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Hyperconvexity of non-smooth pseudoconvex domains

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Abstract. We show that a bounded pseudoconvex domain $D \subset \mathbb{C}^n$ is hyperconvex if its boundary ∂D can be written locally as a complex continuous family of log-Lipschitz curves. We also prove that the graph of a holomorphic motion of a bounded regular domain $\Omega \subset \mathbb{C}$ is hyperconvex provided every component of $\partial\Omega$ contains at least two points. Furthermore, we show that hyperconvexity is a Hölder-homeomorphic invariant for planar domains.

1. Introduction. This paper is an attempt to study hyperconvexity of non-smooth pseudoconvex domains through the variational method. Recall that a domain $D \subset \mathbb{C}^n$ is *hyperconvex* if there is a continuous plurisubharmonic function $\rho < 0$ on D such that $D_c := \{z \in D : \rho(z) < c\}$ is relatively compact in D for each $c < 0$ (see [29]). It is well known that a planar domain is hyperconvex if and only if its boundary is regular for the Dirichlet problem. Thus every planar domain for which all connected components of the boundary are continua is hyperconvex. Hyperconvexity is also a basic concept in pluripotential theory (see e.g., [2], [3]).

The story of characterizing pseudoconvexity in terms of hyperconvexity starts from a fundamental paper of Diederich and Fornæss [10]. They proved that each bounded pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$ is hyperconvex with $-\rho \asymp \delta^\alpha$ for some $\alpha > 0$, where δ denotes the boundary distance. Their result was generalized to the case of C^1 and Lipschitz boundaries by Kerzman and Rosay [18] and Demailly [9] respectively, but with worse estimate $-\rho \asymp |\log\delta|^{-1}$. Only recently, Harrington [16] proved a Diederich–Fornæss type result for pseudoconvex domains with Lipschitz boundaries, basing on an ingenious quantitative analysis of Oka's lemma. Modifying Demailly's technique slightly, Avelin–Hed–Persson [1] showed that every pseudoconvex domain with log-Lipschitz boundary is hyperconvex.

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We also refer to [13], [22] and [11] for related results on complex manifolds.

Throughout this paper (*unless otherwise stated*), *we shall assume the boundedness of the domain considered.*

A pseudoconvex domain in $\mathbb{C}^n$ is called a *continuous pseudoconvex domain* if its boundary can be written locally as the graph of a continuous function. A longstanding problem in several complex variables is the following:

Is every continuous pseudoconvex domain in $\mathbb{C}^n$ *hyperconvex?*

We shall give a partial answer as follows:

THEOREM 1.1. *A pseudoconvex domain in* $\mathbb{C}^n$ *is hyperconvex if its boundary can be written locally as a complex continuous family of log-Lipschitz curves.*

A pseudoconvex domain in $\mathbb{C}^n$ is called a *radial log-Lipschitz pseudoconvex domain* if its boundary can be written locally as a complex continuous family of log-Lipschitz curves. Roughly speaking, a radial log-Lipschitz pseudoconvex domain is a continuous pseudoconvex domain with log-Lipschitz boundary along the "radial" direction (see Section 2 for more details). Since every domain with log-Lipschitz boundary is radial log-Lipschitz, Theorem 1.1 can be seen as an improvement of the result of Avelin–Hed–Persson [1].

A trivial example supporting the above theorem is the *Hartogs domain* defined by

$$D_\phi := \{(z,w) \in D \times \mathbb{C} : |w| < e^{-\phi(z)}\},$$

where D is a smooth hyperconvex domain in $\mathbb{C}^n$ and ϕ is a continuous plurisubharmonic function on D. Less obvious examples are graphs of holomorphic motions of planar domains with continuous boundaries.

