Parametrization of Riemann-measurable selections for multifunctions of two variables with application to differential inclusions

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Abstract. We consider a multifunction $F : T \times X \to 2^E$, where $T$, $X$ and $E$ are separable metric spaces, with $E$ complete. Assuming that $F$ is jointly measurable in the product and a.e. lower semicontinuous in the second variable, we establish the existence of a selection for $F$ which is measurable with respect to the first variable and a.e. continuous with respect to the second one. Our result is in the spirit of [11], where multifunctions of only one variable are considered.

1. Introduction. If $X$ is a topological space, we denote by $\mathcal{B}(X)$ the Borel $\sigma$-algebra of $X$. Moreover, if $\mu$ is measure on $\mathcal{B}(X)$, we denote by $\mathcal{T}_\mu(X)$ the completion of $\mathcal{B}(X)$ with respect to $\mu$. We briefly put $\mathcal{T}_\mu = \mathcal{T}_\mu(X)$ when ambiguities do not occur. For the basic definitions about multifunctions, we refer the reader to [6] and [7].

This note is motivated by the main result of [11], which concerns the existence of Riemann-measurable selections (i.e., selections which are a.e. continuous) for a given multifunction. For the reader’s convenience, we now state the main result of [11] (as usual, by a Polish space we mean a complete separable metric space).

Theorem 1 (Theorem 3 of [11]). Let $X$ be a Polish space equipped with a $\sigma$-finite regular Borel measure, $E$ a metric space and $F : X \to 2^E$ a multifunction with nonempty complete values. If $F$ is lower semicontinuous at almost every point of $X$, then there exists a selection of $F$ which is continuous at almost every point of $X$.

We refer to [11] for motivations leading to Theorem 1. Applications of Theorem 1 to implicit integral equations and to elliptic differential equations can be found in [2] and [8], respectively.

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Our aim in this paper is to prove a parametrized version of Theorem 1 for multifunctions of two variables, obtaining, in particular, sufficient conditions for the existence of a selection which is measurable with respect to the first variable and a.e. continuous with respect to the second one. More precisely, we prove the following result.

**Theorem 2.** Let $T,X$ be two Polish spaces and let $\mu, \psi$ be two positive regular Borel measures on $T$ and $X$, respectively, with $\mu$ finite and $\psi$ $\sigma$-finite. Let $S$ be a separable metric space, $F : T \times X \to 2^S$ a multifunction with nonempty complete values, and let $E \subseteq X$ be a given set. Assume that:

1. $F$ is $T_\mu \otimes \mathcal{B}(X)$-measurable;
2. for a.a. $t \in T$,

\[ \{ x \in X : F(t, \cdot) \text{ is not lower semicontinuous at } x \} \subseteq E. \]

Then there exist a selection $\phi : T \times X \to S$ of $F$ and a set $R \in \mathcal{B}(X)$, with $\psi(R) = 0$, such that

1. $\phi(\cdot, x)$ is $T_\mu$-measurable for each $x \in X \setminus (E \cup R)$;
2. for a.a. $t \in T$,

\[ \{ x \in X : \phi(t, \cdot) \text{ is not continuous at } x \} \subseteq E \cup R. \]

The proof of Theorem 2 will be given in Section 2, while in Section 3 we shall provide an application of Theorem 2 to differential inclusions.

**2. Proof of Theorem 2.** Before proving Theorem 2, we need the following preliminary results.

**Lemma 1.** Let $T,X$ be two Polish spaces and let $\mu, \psi$ be two positive $\sigma$-finite regular Borel measure on $X$ and $Y$, respectively. Then there exist two sets $Q \in \mathcal{B}(T)$ and $R \in \mathcal{B}(X)$, with $\mu(Q) = \psi(R) = 0$, a continuous open function $\pi : \mathbb{N}^T \to T \times X$, and a function $\sigma : T \times X \to \mathbb{N}^T$ which is continuous at each point of $(T \setminus Q) \times (X \setminus R)$ and satisfies $\pi(\sigma(t,x)) = (t,x)$ for all $(t,x) \in T \times X$.

