x \in \mathbb{R}$, because the dual operator of a Markov operator is a nonnegative contraction on $L^\infty$. Moreover, there exists a set $B \subset \mathbb{R}$ with positive Lebesgue measure such that $P^*u(x) < u(x_0)$ for $x \in B$. It follows from (27) and (28) that

\[ \lambda \int_{B} K(y, x_0)u(x_0)\, dy + \lambda \int_{\mathbb{R} \setminus B} K(y, x_0)u(x_0)\, dy = u(x_0). \]

Since $R(\lambda, A)^*P^*u$ is continuous, $u$ does not satisfy (24).

**Case 2:** We have $u(x) < 1$ for all $x \in \mathbb{R}$. From the symmetry of the conditions in (T) and (17) we can assume that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} x_n = +\infty$ such that $\lim_{n \to \infty} u(x_n) = 1$. We take $x_0 > L$ such that $u(x) \leq u(x_0)$ for $x \in [-M, x_0]$. Define a set $U$ as in the statement of Lemma 2, i.e.

\[ U = \{ x \in (x_0, \infty) : u(y) \leq u(x) \text{ for } y \in (x_0, x) \}. \]

There exists a set $V \subset U$ with positive Lebesgue measure such that $A^*u(x) < 0$ for $x \in V$. Otherwise, if

\[ A^*u(x) = a(x)u''(x) + b(x)u'(x) \geq 0 \quad \text{a.e. in } U, \]

then it follows from Lemma 2 that there exists a nondecreasing function $v : (x_0, \infty) \to [0, 1)$ with absolutely continuous derivative such that $\sup_{x \in (x_0, \infty)} v(x) = 1$ and

\begin{equation}
a(x)v''(x) + b(x)v'(x) \geq 0 \quad \text{a.e. in } (x_0, \infty).\end{equation}
Take $x_1 > x_0$ such that $v'(x_1) > 0$. By standard arguments from the theory of differential inequalities it follows from (29) that

$$(30) \quad v(x) \geq v(x_1) + v'(x_1) \int_{x_1}^{x} \exp \left( - \int_{x_1}^{s} \frac{b(r)}{a(r)} \, dr \right) \, ds.$$

By (17) the right side of (30) tends to $\infty$ as $x \to \infty$, which contradicts the boundedness of $v$.

It follows from (16), (T) and (23) that for $x \in V$ we have

$$0 > A^*u(x) = u(x) - P^*u(x) = \int_{-M}^{x} [u(x) - u(y)] \mathcal{P}(x, dy) \geq 0,$$

which implies that $u$ cannot satisfy (23). This completes the proof.

**Example 1.** Suppose that $S : \mathbb{R} \to \mathbb{R}$ is a measurable transformation satisfying

$$l(S^{-1}(B)) = 0 \quad \text{for all } B \in \mathcal{B}(\mathbb{R}) \text{ with } l(B) = 0,$$

where $l : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ is the Lebesgue measure restricted to the Borel subsets of $\mathbb{R}$. If in the definition of the operator $P$ we put

$$\mathcal{P}(x, B) = \delta_{S(x)}(B) \quad \text{for } x \in \mathbb{R} \text{ and } B \in \mathcal{B}(\mathbb{R}),$$

where $\delta_x$ is the Dirac measure at $x$, then $P$ becomes the Frobenius–Perron operator corresponding to $S$, i.e. the unique Markov operator satisfying

$$\int_{S^{-1}(A)} f(x) \, dx = \int_{A} Pf(x) \, dx \quad \text{for all } f \in L^1(\mathbb{R}) \text{ and } A \in \mathcal{B}(\mathbb{R}).$$

Then condition (T) is of the form

(T') There exist nonnegative constants $M$ and $L$ such that

$$-M \leq S(x) \leq x \quad \text{for } x \geq L, \quad x \leq S(x) \leq M \quad \text{for } x \leq -L.$$

In this case equation (1) describes the evolution of densities under the diffusion process perturbed by deterministic jumps induced by the mapping $S$.

**Remark 2.** Condition (17) is strictly connected with the complete mixing property of the semigroup $(T(t))_{t \geq 0}$ generated by $A$. Namely, in [11] it is shown that $(T(t))_{t \geq 0}$ is completely mixing if and only if

$$\int_{\mathbb{R}} \exp \left( - \int_{0}^{x} \frac{b(r)}{a(r)} \, dr \right) \, dx = \infty.$$
References


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