DEFINITION (cf. [20], see also [30], [27]). Let $\triangle$ (resp. $\triangle_r$) be the disc centered at the origin of $\mathbb{C}$ with radius 1 (resp. r) and Ω be a planar domain. A map

$$F : \triangle \times \Omega \to \triangle \times \mathbb{C}, \qquad (\lambda, z) \mapsto (\lambda, f(\lambda, z)), \tag{1.1}$$

is called a *holomorphic motion* of Ω if

(i) $f(0,z) = z$ for all $z \in \mathbb{C}$,
(ii) for every $z \in \Omega$, $\lambda \mapsto f(\lambda, z)$ is holomorphic on $\triangle$,
(iii) for every $\lambda \in \triangle$, $z \mapsto \zeta = f(\lambda, z)$ is injective on Ω.

Indeed, F extends to a holomorphic motion of the whole plane and $z \mapsto \zeta = f(\lambda, z)$ is quasiconformal on $\mathbb{C}$ for every $\lambda \in \triangle$ (cf. [20], [27]).

We call the image of F the *graph* of the holomorphic motion of Ω under F. A *local graph* of the holomorphic motion of Ω under F is defined to be

$F(\triangle_r \times \Omega)$, $0 < r < 1$. In [6] and [7], Chen–Zhang studied complex analytic properties of local graphs of a holomorphic motion of a planar domain. In particular, they asked the following question:

Is the graph of a holomorphic motion of a regular planar domain hyperconvex?

By use of Vâjâitu's theorem (see [31]), the graph of a holomorphic motion is hyperconvex if every local graph is hyperconvex. Thus we can give a partial answer to Chen–Zhang's question as follows:

THEOREM 1.2. *The graph of a holomorphic motion of a planar domain Ω is hyperconvex if every component of $\partial\Omega$ contains at least two points* (*e.g.*, *Ω is simply-connected*).

Our final result is on stability of hyperconvexity for planar domains under Hörder continuous maps. We say that two planar domains D_1 and D_2 are *Hölder-homeomorphic* if there exist two constants $0 < \alpha \le 1$, $\beta > 1$ and a homeomorphism f from a neighborhood U_1 of $\overline{D}_1$ to a neighborhood U_2 of $\overline{D}_2$ such that $f(D_1) = D_2$ and

$$\frac{1}{\beta}|z-w|^{1/\alpha} \le |f(z)-f(w)| \le \beta|z-w|^{\alpha}, \qquad \forall z, w \in U_1. \tag{1.2}$$

THEOREM 1.3. *Hyperconvexity of planar domains is a Hölder-homeomorphic invariant.*

In particular, every quasiconformal deformation of a hyperconvex planar domain is still hyperconvex.

If two planar domains D_1 and D_2 are Hölder-homeomorphic and D_1 is an L^2_h-domain of holomorphy, then so is D_2 (see [8, Theorem 9.9] or [24, Theorem 2]). Notice that the graph of a holomorphic motion of a planar domain is a Levi flat pseudoconvex domain (maybe unbounded). Making use of Theorem 1 of [24] and the Ohsawa–Takegoshi L^2 extension theorem (see [23]), we also infer that the graph of a holomorphic motion of an L^2_h-planar domain of holomorphy is an L^2_h-domain of holomorphy as long as the graph is bounded.

For other related results concerning variation of the Green's function and some other functions, one may consult [28], [15] and [19]. Since hyperconvexity is closely related to the behavior of the Bergman kernel and the Bergman metric (see [12], [14], [21], [4], [17], [5], [25], [34]), one may ask whether the previous theorem is correct or not for these two notions. Unfortunately the answer is negative. In fact, if we put $f(\alpha, 0) = 0$ and

$$f(\alpha, z) = e^{2\alpha \log|z|}/\bar{z}, \qquad \forall z \in \mathbb{C} \setminus \{0\},$$

for every complex number α such that $\mathrm{Re}(\alpha) > 1/2$, then we get a holomorphic motion of the whole plane

$$(\alpha, z) \mapsto (\alpha, f(\alpha, z)), \qquad (\alpha, z) \in \{\mathrm{Re}(\alpha) > 1/2\} \times \mathbb{C}.$$

But $z \mapsto f(\alpha, z)$ may map a Bergman complete (resp. exhaustive) planar domain to a Bergman non-complete (resp. non-exhaustive) planar domain (see [32]).