**Proof.** By Lemma 1 of [11], there exist $Q \in \mathcal{B}(T)$ and $R \in \mathcal{B}(X)$ with $\mu(Q) = \psi(R) = 0$, two continuous open functions $\pi_1 : \mathbb{N}^T \to T$, $\pi_2 : \mathbb{N}^T \to X$, a function $\sigma_1 : T \to \mathbb{N}^T$ which is continuous at each point of $T \setminus Q$, and a function $\sigma_2 : X \to \mathbb{N}^T$ which is continuous at each point of $X \setminus R$, such that $\pi_1(\sigma_1(t)) = t$ and $\pi_2(\sigma_2(x)) = x$ for all $(t,x) \in T \times X$. For each $\alpha := \{ n_k \}_{k \in \mathbb{N}}$, we denote by $\alpha_e$ and $\alpha_o$ the sequences $\{ n_{2k} \}_k$ and $\{ n_{2k-1} \}_k$, respectively. If we put $\pi(\sigma) = (\pi_1(\alpha_e), \pi_2(\alpha_o))$ for all $\alpha \in \mathbb{N}^T$, then $\pi : \mathbb{N}^T \to T \times X$ is a continuous open function. Moreover, if we put $\sigma(t,x) = \{ n(t,x)_k \}_k$, where $\{ n(t,x)_k \}_k = \sigma_1(t)$ and $\{ n(t,x)_{2k-1} \}_k = \sigma_2(x)$, then $\sigma : T \times X \to \mathbb{N}^T$ is continuous at each point of $(T \setminus Q) \times (X \setminus R)$ and one has $\pi(\sigma(t,x)) = (t,x)$ for all $(t,x) \in T \times X$. 


LEMMA 2. Let $T, X, Q, R, \mu, \psi, \pi, \sigma$ be as in Lemma 1. Let $E$ be a metric space, $B \subseteq T \times X$ and $V \subseteq B$ two given sets, and $F : B \to 2^E$ a multifunction with nonempty complete values which is lower semicontinuous at each point of $B \setminus V$. Then there exists a selection $g$ of $F$ which is continuous at each point of $[B \cap ((T \setminus Q) \times (X \setminus R))] \setminus V$.

Proof. Put $Z = \pi^{-1}(B)$ and $G = F \circ \pi|_Z$. Observe that $Z$ is 0-dimensional and $G$ is lower semicontinuous at each point of $Z \setminus \pi^{-1}(V)$. Consequently, by the proof of Lemma 2 of [11], there exists a selection $s$ of $G$ which is continuous at each point of $Z \setminus \pi^{-1}(V)$. Since $\sigma(t, x) \in \pi^{-1}(t, x) \subseteq Z$ for all $(t, x) \in B$, we can put $g(t, x) = s(\sigma(t, x))$ for all $(t, x) \in B$. Then $g(t, x) \in F(\pi(\sigma(t, x))) = F(t, x)$ for all $(t, x) \in B$. Further, it is easily seen that $g$ is continuous at each point of $[B \cap ((T \setminus Q) \times (X \setminus R))] \setminus V$.

The next lemma follows from the proof of Lemma 2.3 of [1].

LEMMA 3. Let $X$ and $S$ be metric spaces, with $S$ separable, $F : X \to 2^S$ a multifunction with nonempty values, $\{s_n\}$ a dense sequence in $S$, and $x_0 \in X$. Denote by $d$ the distance in $S$.

(i) If $F$ is lower semicontinuous at $x_0$, then for each $s \in S$ the function $x \in X \mapsto d(s, F(x))$ is upper semicontinuous at $x_0$.

(ii) If for each $n \in \mathbb{N}$ the function $x \in X \mapsto d(s_n, F(x))$ is upper semicontinuous at $x_0$, then $F$ is lower semicontinuous at $x_0$.

LEMMA 4. Let $T, X, \mu, \psi$ be as in Lemma 1, with $\mu$ finite. Let $f : T \times X \to \mathbb{R}$ be a single-valued function and $E \subseteq X$ a given set. Assume that:

(i) $f$ is $T_\mu \otimes \mathcal{B}(X)$-measurable;

(ii) $\inf_{T \times X} f > -\infty$;

(iii) for a.a. $t \in T$,

$\{x \in X : f(t, \cdot) \text{ is not lower semicontinuous at } x\} \subseteq E.$

Then for each $\varepsilon > 0$ there exists a $T_\mu$-measurable set $K \subseteq T$ such that $\mu(T \setminus K) \leq \varepsilon$ and the function $f|_{K \times X}$ is lower semicontinuous at each point $(t, x) \in K \times (X \setminus E)$.

