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Parametrization of Riemann-measurable selections for multifunctions of two variables with application to differential inclusions

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Abstract. We consider a multifunction $F : T \times X \to 2^E$, where T, X and E are separable metric spaces, with E complete. Assuming that F is jointly measurable in the product and a.e. lower semicontinuous in the second variable, we establish the existence of a selection for F which is measurable with respect to the first variable and a.e. continuous with respect to the second one. Our result is in the spirit of [11], where multifunctions of only one variable are considered.

1. Introduction. If X is a topological space, we denote by $\mathcal{B}(X)$ the Borel σ-algebra of X. Moreover, if μ is measure on $\mathcal{B}(X)$, we denote by $\mathcal{T}_\mu(X)$ the completion of $\mathcal{B}(X)$ with respect to μ. We briefly put $\mathcal{T}_\mu = \mathcal{T}_\mu(X)$ when ambiguities do not occur. For the basic definitions about multifunctions, we refer the reader to [6] and [7].

This note is motivated by the main result of [11], which concerns the existence of Riemann-measurable selections (i.e., selections which are a.e. continuous) for a given multifunction. For the reader's convenience, we now state the main result of [11] (as usual, by a *Polish space* we mean a complete separable metric space).

Theorem 1 (Theorem 3 of [11]). *Let X be a Polish space equipped with a σ-finite regular Borel measure, E a metric space and $F : X \to 2^E$ a multifunction with nonempty complete values. If F is lower semicontinuous at almost every point of X, then there exists a selection of F which is continuous at almost every point of X.*

We refer to [11] for motivations leading to Theorem 1. Applications of Theorem 1 to implicit integral equations and to elliptic differential equations can be found in [2] and [8], respectively.

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Our aim in this paper is to prove a parametrized version of Theorem 1 for multifunctions of two variables, obtaining, in particular, sufficient conditions for the existence of a selection which is measurable with respect to the first variable and a.e. continuous with respect to the second one. More precisely, we prove the following result.

THEOREM 2. *Let T, X be two Polish spaces and let μ, ψ be two positive regular Borel measures on T and X, respectively, with μ finite and ψ σ-finite. Let S be a separable metric space, $F : T \times X \to 2^S$ a multifunction with nonempty complete values, and let $E \subseteq X$ be a given set. Assume that*:

(i) *F is $\mathcal{T}_\mu \otimes \mathcal{B}(X)$-measurable*;
(ii) *for a.a. $t \in T$,*

$$\{x \in X : F(t,\cdot) \text{ is not lower semicontinuous at } x\} \subseteq E. \tag{1}$$

Then there exist a selection $\phi : T \times X \to S$ of F and a set $R \in \mathcal{B}(X)$, with $\psi(R) = 0$, such that

(i)′ *$\phi(\cdot, x)$ is $\mathcal{T}_\mu$-measurable for each $x \in X \setminus (E \cup R)$;*
(ii)′ *for a.a. $t \in T$,*

$$\{x \in X : \phi(t,\cdot) \text{ is not continuous at } x\} \subseteq E \cup R.$$

The proof of Theorem 2 will be given in Section 2, while in Section 3 we shall provide an application of Theorem 2 to differential inclusions.

2. Proof of Theorem 2. Before proving Theorem 2, we need the following preliminary results.

LEMMA 1. *Let T, X be two Polish spaces and let μ, ψ be two positive σ-finite regular Borel measure on X and Y, respectively. Then there exist two sets $Q \in \mathcal{B}(T)$ and $R \in \mathcal{B}(X)$, with $\mu(Q) = \psi(R) = 0$, a continuous open function $\pi : \mathbb{N}^{\mathbb{N}} \to T \times X$, and a function $\sigma : T \times X \to \mathbb{N}^{\mathbb{N}}$ which is continuous at each point of $(T \setminus Q) \times (X \setminus R)$ and satisfies $\pi(\sigma(t,x)) = (t,x)$ for all $(t,x) \in T \times X$.*