2. Proof of Theorem 1.1. Let D be a domain in $\mathbb{C}^n$. Denote by $\mathrm{PSH}(D)$ the space of plurisubharmonic functions on D. The key point of the proof is to replace Oka's lemma in previous works by the following variation:

D is pseudoconvex if and only if $-\log \delta_D(z, X) \in \mathrm{PSH}(D \times \mathbb{C}^n)$, where

$$\delta_D(z, X) := \delta_{D \cap (z + \mathbb{C}X)}(z) = \sup\{r > 0 : z + aX \subset D,\ \forall a \in \mathbb{C},\ |a| < r\}.$$

The proof relies heavily on a quantitative analysis of the boundary. Before we do it, we shall give a precise description of radial log-Lipschitz domains.

That the boundary of a domain D in $\mathbb{C}^n$ can be written locally as the graph of a continuous function means that for any $p \in \partial D$, there exists a ball $B(p, 2r_p)$ at p with radius $2r_p$ and a complex affine transformation $\Phi_p : w \mapsto z = A_p(w - p)$, $A_p \in U(n)$, such that

$$\Phi_p(D \cap B(p, 2r_p)) = \{z = (z', z_n) \in B(0, 2r_p) : \mathrm{Im}\, z_n > \varphi_p(z', \mathrm{Re}\, z_n)\},$$

where φ_p is continuous on $B(0, 3r_p) \cap \{\mathrm{Im}\, z_n = 0\}$ and $\varphi_p(0) = 0$. Generally speaking $D \cap B(p, 2r_p)$ is not connected. We claim that $\Phi_p(\partial D \cap B(p, 2r_p)) = \Phi_p(\partial D) \cap B(0, 2r_p)$ can be written as

$$\{z \in B(0, 2r_p) : \mathrm{Im}\, z_n = \varphi_p(z', \mathrm{Re}\, z_n)\}. \tag{2.1}$$

In fact, if $z \in \{z \in B(0, 2r_p) : \mathrm{Im}\, z_n = \varphi_p(z', \mathrm{Re}\, z_n)\}$, then $z + (0', \sqrt{-1}\,\varepsilon) \in \Phi_p(D \cap B(p, 2r_p))$ for sufficiently small $\varepsilon > 0$. Thus

$$\{z \in B(0, 2r_p) : \mathrm{Im}\, z_n = \varphi_p(z', \mathrm{Re}\, z_n)\} \subset \Phi_p(\partial D \cap B(p, 2r_p)). \tag{2.2}$$

If $z \in \Phi_p(\partial D \cap B(p, 2r_p))$, then there exists $z^{(j)} \in \Phi_p(D \cap B(p, 2r_p))$ such that $|z^{(j)} - z| \to 0$. Since φ_p is continuous, we have

$$z \in \{z \in B(0, 2r_p) : \mathrm{Im}\, z_n \geq \varphi_p(z', \mathrm{Re}\, z_n)\}.$$

Thus

$$\{z \in B(0, 2r_p) : \mathrm{Im}\, z_n = \varphi_p(z', \mathrm{Re}\, z_n)\} \supset \Phi_p(\partial D \cap B(p, 2r_p)) \tag{2.3}$$

and the proof of our claim is complete. Since ∂D is compact, we may choose a finite set of points $\{p_j\} \subset \partial D$ such that $\bigcup_j B(p_j, r_{p_j}) \supset \partial D$. Thus our definition is equivalent to Definition 5 in [1]. Viewing

$$\{\Phi_p(D \cap B(p, 2r_p)) \cap (\{z'\} \times \mathbb{C})\}_{z'}$$

as a complex continuous family of planar domains with continuous boundaries, we see that ∂D can be written locally as a complex continuous family of continuous curves. Suppose furthermore that there exist constants $C, N > 0$ such that

$$|\varphi_p(z', a) - \varphi_p(z', b)| \le C|a-b| \cdot \big|\log|a-b|\big|^N \tag{2.4}$$

for all $(z', a), (z', b) \in B(0, 2r_p) \cap \{\operatorname{Im} z_n = 0\}$. Then we say that ∂D can be written locally as a complex continuous family of log-Lipschitz curves.