Proof. Without loss of generality, we can suppose $f \geq 0$ in $T \times X$. Let $T_0 \in \mathcal{B}(X)$ be such that $\mu(T \setminus T_0) = 0$ and (2) holds for all $t \in T_0$. For each $n \in \mathbb{N}$, let $f_n : T \times X \to [0, \infty[$ be the function defined by putting, for all $(t, x) \in T \times X$,

$\begin{equation}
 f_n(t, x) := \inf_{y \in X} [nd(x, y) + f(t, y)].
\end{equation}$

We observe the following facts.

(a) For each $x \in X$, the function $f_n(\cdot, x)$ is $T_\mu$-measurable over $T$. This follows from Lemma III.39 of [3], since the function
\[(t, y) \mapsto nd(x, y) + f(t, y)\]
is $T_\mu \otimes \mathcal{B}(X)$-measurable for each fixed $n \in \mathbb{N}$ and $x \in X$.

(b) For each $n \in \mathbb{N}$ and each $(t, x) \in T \times X$, one has
\[
f_n(t, x) \leq f(t, x).
\]
Indeed, it is enough to put $y = x$ in (3).

(c) For each $n \in \mathbb{N}$ and each $(t, x) \in T \times X$, one has
\[
f_n(t, x) = \inf_{y \in X} [nd(x, z) + nd(z, y) + f(t, y)] = nd(x, z) + f_n(t, z),
\]
hence
\[
f_n(t, x) - f_n(t, z) \leq nd(x, z).
\]
By the latter inequality, upon interchanging the roles of $x$ and $z$, our assertion follows.

(d) For all $(t, x) \in T \times X$, set
\[
f^*(t, x) := \sup_{n \in \mathbb{N}} f_n(t, x).
\]
Then
\[
f^*(t, x) = f(t, x) \quad \text{for all } (t, x) \in T_0 \times (X \setminus E).
\]
To see this, let $(t, x) \in T_0 \times (X \setminus E)$ and $\eta > 0$. Since $f(t, \cdot)$ is lower semicontinuous at $x$, there exists $\delta > 0$ such that for each $y \in X$ with $d(x, y) < \delta$ one has
\[
f(t, y) > \beta := f(t, x) - \eta.
\]
Pick $n^* > \beta/\delta$. For each $y \in X$ we get
\[
n^*d(x, y) + f(t, y) \geq \begin{cases} f(t, y) > \beta & \text{if } d(x, y) < \delta, \\ n^*\delta + f(t, y) > \beta + f(t, y) \geq \beta & \text{if } d(x, y) \geq \delta. \end{cases}
\]
It follows that $f_{n^*}(t, x) \geq \beta$ and thus, by taking into account (4), the equality (5) holds.

At this point, fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, by the Scorza Dragoni Theorem, there exists a $T_\mu$-measurable set $K_n \subseteq T_0$ such that
\[
\mu(T \setminus K_n) \leq \varepsilon/2^n
\]
and $f_n|_{K_n \times X}$ is continuous. The set $K := \bigcap_{n \in \mathbb{N}} K_n$ is $T_\mu$-measurable and
\[
\mu(T \setminus K) = \mu\left( \bigcup_{n \in \mathbb{N}} (T \setminus K_n) \right) \leq \sum_{n=1}^{\infty} \mu(T \setminus K_n) \leq \varepsilon.
\]
Since each $f_n|_{K \times X}$ is continuous, $f^*|_{K \times X}$ is lower semicontinuous (as the upper envelope of a sequence of continuous functions). Now, choose any $(t^*, x^*) \in K \times (X \setminus E)$, and let us show that $f|_{K \times X}$ is lower semicontinuous.
at \((t^*, x^*)\). To this end, fix \(\gamma > 0\). Since \(f^*|_{K \times X}\) is lower semicontinuous, there exists a neighborhood \(U\) of \((t^*, x^*)\) in \(K \times X\) such that
\[
 f^*(t^*, x^*) - \gamma < f^*(t, x) \quad \text{for all } (t, x) \in U.
\]
Taking into account (4) and (5), it follows that for all \((t, x) \in U\) one has
\[
 f(t, x) \geq f^*(t, x) > f^*(t^*, x^*) - \gamma = f(t^*, x^*) - \gamma.
\]
Hence, \(f|_{K \times X}\) is lower semicontinuous at \((t^*, x^*)\), as claimed. The proof is complete.