Proof. By Lemma 1 of [11], there exist $Q \in \mathcal{B}(T)$ and $R \in \mathcal{B}(X)$ with $\mu(Q) = \psi(R) = 0$, two continuous open functions $\pi_1 : \mathbb{N}^{\mathbb{N}} \to T$, $\pi_2 : \mathbb{N}^{\mathbb{N}} \to X$, a function $\sigma_1 : T \to \mathbb{N}^{\mathbb{N}}$ which is continuous at each point of $T \setminus Q$, and a function $\sigma_2 : X \to \mathbb{N}^{\mathbb{N}}$ which is continuous at each point of $X \setminus R$, such that $\pi_1(\sigma_1(t)) = t$ and $\pi_2(\sigma_2(x)) = x$ for all $(t,x) \in T \times X$. For each $\alpha := \{n_k\}_k \in \mathbb{N}^{\mathbb{N}}$, we denote by α_e and α_o the sequences $\{n_{2k}\}_k$ and $\{n_{2k-1}\}_k$, respectively. If we put $\pi(\alpha) = (\pi_1(\alpha_e), \pi_2(\alpha_o))$ for all $\alpha \in \mathbb{N}^{\mathbb{N}}$, then $\pi : \mathbb{N}^{\mathbb{N}} \to T \times X$ is a continuous open function. Moreover, if we put $\sigma(t,x) = \{n(t,x)_k\}_k$, where $\{n(t,x)_{2k}\}_k = \sigma_1(t)$ and $\{n(t,x)_{2k-1}\}_k = \sigma_2(x)$, then $\sigma : T \times X \to \mathbb{N}^{\mathbb{N}}$ is continuous at each point of $(T \setminus Q) \times (X \setminus R)$ and one has $\pi(\sigma(t,x)) = (t,x)$ for all $(t,x) \in T \times X$.

LEMMA 2. *Let $T, X, Q, R, \mu, \psi, \pi, \sigma$ be as in Lemma* 1. *Let E be a metric space, $B \subseteq T \times X$ and $V \subseteq B$ two given sets, and $F : B \to 2^E$ a multifunction with nonempty complete values which is lower semicontinuous at each point of $B \setminus V$. Then there exists a selection g of F which is continuous at each point of $[B \cap ((T \setminus Q) \times (X \setminus R))] \setminus V$.*

Proof. Put $Z = \pi^{-1}(B)$ and $G = F \circ \pi|_Z$. Observe that Z is 0-dimensional and G is lower semicontinuous at each point of $Z \setminus \pi^{-1}(V)$. Consequently, by the proof of Lemma 2 of [11], there exists a selection s of G which is continuous at each point of $Z \setminus \pi^{-1}(V)$. Since $\sigma(t, x) \in \pi^{-1}(t, x) \subseteq Z$ for all $(t, x) \in B$, we can put $g(t, x) = s(\sigma(t, x))$ for all $(t, x) \in B$. Then $g(t, x) \in F(\pi(\sigma(t, x)) = F(t, x)$ for all $(t, x) \in B$. Further, it is easily seen that g is continuous at each point of $[B \cap ((T \setminus Q) \times (X \setminus R))] \setminus V$.

The next lemma follows from the proof of Lemma 2.3 of [1].

LEMMA 3. *Let X and S be metric spaces, with S separable, $F : X \to 2^S$ a multifunction with nonempty values, $\{s_n\}$ a dense sequence in S, and $x_0 \in X$. Denote by d the distance in S.*

(i) *If F is lower semicontinuous at x_0, then for each $s \in S$ the function $x \in X \mapsto d(s, F(x))$ is upper semicontinuous at x_0.*
(ii) *If for each $n \in \mathbb{N}$ the function $x \in X \mapsto d(s_n, F(x))$ is upper semicontinuous at x_0, then F is lower semicontinuous at x_0.*

LEMMA 4. *Let T, X, μ, ψ be as in Lemma* 1, *with μ finite. Let $f : T \times X \to \mathbb{R}$ be a single-valued function and $E \subseteq X$ a given set. Assume that*:

(i) *f is $\mathcal{T}_\mu \otimes \mathcal{B}(X)$-measurable*;
(ii) $\inf_{T\times X} f > -\infty$;
(iii) *for a.a. $t \in T$*,

$$\{x \in X : f(t, \cdot) \text{ is not lower semicontinuous at } x\} \subseteq E. \tag{2}$$

Then for each $\varepsilon > 0$ there exists a $\mathcal{T}_\mu$-measurable set $K \subseteq T$ such that $\mu(T \setminus K) \le \varepsilon$ and the function $f|_{K\times X}$ is lower semicontinuous at each point $(t, x) \in K \times (X \setminus E)$.