By [18], we know that hyperconvexity is a local property (see [11] for counterexamples in $\mathbb{P}^n$), that is, we need only show that every point $p \in \partial D$ has a neighborhood U_p such that every component of $D \cap U_p$ is hyperconvex.

In general we say that an open set D is hyperconvex if every component of D is hyperconvex.

We may assume that $p = 0 \in \mathbb{C}^n$. Denote by B'_r the ball centered at $0' \in \mathbb{C}^{n-1}$ with radius r. Suppose Φ_0 is the identity mapping and $0 < r_0 < e^{-N}$. Since $x\big(\log\frac{1}{x}\big)^N$ is a strictly increasing function on $\big(0, \frac{1}{4}e^{-N}\big)$ and $x\big(\log\frac{1}{x}\big)^N > x$ on $\big(0, \frac{1}{4}e^{-N}\big)$, for every $0 < \varepsilon \le r_0/4$ there exists a unique $0 < g(\varepsilon) \le \varepsilon$ such that

$$(C+1)g(\varepsilon)\bigg(\log\frac{1}{g(\varepsilon)}\bigg)^N = \varepsilon. \tag{2.5}$$

Clearly $\lim_{\varepsilon\to 0+} g(\varepsilon) = 0$. Put $g(0) = 0$; then $g(\varepsilon)$ is a strictly increasing function on $[0, r_0/4)$. Choose a sufficiently small $0 < r < r_0/4$ such that

$$|\varphi_0(z', 0)| < r_0/4, \qquad \forall z' \in B'_r. \tag{2.6}$$

For every $0 \le \varepsilon \le r_0/4$, put

$$D_\varepsilon = \{z \in B'_r \times \triangle_{r_0/2+g(\varepsilon)} : \operatorname{Im} z_n + \varepsilon > \varphi_0(z', \operatorname{Re} z_n)\}, \tag{2.7}$$

$$D_{\varepsilon,z'} = D_\varepsilon \cap (\{z'\} \times \mathbb{C}), \qquad z' \in B'_r. \tag{2.8}$$

It suffices to show that D_0 is hyperconvex.

Lemma 2.1. *D_0 is hyperconvex if there exists a continuous plurisubharmonic function $\psi : D_0 \to (-\infty, 0)$ such that*

$$\psi(z) \to 0 \quad \text{as } \delta_{D_0,z'}(z_n) \to 0. \tag{2.9}$$

Proof. We claim that (2.9) implies

$$\lim_{D_0 \ni z \to \zeta} \psi(z) = 0, \qquad \forall \zeta \in \partial D_0 \cap \partial D. \tag{2.10}$$

To see this, let $\rho(z) = \varphi_0(z', \operatorname{Re} z_n) - \operatorname{Im} z_n$ and fix $\zeta \in \partial D_0 \cap \partial D$. Since ρ is continuous and $\rho(\zeta) = 0$, we have $\rho(z) \to 0$ $(z \to \zeta)$. Thus $\delta_{D_0,z'}(z_n) \le |\rho(z)| \to 0$ $(z \to \zeta)$, from which (2.10) immediately follows.

