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A note on Costara's paper

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Abstract. We show that the symmetrized bidisc $\mathbb{G}_2 = \{(\lambda_1 + \lambda_2, \lambda_1\lambda_2) : |\lambda_1|, |\lambda_2| < 1\} \subset \mathbb{C}^2$ cannot be exhausted by domains biholomorphic to convex domains.

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$. The open symmetrized bidisc $\mathbb{G}_2$ is the image of the bidisc $\mathbb{D}^2$ under the "symmetrization map" $\pi : (\lambda_1, \lambda_2) \mapsto (\lambda_1 + \lambda_2, \lambda_1\lambda_2)$.

A well-known theorem of L. Lempert states that on convex domains Carathéodory and Kobayashi pseudodistances coincide (see [4], and also [3, 2]). It turns out that the same is true on $\mathbb{G}_2$. So, it is important to know whether $\mathbb{G}_2$ can be presented as an exhaustion of domains biholomorphic to convex domains.

In [1] C. Costara proved that $\mathbb{G}_2$ is not biholomorphic to a convex domain. Using similar arguments we show the following improvement.

THEOREM 1. *$\mathbb{G}_2$ cannot be exhausted by domains biholomorphic to convex domains.*

Proof. Note that π is a proper holomorphic mapping. Let $\varrho(s,p) = \max\{|\lambda_1|, |\lambda_2|\}$, where λ_1, λ_2 are such that $\pi(\lambda_1, \lambda_2) = (s,p)$. It is easy to see that ϱ is a continuous plurisubharmonic function in $\mathbb{C}^2$. Moreover, $\varrho(\lambda s, \lambda^2 p) = |\lambda|\varrho(s,p)$ for any $\lambda \in \mathbb{C}$ and any $(s,p) \in \mathbb{C}^2$. We put $\varphi_\lambda(z_1, z_2) = (\lambda z_1, \lambda^2 z_2)$. Then $\varrho(\varphi_\lambda(z)) = |\lambda|\varrho(z)$.

One can check that $\mathbb{G}_2 = \{(s,p) \in \mathbb{C}^2 : \varrho(s,p) < 1\}$. For any $\varepsilon > 0$ we put $G_\varepsilon := \{(s,p) \in \mathbb{C}^2 : \varrho(s,p) < 1 - \varepsilon\}$.

Assume that U_ε is a neighborhood of $\overline{G}_\varepsilon$ and $f_\varepsilon : U_\varepsilon \to V_\varepsilon$ is a biholomorphic mapping, where V_ε is a convex domain. We may assume that $f_\varepsilon(0) = 0$ and that $f'_\varepsilon(0) = \mathrm{id}$ (see [1]).

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Fix $(s_1,p_1),(s_2,p_2)\in\mathbb{C}^2$ and $r\in[0,1]$. Put

$$R:=\max\{\varrho(s_1,p_1),\varrho(s_2,p_2)\},$$
$$g_\varepsilon(\lambda):=f_\varepsilon^{-1}(rf_\varepsilon(\lambda s_1,\lambda^2p_1)+(1-r)f_\varepsilon(\lambda s_2,\lambda^2p_2)).$$

We have $g_\varepsilon(0)=0$. Note that g_ε is well defined for $|\lambda|<(1-\varepsilon)/R$. Indeed, $\varrho(\varphi_\lambda(s_j,p_j))=|\lambda|\varrho(s_j,p_j)\le R|\lambda|<1-\varepsilon$ for $j=1,2$. Moreover, we have $\varrho(g_\varepsilon(\lambda))\le 1$ for any $|\lambda|<(1-\varepsilon)/R$. Let $h_\varepsilon(\lambda)=\varphi_{1/\lambda}(g_\varepsilon(\lambda))$. Then $h_\varepsilon:\mathbb{D}(0,(1-\varepsilon)/R)\setminus\{0\}\to\mathbb{C}^2$ is a holomorphic mapping. Moreover, it extends holomorphically to 0. Set $g_\varepsilon=((g_\varepsilon)_1,(g_\varepsilon)_2)$. Simple calculations show

(1) $((g_\varepsilon)_1)'(0)=rs_1+(1-r)s_2$;
(2) $((g_\varepsilon)_2)'(0)=0$;
(3) $((g_\varepsilon)_2)''(0)=2(rp_1+(1-r)p_2)$
$$+\frac{\partial^2((f_\varepsilon)_2)}{\partial s^2}(0)(rs_1^2+(1-r)s_2^2-(rs_1+(1-r)s_2)^2).$$

Put

$$t_\varepsilon=\frac{1}{2}\cdot\frac{\partial^2((f_\varepsilon)_2)}{\partial s^2}(0).$$

Then

$$h_\varepsilon(0)=(rs_1+(1-r)s_2,rp_1+(1-r)p_2+t_\varepsilon r(1-r)(s_1-s_2)^2).$$

By the maximum principle $\varrho(h_\varepsilon(\lambda))\le\max_{|\mu|=t}\varrho(h_\varepsilon(\mu))$. But for $\lambda\ne 0$ we have

$$\varrho(h_\varepsilon(\lambda))=\varrho(\varphi_{1/\lambda}(g_\varepsilon(\lambda)))=\frac{1}{|\lambda|}\,\varrho(g_\varepsilon(\lambda))\le\frac{1}{|\lambda|}.$$

Hence,

$$\varrho(h_\varepsilon(0))\le\frac{R}{1-\varepsilon}. \tag{1}$$

Write $t_\varepsilon=e^{i\theta}|t_\varepsilon|$. Take $r=1/2$, $(s_1,p_1)=\pi(\zeta,-1)$, and $(s_2,p_2)=\pi(\zeta,1)$, where $\zeta=e^{i(\theta+\pi)/2}$. Note that $t_\varepsilon=-\zeta^2|t_\varepsilon|$. We have $\varrho(1,-|t_\varepsilon|)=\varrho(\zeta,t_\varepsilon)\le 1/(1-\varepsilon)$. From this we get

$$\frac{1+\sqrt{1+4|t_\varepsilon|}}{2}=\varrho(1,-|t_\varepsilon|)\le\frac{1}{1-\varepsilon}.$$

So, $t_\varepsilon\to 0$ as $\varepsilon\to 0$. Letting $\varepsilon\to 0$ in (1) we get

$$\varrho(rs_1+(1-r)s_2,rp_1+(1-r)p_2)\le\max\{\varrho(s_1,p_1),\varrho(s_2,p_2)\}, \tag{2}$$

which contradicts the non-convexity of $\mathbb{G}_2$. ∎

References

[1] C. Costara, *The symmetrized bidisc as a counterexample to the converse of Lempert's theorem*, Bull. London Math. Soc., to appear (2003).

[2] A. Edigarian, *A remark on the Lempert theorem*, Univ. Iag. Acta Math. 32 (1995), 83–86.
[3] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, de Gruyter, 1993.
[4] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France 109 (1981), 427–474.

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