Proof. Without loss of generality, we can suppose $f \ge 0$ in $T \times X$. Let $T_0 \in \mathcal{B}(X)$ be such that $\mu(T \setminus T_0) = 0$ and (2) holds for all $t \in T_0$. For each $n \in \mathbb{N}$, let $f_n : T \times X \to [0, \infty[$ be the function defined by putting, for all $(t, x) \in T \times X$,

$$f_n(t, x) := \inf_{y\in X} [nd(x, y) + f(t, y)]. \tag{3}$$

We observe the following facts.

(a) For each $x \in X$, the function $f_n(\cdot, x)$ is $\mathcal{T}_\mu$-measurable over T. This follows from Lemma III.39 of [3], since the function

$$(t, y) \mapsto nd(x, y) + f(t, y)$$

is $\mathcal{T}_\mu \otimes \mathcal{B}(X)$-measurable for each fixed $n \in \mathbb{N}$ and $x \in X$.

(b) For each $n \in \mathbb{N}$ and each $(t, x) \in T \times X$, one has

$$f_n(t, x) \leq f(t, x). \tag{4}$$

Indeed, it is enough to put $y = x$ in (3).

(c) For each $n \in \mathbb{N}$ and each $t \in T$, the function $f_n(t, \cdot)$ is n-Lipschitzian over X. Indeed, if $n \in \mathbb{N}$ and $t \in T$ are fixed, then for each $x, z \in X$ one has

$$f_n(t, x) \leq \inf_{y \in X} [nd(x, z) + nd(z, y) + f(t, y)] = nd(x, z) + f_n(t, z),$$

hence

$$f_n(t, x) - f_n(t, z) \leq nd(x, z).$$

By the latter inequality, upon interchanging the roles of x and z, our assertion follows.

(d) For all $(t, x) \in T \times X$, set

$$f^*(t, x) := \sup_{n \in \mathbb{N}} f_n(t, x)\,.$$

Then

$$f^*(t, x) = f(t, x) \quad \text{for all } (t, x) \in T_0 \times (X \setminus E). \tag{5}$$

To see this, let $(t, x) \in T_0 \times (X \setminus E)$ and $\eta > 0$. Since $f(t, \cdot)$ is lower semicontinuous at x, there exists $\delta > 0$ such that for each $y \in X$ with $d(x, y) < \delta$ one has

$$f(t, y) > \beta := f(t, x) - \eta.$$

Pick $n^* > \beta/\delta$. For each $y \in X$ we get

$$n^* d(x, y) + f(t, y) \geq \begin{cases} f(t, y) > \beta & \text{if } d(x, y) < \delta, \\ n^*\delta + f(t, y) > \beta + f(t, y) \geq \beta & \text{if } d(x, y) \geq \delta. \end{cases}$$

It follows that $f_{n^*}(t, x) \geq \beta$ and thus, by taking into account (4), the equality (5) holds.

At this point, fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, by the Scorza Dragoni Theorem, there exists a $\mathcal{T}_\mu$-measurable set $K_n \subseteq T_0$ such that

$$\mu(T \setminus K_n) \leq \varepsilon/2^n$$

and $f_n|_{K_n \times X}$ is continuous. The set $K := \bigcap_{n \in \mathbb{N}} K_n$ is $\mathcal{T}_\mu$-measurable and

$$\mu(T \setminus K) = \mu\Big(\bigcup_{n \in \mathbb{N}} (T \setminus K_n) \Big) \leq \sum_{n=1}^{\infty} \mu(T \setminus K_n) \leq \varepsilon.$$