Now $\max\{\psi(z), |z'| - r, |z_n| - r_0/2\}$ is a bounded plurisubharmonic exhaustion function on D_0, thus D_0 is hyperconvex. ∎

Take $X_0 = (0', \sqrt{-1})$; we claim that

$$D_\varepsilon = (D - \varepsilon X_0) \cap (B'_r \times \triangle_{r_0/2+g(\varepsilon)}). \tag{2.11}$$

In fact, if $z \in D_\varepsilon$, then $w := z + \varepsilon X_0$ satisfies $\operatorname{Im} w_n > \varphi_0(w', \operatorname{Re} w_n)$ and $|w| < 2r_0$. Thus $w \in D$ and $D_\varepsilon \subset (D - \varepsilon X_0) \cap (B'_r \times \triangle_{r_0/2+g(\varepsilon)})$. If $z \in (D - \varepsilon X_0) \cap (B'_r \times \triangle_{r_0/2+g(\varepsilon)})$, then $z + \varepsilon X_0 \in D$ and $|z + \varepsilon X_0| < 2r_0$. Thus $\operatorname{Im} z_n + \varepsilon > \varphi_0(z', \operatorname{Re} z_n)$ and $z \in D_\varepsilon$.

By (2.11), D_ε is pseudoconvex so that

$$-\log \delta_{D_\varepsilon}(z, X_0) = -\log \delta_{D_{\varepsilon,z'}}(z_n) \in \operatorname{PSH}(D_\varepsilon). \tag{2.12}$$

Furthermore, we have

Lemma 2.2.

$$\delta_{D_0}(z, X_0) + g(\varepsilon) \le \delta_{D_\varepsilon}(z, X_0) \le \delta_{D_0}(z, X_0) + \varepsilon, \qquad \forall z \in D_0. \tag{2.13}$$

Proof. It suffices to show that

$$g(\varepsilon) \le \delta_{D_\varepsilon}(z, X_0) \le \varepsilon, \qquad \forall z \in \bigcup_{z' \in B'_r} \partial D_{0,z'}. \tag{2.14}$$

Put

$$C^1_{\varepsilon,z'} = \{(z', z_n) : |z_n| < r_0/2 + g(\varepsilon),\ \operatorname{Im} z_n + \varepsilon = \varphi_0(z', \operatorname{Re} z_n)\} \cap \partial D_{\varepsilon,z'},$$
$$C^2_{\varepsilon,z'} = \{(z', z_n) : |z_n| = r_0/2 + g(\varepsilon),\ \operatorname{Im} z_n + \varepsilon \ge \varphi_0(z', \operatorname{Re} z_n)\} \cap \partial D_{\varepsilon,z'},$$

for every $0 \le \varepsilon \le r_0/4$ and $z' \in B'_r$. Thus $\partial D_{\varepsilon,z'} = C^1_{\varepsilon,z'} \cup C^2_{\varepsilon,z'}$. By (2.6),

$$(0', (\varphi_0(z', 0) - \varepsilon)\sqrt{-1}) \in C^1_{\varepsilon,z'},$$

thus $C^1_{\varepsilon,z'} \neq \emptyset$. For every $z \in C^1_{0,z'}$, we have $z - \varepsilon X_0 \notin D_\varepsilon$. Thus there exists $0 \le t \le \varepsilon$ such that $z - tX_0 \in \partial D_\varepsilon$. Hence

$$\delta_{D_\varepsilon}(z, X_0) \le \varepsilon.$$

Next we prove the other inequality. Take $s \in \mathbb{C}$ such that $(z', z_n + s) \in C^1_{\varepsilon,z'}$ and $|s| = d(z, C^1_{\varepsilon,z'})$. Since $z \in C^1_{0,z'}$ and $\varphi_0(z', \operatorname{Re} z_n) = \operatorname{Im} z_n$, we have

$$\operatorname{Im}(z_n + s) + \varepsilon = \varphi_0(z', \operatorname{Re}(z_n + s)) - \varphi_0(z', \operatorname{Re} z_n) + \operatorname{Im} z_n.$$

By (2.4), we have

$$\varepsilon \le |s| + C|s| \cdot \big|\log |s|\big|^N,$$

thus $|s| \ge g(\varepsilon)$ by virtue of (2.5). Clearly, $d(z, C^2_{\varepsilon,z'}) \ge g(\varepsilon)$. Since

$$\delta_{D_\varepsilon}(z, X_0) = \delta_{D_{\varepsilon,z'}}(z_n) = \min\{d(z, C^1_{\varepsilon,z'}), d(z, C^2_{\varepsilon,z'})\},$$

we get (2.14) for every $z \in C^1_{0,z'}$.