**Lemma 5.** Let \(T, X, \mu, \psi\) be as in Lemma 1, with \(\mu\) finite, \(S\) a separable metric space, \(F : T \times X \to 2^S\) a multifunction with nonempty closed values, and \(E \subseteq X\) a given set. Assume that:

(i) \(F\) is \(T \otimes \mathcal{B}(X)\)-measurable;

(ii) for a.a. \(t \in T\),

\[
 \{x \in X : F(t, \cdot) \text{ is not lower semicontinuous at } x\} \subseteq E.
\]

Then for each \(\varepsilon > 0\) there exists a \(T \mu\)-measurable set \(K \subseteq T\) such that \(\mu(T \setminus K) \leq \varepsilon\) and the multifunction \(F|_{K \times X}\) is lower semicontinuous at each \((t, x) \in K \times (X \setminus E)\).

**Proof.** Let \(\varrho\) be an equivalent distance over \(S\) such that \(\varrho \leq 1\), and let \(\{y_n\}\) be a dense sequence in \(S\). By Proposition 13.2.2 of [7], for each \(y \in S\) the function \(\varrho(y, F(\cdot, \cdot))\) is \(T \otimes \mathcal{B}(X)\)-measurable. Moreover, by Lemma 3, for each \(y \in E\) and for a.a. \(t \in T\) the function \(\varrho(y, F(t, \cdot))\) is upper semicontinuous at each \(x \in X \setminus E\). Now, fix \(\varepsilon > 0\). For each \(n \in \mathbb{N}\), by Lemma 4 applied to the function \(-\varrho(y_n, F(\cdot, \cdot))\), there exists a \(T \mu\)-measurable set \(K_n \subseteq T\) such that
\[
 \mu(T \setminus K_n) \leq \varepsilon / 2^n
\]
and the function \(\varrho(y_n, F(\cdot, \cdot))|_{K_n \times X}\) is upper semicontinuous at each \((t, x) \in K_n \times (X \setminus E)\). Putting \(K := \bigcap_{n \in \mathbb{N}} K_n\), we see that \(K\) is \(T \mu\)-measurable, \(\mu(T \setminus K) \leq \varepsilon\), and for each \(n \in \mathbb{N}\) the function \(\varrho(y_n, F(\cdot, \cdot))|_{K \times X}\) is upper semicontinuous at each \((t, x) \in K \times (X \setminus E)\). By Lemma 3, this implies our conclusion.

**Proof of Theorem 2.** By Lemma 5, there exists a sequence \(\{K_n\}_{n \in \mathbb{N}}\) of pairwise disjoint \(T \mu\)-measurable subsets of \(T\) such that the set
\[
 Y := T \setminus \bigcup_{n \in \mathbb{N}} K_n
\]
is negligible and, for each \(n \in \mathbb{N}\), the multifunction \(F|_{K_n \times X}\) is lower semicontinuous at each point of \(K_n \times (X \setminus E)\). We can assume that inclusion (1) holds for all \(t \in \bigcup_{n \in \mathbb{N}} K_n\). Let \(Q\) and \(R\) be as in Lemma 1. By Lemma 2,
for each \( n \in \mathbb{N} \) there exists a selection \( g_n : K_n \times X \to S \) of \( F|_{K_n \times X} \) which is continuous at each point of
\[
(K_n \setminus Q) \times (X \setminus (R \cup E)).
\]
For all \( t \in Y \), let \( h_t : X \to S \) be any selection of the multifunction \( F(t, \cdot) \).
Define \( \phi : T \times X \to S \) by putting
\[
\phi(t, x) = \begin{cases} 
    g_n(t, x) & \text{if } t \in K_n, \\
    h_t(x) & \text{if } t \in Y.
\end{cases}
\]
Clearly, \( \phi \) is a selection of \( F \). Let us show that \( \phi \) satisfies our conclusion. To this end, choose any \( x^* \in X \setminus (E \cup R) \). Since by construction each function \( g_n(\cdot, x^*)|_{K_n \setminus Q} \) is continuous, it is \( T_n \)-measurable. Since \( Y \cup Q \) is negligible, \( \phi \) satisfies (i). In order to prove (ii), choose any \( t^* \in T \setminus (Y \cup Q) \), and let \( n \in \mathbb{N} \) be such that \( t^* \in K_n \). Since
\[
\{ x \in X : g_n(t^*, \cdot) \text{ is discontinuous at } x \} \subseteq E \cup R
\]
and \( g_n(t^*, \cdot) = \phi(t^*, \cdot) \), our claim follows. This completes the proof.