Since each $f_n|_{K \times X}$ is continuous, $f^*|_{K \times X}$ is lower semicontinuous (as the upper envelope of a sequence of continuous functions). Now, choose any $(t^*, x^*) \in K \times (X \setminus E)$, and let us show that $f|_{K \times X}$ is lower semicontinuous at (t^*, x^*). To this end, fix $\gamma > 0$. Since $f^*|_{K\times X}$ is lower semicontinuous, there exists a neighborhood U of (t^*, x^*) in $K \times X$ such that

$$f^*(t^*, x^*) - \gamma < f^*(t, x) \quad \text{for all } (t, x) \in U.$$

Taking into account (4) and (5), it follows that for all $(t, x) \in U$ one has

$$f(t, x) \geq f^*(t, x) > f^*(t^*, x^*) - \gamma = f(t^*, x^*) - \gamma.$$

Hence, $f|_{K\times X}$ is lower semicontinuous at (t^*, x^*), as claimed. The proof is complete.

LEMMA 5. *Let T, X, μ, ψ be as in Lemma* 1, *with μ finite*, *S a separable metric space*, *$F : T \times X \to 2^S$ a multifunction with nonempty closed values*, *and $E \subseteq X$ a given set. Assume that*:

(i) *F is $\mathcal{T}_\mu \otimes \mathcal{B}(X)$-measurable*;
(ii) *for a.a. $t \in T$*,

$$\{x \in X : F(t, \cdot) \text{ is not lower semicontinuous at } x\} \subseteq E.$$

Then for each $\varepsilon > 0$ there exists a $\mathcal{T}_\mu$-measurable set $K \subseteq T$ such that $\mu(T \setminus K) \leq \varepsilon$ and the multifunction $F|_{K\times X}$ is lower semicontinuous at each $(t, x) \in K \times (X \setminus E)$.

Proof. Let ϱ be an equivalent distance over S such that $\varrho \leq 1$, and let $\{y_n\}$ be a dense sequence in S. By Proposition 13.2.2 of [7], for each $y \in S$ the function $\varrho(y, F(\cdot, \cdot))$ is $\mathcal{T}_\mu \otimes \mathcal{B}(X)$-measurable. Moreover, by Lemma 3, for each $y \in E$ and for a.a. $t \in T$ the function $\varrho(y, F(t, \cdot))$ is upper semicontinuous at each $x \in X \setminus E$. Now, fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, by Lemma 4 applied to the function $-\varrho(y_n, F(\cdot, \cdot))$, there exists a $\mathcal{T}_\mu$-measurable set $K_n \subseteq T$ such that

$$\mu(T \setminus K_n) \leq \varepsilon/2^n$$

and the function $\varrho(y_n, F(\cdot, \cdot))|_{K_n\times X}$ is upper semicontinuous at each $(t, x) \in K_n \times (X \setminus E)$. Putting $K := \bigcap_{n\in\mathbb{N}} K_n$, we see that K is $\mathcal{T}_\mu$-measurable, $\mu(T \setminus K) \leq \varepsilon$, and for each $n \in \mathbb{N}$ the function $\varrho(y_n, F(\cdot, \cdot))|_{K\times X}$ is upper semicontinuous at each $(t, x) \in K \times (X \setminus E)$. By Lemma 3, this implies our conclusion.

Proof of Theorem 2. By Lemma 5, there exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ of pairwise disjoint $\mathcal{T}_\mu$-measurable subsets of T such that the set

$$Y := T \setminus \bigcup_{n\in\mathbb{N}} K_n$$

is negligible and, for each $n \in \mathbb{N}$, the multifunction $F|_{K_n\times X}$ is lower semicontinuous at each point of $K_n \times (X \setminus E)$. We can assume that inclusion (1) holds for all $t \in \bigcup_{n\in\mathbb{N}} K_n$. Let Q and R be as in Lemma 1. By Lemma 2, for each $n \in \mathbb{N}$ there exists a selection $g_n : K_n \times X \to S$ of $F|_{K_n \times X}$ which is continuous at each point of

$$(K_n \setminus Q) \times (X \setminus (R \cup E)).$$

For all $t \in Y$, let $h_t : X \to S$ be any selection of the multifunction $F(t, \cdot)$. Define $\phi : T \times X \to S$ by putting

$$\phi(t,x) = \begin{cases} g_n(t,x) & \text{if } t \in K_n, \\ h_t(x) & \text{if } t \in Y. \end{cases}$$