For every $z \in C^2_{0,z'}$, clearly

$$\delta_{D_\varepsilon}(z, X_0) \le g(\varepsilon) \le \varepsilon.$$

Take $s \in \mathbb{C}$ such that $(z', z_n + s) \in C^1_{\varepsilon,z'}$ and $|s| = d(z, C^1_{\varepsilon,z'})$. Since $z \in C^2_{0,z'}$ and $\varphi_0(z', \operatorname{Re} z_n) \le \operatorname{Im} z_n$, we have

$$\operatorname{Im}(z_n + s) + \varepsilon \le \varphi_0(z', \operatorname{Re}(z_n + s)) - \varphi_0(z', \operatorname{Re} z_n) + \operatorname{Im} z_n,$$

and we get (2.14) for every $z \in C^2_{0,z'}$ exactly as before. ∎

Now consider the following family of functions:

$$\psi_\varepsilon = \frac{\log \frac{g(\varepsilon)}{\delta_{D_\varepsilon}(z, X_0)} - 1}{\log \frac{1}{g(\varepsilon)}}, \quad 0 < \varepsilon \le r_0/4. \tag{2.15}$$

By (2.12), we have $\psi_\varepsilon \in \mathrm{PSH}(D_\varepsilon)$. By the previous lemma,

$$\frac{\log \frac{g(\varepsilon)}{\delta_{D_0}(z, X_0) + \varepsilon} - 1}{\log \frac{1}{g(\varepsilon)}} \le \psi_\varepsilon \le \frac{\log \frac{g(\varepsilon)}{\delta_{D_0}(z, X_0) + g(\varepsilon)} - 1}{\log \frac{1}{g(\varepsilon)}}, \quad \forall z \in D_0.$$

Put

$$\psi = \sup_{0 < \varepsilon \le r_0/4} \psi_\varepsilon.$$

Since

$$\frac{\log \frac{g(\varepsilon)}{\delta_{D_0}(z, X_0) + g(\varepsilon)} - 1}{\log \frac{1}{g(\varepsilon)}} \le - \frac{1 + \log\bigl(1 + \frac{\delta_{D_0}(z, X_0)}{g(\varepsilon)}\bigr)}{\log \frac{1}{\delta_{D_0}(z, X_0)} + \log\bigl(1 + \frac{\delta_{D_0}(z, X_0)}{g(\varepsilon)}\bigr)},$$

we get

$$\psi(z) \le \frac{-1}{\log \frac{1}{\delta_{D_0}(z, X_0)}}$$

for all z with $\delta_{D_0}(z, X_0) \le e^{-1}$. If $\delta_{D_0}(z, X_0) \le g(r_0/4)$, we may take $0 < \varepsilon \le r_0/4$ satisfying $g(\varepsilon) = \delta_{D_0}(z, X_0)$ so that

$$\psi(z) \ge - \frac{1 + \log\bigl(1 + (C+1)\bigl(\log \frac{1}{\delta_{D_0}(z, X_0)}\bigr)^N\bigr)}{\log \frac{1}{\delta_{D_0}(z, X_0)}}$$

for all z with $\delta_{D_0}(z, X_0) \le g(r_0/4)$. Thus D_0 is hyperconvex by virtue of Lemma 2.1.

3. Proof of Theorem 1.2. It suffices to prove that every local graph is hyperconvex.

Fix $0 < r < 1$. Since every point in $F((\partial \triangle_r) \times \overline{\Omega})$ admits a natural plurisubharmonic barrier $|\lambda|^2 - r^2$, it suffices to show that every point in $F(\triangle_r \times \partial \Omega)$ admits a plurisubharmonic barrier.