3. An application to differential inclusions. In this section we provide an application of Theorem 2 to differential inclusions. In particular we stress that a multifunction \( F \) satisfying the assumptions of Theorem 3 below can fail to be lower semicontinuous in the second variable at each \( x \in \mathbb{R} \). Moreover, unlike other recent results in the field (see, for instance, [10] and references therein), the convexity of the values of \( F \) is not assumed. Our result is as follows (as usual, we denote by \( m \) the Lebesgue measure in \( \mathbb{R} \)).

**Theorem 3.** Let \([a, b]\) be a real interval, \( F : [a, b] \times \mathbb{R} \to 2^{\mathbb{R}} \) a multifunction and \( p \in [1, \infty[ \). Assume that there exists a multifunction \( G : [a, b] \times \mathbb{R} \to 2^{\mathbb{R}} \), with nonempty closed values, satisfying the following conditions:

(i) \( G \) is \( \mathcal{L}([a, b]) \otimes \mathcal{B}(\mathbb{R}) \)-measurable;

(ii) there exists \( E_0 \subseteq \mathbb{R} \), with \( m(E_0) = 0 \), such that for a.a. \( t \in [a, b] \),
\[
\{ x \in \mathbb{R} : G(t, \cdot) \text{ is not lower semicontinuous at } x \}
\]
\[
\cup \{ x \in \mathbb{R} : G(t, x) \not\subseteq F(t, x) \} \subseteq E_0;
\]

(iii) there exist \( \beta \in \mathcal{L}^p([a, b]) \) and \( \alpha : [a, b] \to ]0, \infty[ \) such that for a.a. \( t \in [a, b] \) and all \( x \in \mathbb{R} \)
\[
G(t, x) \subseteq [\alpha(t), \beta(t)].
\]
Then there exists \( u \in W^{2,p}([a, b]) \) such that
\[
\begin{cases} 
    u''(t) \in F(t, u(t)) & \text{for a.a. } t \in [a, b], \\
    u(a) = u(b) = 0.
\end{cases}
\]

**Proof.** By Theorem 2, there exist a selection \( \phi : [a, b] \times \mathbb{R} \to \mathbb{R} \) of the multifunction \( G \), and two \( m \)-negligible sets \( K_0 \subseteq [a, b] \) and \( E \subseteq \mathbb{R} \), with
$E_0 \subseteq E$, such that for all $x \in \mathbb{R} \setminus E$ the function $\phi(\cdot, x)$ is measurable and for all $t \in [a, b] \setminus K_0$ one has

$$ \{ x \in \mathbb{R} : \phi(t, \cdot) \text{ is discontinuous at } x \} \subseteq E. $$

Since $\mathbb{R} \setminus E$ a separable dense subset of $\mathbb{R}$, we can find a countable set $P \subset \mathbb{R} \setminus E$ which is dense in $\mathbb{R}$.

Without loss of generality we can assume that

$$ G(t, x) \subseteq [\alpha(t), \beta(t)] \quad \text{for all } x \in \mathbb{R}, t \in [a, b] \setminus K_0. $$

Let $\phi^* : [a, b] \times \mathbb{R} \to \mathbb{R}$ be defined by

$$ \phi^*(t, x) = \begin{cases} \phi(t, x) & \text{if } t \in [a, b] \setminus K_0, \\ \beta(t) & \text{if } t \in K_0. \end{cases} $$

Since $\phi$ is a selection of $G$, by assumption (iii) we have

$$ \alpha(t) \leq \phi^*(t, x) \leq \beta(t) \quad \text{for all } x \in \mathbb{R}, t \in [a, b]. $$

In particular, observe that $\phi^*(t, \cdot)$ is bounded for each $t \in [a, b]$, and $\phi^*(\cdot, x)$ is measurable for each $x \in P$. Consequently, by Proposition 2 of [4], the multifunction $H : [a, b] \times \mathbb{R} \to 2^\mathbb{R}$ defined by setting

$$ H(t, x) = \bigcap_{m \in \mathbb{N}} \overline{\operatorname{co}} \left( \bigcup_{y \in D, |y-x| \leq 1/m} \{ \phi^*(t, y) \} \right) $$

satisfies the following conditions:

(a) $H$ has nonempty closed convex values;
(b) for all $x \in \mathbb{R}$, the multifunction $H(\cdot, x)$ is measurable;
(c) for each $t \in [a, b]$, the multifunction $H(t, \cdot)$ has closed graph;
(d) for all $t \in [a, b] \setminus K_0$ and all $x \in \mathbb{R} \setminus E$,

$$ H(t, x) = \{ \phi(t, x) \}. $$

Moreover, by the above construction it follows that

$$ H(t, x) \subseteq [\alpha(t), \beta(t)] \quad \text{for all } x \in \mathbb{R}, t \in [a, b]. $$

Now we want to apply Theorem 1 of [10] to the multifunction $H$, taking $T = [a, b], X = Y = \mathbb{R}$, $s = q = p$,

$$ V = \left\{ u \in W^{1,p}([a, b]) : \int_a^b u(t) \, dt = 0 \right\}, $$

$$ \Psi(u) = u', \Phi(u)(t) = \int_a^t u(\tau) \, d\tau, \varphi \equiv +\infty \text{ and } r = ||\beta||_{L^p([a, b])}. $$

To this end, we observe that the operators $\Psi$ and $\Phi$ satisfy all the conditions of Theorem 1 of [10] (see the proof of Theorem 3 of [10]) as does the multifunction $H$. Consequently, there exist a function $v \in V$ and a negligible set $K \subseteq [a, b]$, with $K_0 \subseteq K$, such that

$$ \Psi(v)(t) \in H(t, \Phi(v)(t)) \quad \text{for all } t \in [a, b] \setminus K. $$

In particular, observe that $\phi^*(t, \cdot)$ is bounded for each $t \in [a, b]$, and $\phi^*(\cdot, x)$ is measurable for each $x \in P$. Consequently, by Proposition 2 of [4], the multifunction $H : [a, b] \times \mathbb{R} \to 2^\mathbb{R}$ defined by setting

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satisfies the following conditions:

(a) $H$ has nonempty closed convex values;
(b) for all $x \in \mathbb{R}$, the multifunction $H(\cdot, x)$ is measurable;
(c) for each $t \in [a, b]$, the multifunction $H(t, \cdot)$ has closed graph;
(d) for all $t \in [a, b] \setminus K_0$ and all $x \in \mathbb{R} \setminus E$,

$$ H(t, x) = \{ \phi(t, x) \}. $$

Moreover, by the above construction it follows that

$$ H(t, x) \subseteq [\alpha(t), \beta(t)] \quad \text{for all } x \in \mathbb{R}, t \in [a, b]. $$

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$$ V = \left\{ u \in W^{1,p}([a, b]) : \int_a^b u(t) \, dt = 0 \right\}, $$

$$ \Psi(u) = u', \Phi(u)(t) = \int_a^t u(\tau) \, d\tau, \varphi \equiv +\infty \text{ and } r = ||\beta||_{L^p([a, b])}. $$

To this end, we observe that the operators $\Psi$ and $\Phi$ satisfy all the conditions of Theorem 1 of [10] (see the proof of Theorem 3 of [10]) as does the multifunction $H$. Consequently, there exist a function $v \in V$ and a negligible set $K \subseteq [a, b]$, with $K_0 \subseteq K$, such that

$$ \Psi(v)(t) \in H(t, \Phi(v)(t)) \quad \text{for all } t \in [a, b] \setminus K. $$
Without loss of generality we can assume that $v$ is continuous. Put $\gamma = \Phi(v)$. By (6) and (7) we have $\gamma''(t) = v'(t) \geq \alpha(t) > 0$ a.e. in $[a, b]$, hence $\gamma' = v$ is strictly increasing. Since $\gamma(a) = \gamma(b)$, there exists $c \in ]a, b]$ such that
\[
\gamma'(t) < 0 \quad \text{for all } t \in [a, c],
\]
\[
\gamma'(t) > 0 \quad \text{for all } t \in [c, b].
\]
By Theorem 2 of [12], the functions $(\gamma|_{[a,c]})^{-1}$ and $(\gamma|_{[c,b]})^{-1}$ are absolutely continuous. By assumption (ii), there exists $K_1 \subseteq [a, b]$, with $m(K_1) = 0$, such that for all $t \in [a, b] \setminus K_1$ one has
\[
E \ni x < \overline{G(t, x)} \subseteq F(t, x).
\]

Then our conclusion follows by taking $u(t) = \int_a^t v(\tau) \, d\tau$.

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