Clearly, ϕ is a selection of F. Let us show that ϕ satisfies our conclusion. To this end, choose any $x^* \in X \setminus (E \cup R)$. Since by construction each function $g_n(\cdot, x^*)|_{K_n \setminus Q}$ is continuous, it is $\mathcal{T}_\mu$-measurable. Since $Y \cup Q$ is negligible, ϕ satisfies (i)$'$. In order to prove (ii)$''$, choose any $t^* \in T \setminus (Y \cup Q)$, and let $n \in \mathbb{N}$ be such that $t^* \in K_n$. Since

$$\{x \in X : g_n(t^*, \cdot) \text{ is discontinuous at } x\} \subseteq E \cup R$$

and $g_n(t^*, \cdot) = \phi(t^*, \cdot)$, our claim follows. This completes the proof.

3. An application to differential inclusions. In this section we provide an application of Theorem 2 to differential inclusions. In particular we stress that a multifunction F satisfying the assumptions of Theorem 3 below can fail to be lower semicontinuous in the second variable at each $x \in \mathbb{R}$. Moreover, unlike other recent results in the field (see, for instance, [10] and references therein), the convexity of the values of F is not assumed. Our result is as follows (as usual, we denote by m the Lebesgue measure in $\mathbb{R}$).

THEOREM 3. *Let $[a,b]$ be a real interval, $F : [a,b] \times \mathbb{R} \to 2^{\mathbb{R}}$ a multifunction and $p \in [1, \infty[$. Assume that there exists a multifunction $G : [a,b] \times \mathbb{R} \to 2^{\mathbb{R}}$, with nonempty closed values, satisfying the following conditions*:

(i) *G is $\mathcal{L}([a,b]) \otimes \mathcal{B}(\mathbb{R})$-measurable*;

(ii) *there exists $E_0 \subseteq \mathbb{R}$, with $m(E_0) = 0$, such that for a.a. $t \in [a,b]$,*

$$\{x \in \mathbb{R} : G(t, \cdot) \text{ is not lower semicontinuous at } x\} \cup \{x \in \mathbb{R} : G(t,x) \not\subseteq F(t,x)\} \subseteq E_0;$$

(iii) *there exist $\beta \in L^p([a,b])$ and $\alpha : [a,b] \to]0, \infty[$ such that for a.a. $t \in [a,b]$ and all $x \in \mathbb{R}$*

$$G(t,x) \subseteq [\alpha(t), \beta(t)].$$

Then there exists $u \in W^{2,p}([a,b])$ such that

$$\begin{cases} u''(t) \in F(t, u(t)) & \text{for a.a. } t \in [a,b], \\ u(a) = u(b) = 0. \end{cases}$$

Proof. By Theorem 2, there exist a selection $\phi : [a,b] \times \mathbb{R} \to \mathbb{R}$ of the multifunction G, and two m-negligible sets $K_0 \subseteq [a,b]$ and $E \subseteq \mathbb{R}$, with

$E_0 \subseteq E$, such that for all $x \in \mathbb{R} \setminus E$ the function $\phi(\cdot, x)$ is measurable and for all $t \in [a,b] \setminus K_0$ one has

$$\{x \in \mathbb{R} : \phi(t,\cdot) \text{ is discontinuous at } x\} \subseteq E.$$

Since $\mathbb{R} \setminus E$ a separable dense subset of $\mathbb{R}$, we can find a countable set $P \subset \mathbb{R} \setminus E$ which is dense in $\mathbb{R}$.

Without loss of generality we can assume that

$$G(t,x) \subseteq [\alpha(t), \beta(t)] \quad \text{for all } x \in \mathbb{R},\ t \in [a,b] \setminus K_0.$$

Let $\phi^* : [a,b] \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi^*(t,x) = \begin{cases} \phi(t,x) & \text{if } t \in [a,b] \setminus K_0, \\ \beta(t) & \text{if } t \in K_0. \end{cases}$$

Since ϕ is a selection of G, by assumption (iii) we have

$$\alpha(t) \le \phi^*(t,x) \le \beta(t) \quad \text{for all } x \in \mathbb{R},\ t \in [a,b].$$