Let z_1 be a boundary point of Ω, and E be the component of $\partial \Omega$ containing z_1. Since E contains at least two points, we can choose $z_2 \in E \setminus \{z_1\}$. Since the connected component Γ of $\mathbb{P} \setminus E$ containing Ω is simply connected, $F(\triangle \times \Gamma)$ is simply connected. Thus we may take a single-valued branch of

$w = \log \frac{\zeta - f(\lambda, z_1)}{\zeta - f(\lambda, z_2)}$ so that it is a well defined zero-free holomorphic function on $F(\triangle \times \Omega)$. In fact, if we put

$$\frac{\zeta - f(\lambda, z_1)}{\zeta - f(\lambda, z_2)} = \left| \frac{\zeta - f(\lambda, z_1)}{\zeta - f(\lambda, z_2)} \right| e^{\sqrt{-1}\,\theta} \tag{3.1}$$

with $\theta = \operatorname{Im} w$ (thus θ is continuous on $F(\triangle \times \Omega)$), then for every $k \in \mathbb{Z}$,

$$\phi_k(\lambda, \zeta) := \log \left| \frac{\zeta - f(\lambda, z_1)}{\zeta - f(\lambda, z_2)} \right| + \sqrt{-1}\,(\theta + 2k\pi) \tag{3.2}$$

is a single-valued branch of $\log \frac{\zeta - f(\lambda, z_1)}{\zeta - f(\lambda, z_2)}$. Put

$$\varphi(\lambda, \zeta) = \operatorname{Re}(1/\phi_0(\lambda, \zeta)). \tag{3.3}$$

By (3.2), we have

$$\varphi(\lambda, \zeta) = \frac{\log \left| \frac{\zeta - f(\lambda, z_1)}{\zeta - f(\lambda, z_2)} \right|}{\left(\log \left| \frac{\zeta - f(\lambda, z_1)}{\zeta - f(\lambda, z_2)} \right|\right)^2 + \theta^2}. \tag{3.4}$$

By [27], $z \mapsto \zeta = f(\lambda, z),\ z \in \mathbb{C}$, is a quasiconformal self-homeomorphism with finite dilatation $\big($no more than $\frac{1+|\lambda|}{1-|\lambda|}\big)$ for every $\lambda \in \triangle_r$ (see also λ-Lemma in [20] for a simpler proof that only relies on Schwarz's Lemma). Thus

$$\left(\frac{1-|\lambda|}{1+|\lambda|}\right)^2 \left| \frac{z - z_1}{z - z_2} \right| \le \left| \frac{f(\lambda, z) - f(\lambda, z_1)}{f(\lambda, z) - f(\lambda, z_2)} \right| \le \left(\frac{1+|\lambda|}{1-|\lambda|}\right)^2 \left| \frac{z - z_1}{z - z_2} \right|. \tag{3.5}$$

Clearly φ is negative near the points $F(\lambda, z_1)$. What is more, we have

$$\lim_{F(\triangle\times\Omega)\ni(\lambda,\zeta)\to F(\lambda_0,z_1)=(\lambda_0,f(\lambda_0,z_1))} \varphi(\lambda, \zeta) = 0, \quad \forall |\lambda_0| < 1. \tag{3.6}$$

Thus there exist $0 < \delta_1 < \delta_2 < 1$ and $\varepsilon > 0$ such that

$$\varphi(\lambda, \zeta) \ge \frac{\log \left| \frac{\zeta - f(\lambda, z_1)}{\zeta - f(\lambda, z_2)} \right|}{\left(\log \left| \frac{\zeta - f(\lambda, z_1)}{\zeta - f(\lambda, z_2)} \right|\right)^2} > -\varepsilon$$

on $F(\triangle_r \times \{z \in \Omega : |z - z_1| \le \delta_1\})$, and

$$\varphi(\lambda, \zeta) \le \frac{\log \left| \frac{\zeta - f(\lambda, z_1)}{\zeta - f(\lambda, z_2)} \right|}{\left(\log \left| \frac{\zeta - f(\lambda, z_1)}{\zeta - f(\lambda, z_2)} \right|\right)^2 + C} < -3\varepsilon$$