In particular, observe that $\phi^*(t, \cdot\,)$ is bounded for each $t \in [a,b]$, and $\phi^*(\cdot, x)$ is measurable for each $x \in P$. Consequently, by Proposition 2 of [4], the multifunction $H : [a,b] \times \mathbb{R} \to 2^{\mathbb{R}}$ defined by setting

$$H(t,x) = \bigcap_{m \in \mathbb{N}} \overline{\mathrm{co}}\Big(\overline{\bigcup_{y \in D,\, |y-x| \le 1/m} \{\phi^*(t,y)\}} \Big)$$

satisfies the following conditions:

(a) H has nonempty closed convex values;
(b) for all $x \in \mathbb{R}$, the multifunction $H(\cdot, x)$ is measurable;
(c) for each $t \in [a,b]$, the multifunction $H(t,\cdot)$ has closed graph;
(d) for all $t \in [a,b] \setminus K_0$ and all $x \in \mathbb{R} \setminus E$,

$$H(t,x) = \{\phi(t,x)\}.$$

Moreover, by the above construction it follows that

$$H(t,x) \subseteq [\alpha(t), \beta(t)] \quad \text{for all } x \in \mathbb{R},\ t \in [a,b]. \tag{6}$$

Now we want to apply Theorem 1 of [10] to the multifunction H, taking $T = [a,b]$, $X = Y = \mathbb{R}$, $s = q = p$,

$$V = \Big\{ u \in W^{1,p}([a,b]) : \int_a^b u(t)\,dt = 0 \Big\},$$

$\Psi(u) = u'$, $\Phi(u)(t) = \int_a^t u(\tau)\,d\tau$, $\varphi \equiv +\infty$ and $r = \|\beta\|_{L^p([a,b])}$. To this end, we observe that the operators Ψ and Φ satisfy all the conditions of Theorem 1 of [10] (see the proof of Theorem 3 of [10]) as does the multifunction H. Consequently, there exist a function $v \in V$ and a negligible set $K \subseteq [a,b]$, with $K_0 \subseteq K$, such that

$$\Psi(v)(t) \in H(t, \Phi(v)(t)) \quad \text{for all } t \in [a,b] \setminus K. \tag{7}$$

Without loss of generality we can assume that v is continuous. Put $\gamma = \Phi(v)$. By (6) and (7) we have $\gamma''(t) = v'(t) \geq \alpha(t) > 0$ a.e. in $[a, b]$, hence $\gamma' = v$ is strictly increasing. Since $\gamma(a) = \gamma(b)$, there exists $c \in]a, b[$ such that

$$\gamma'(t) < 0 \quad \text{for all } t \in [a, c[,$$
$$\gamma'(t) > 0 \quad \text{for all } t \in]c, b].$$

By Theorem 2 of [12], the functions $(\gamma|_{[a,c]})^{-1}$ and $(\gamma|_{[c,b]})^{-1}$ are absolutely continuous. By assumption (ii), there exists $K_1 \subseteq [a, b]$, with $m(K_1) = 0$, such that for all $t \in [a, b] \setminus K_1$ one has

$$\{ x \in \mathbb{R} : G(t, x) \nsubseteq F(t, x)\} \subseteq E_0.$$

Clearly, we can assume that $K \subseteq K_1$. Put

$$S := \gamma^{-1}(E) \cup K_1.$$

Since by Theorem 18.25 of [5] the sets $(\gamma|_{[a,c]})^{-1}(E)$ and $(\gamma|_{[c,b]})^{-1}(E)$ are negligible, it follows that $m(S) = 0$. Now, observe that for all $t \in [a, b] \setminus S$ one has $\Phi(v)(t) \in \mathbb{R} \setminus E$, hence,

$$H(t, \Phi(v)(t)) = \{\phi(t, \Phi(v)(t))\} \subseteq G(t, \Phi(v)(t)) \subseteq F(t, \Phi(v)(t)).$$

Consequently, by (7) we get

$$v'(t) \in F\Big(t, \int_a^t v(\tau)\, d\tau\Big) \quad \text{for all } t \in [a, b] \setminus S.$$

Then our conclusion follows by taking $u(t) = \int_a^t v(\tau)\, d\tau$.

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