on $F(\triangle_r \times \{z \in \Omega : |z - z_1| = \delta_2\})$, where

$$C = \sup_{(\lambda,\zeta)\in F(\triangle_r\times\{z\in\Omega :\, |z-z_1|=\delta_2\})} \theta^2 < \infty,$$

since θ is continuous. Put

$$\psi(\lambda, \zeta) = \max\{-2\varepsilon, \varphi(\lambda, \zeta)\}$$

for $(\lambda, \zeta) \in F(\triangle_r \times \{z \in \Omega : |z - z_1| < \delta_2\})$, and

$$\psi(\lambda, \zeta) = -2\varepsilon$$

for $(\lambda,\zeta) \in F(\triangle_r \times \{z \in \Omega : |z - z_1| \geq \delta_2\})$. Then $\psi \in \mathrm{PSH}(F(\triangle_r \times \Omega))$ is a plurisubharmonic barrier at every point $(\lambda,\zeta) \in F(\triangle_r \times \{z_1\})$. The proof is complete.

4. Proof of Theorem 1.3. Let F be an F_σ subset of $\mathbb{C}$ (i.e., a countable union of closed sets). Put

$$A_n = \{z \in \mathbb{C} : \gamma^n < |z| \leq \gamma^{n-1}\}, \qquad n \in \mathbb{Z}_+, \tag{4.1}$$

where $0 < \gamma < 1$ is a constant. Denote by $c(A)$ the logarithmic capacity of a set $A \subset \mathbb{C}$. A basic result in potential theory is Wiener's criterion for thinness (see Wiener [33], also Ransford [26, p. 146]):

F is thin at 0 *if and only if* $\sum_{n=1}^{\infty} \frac{n}{\log(2/c(A_n \cap F))} < \infty$.

To prove Theorem 1.3, we need a modification of the "if" part of Wiener's criterion:

Lemma 4.1. *F is thin at* 0 *if there exist a family* $\{B_n\}_{n\in\mathbb{Z}_+}$ *of* F_σ *subsets of* $\overline{\Delta}$ *and* $0 < \varepsilon, \gamma < 1$ *such that*

$$\bigcup_{n=1}^{\infty} B_n \supset \{0 < |z| < \varepsilon\}, \quad B_n \cap \{|z| < \gamma^n\} = \emptyset, \quad \sum_{n=1}^{\infty} \frac{n}{\log(2/c(B_n \cap F))} < \infty.$$

The proof of this lemma is essentially the same as that of Wiener's criterion. The readers may find the proof in [26, pp. 147–149].

Proof of Theorem 1.3. Let $f : D_1 \to D_2$ be a Hölder homeomorphism. It suffices to show that D_2 is non-hyperconvex if D_1 is non-hyperconvex. We may assume that the complement of D_1 is thin at $0 \in \partial D_1$ and $f(0) = 0$. It suffices to show that the complement of D_2 is thin at $0 \in \partial D_2$. We may also assume that U_1 contains $\overline{\triangle}$ and $f(\overline{\triangle}) \subset \overline{\triangle}$. Put $B_n = f(A_n)$. Since $\bigcup_{n=1}^{\infty} A_n = \triangle \setminus \{0\}$, we have $\bigcup_{n=1}^{\infty} B_n \supset \{0 < |z| < \varepsilon\}$. Furthermore,

$$|z| \geq \frac{1}{\beta}(\gamma^n)^{1/\alpha} \geq (\gamma^N)^n, \qquad \forall z \in B_n,$$

where $N > 1/\alpha$ is sufficiently large. In view of Wiener's criterion, we have

$$\sum_{n=1}^{\infty} \frac{n}{\log(2/c(B_n \setminus D_2))} = \sum_{n=1}^{\infty} \frac{n}{\log(2/c(f(A_n \setminus D_1)))}$$
$$\leq \sum_{n=1}^{\infty} \frac{n}{\log(2/(\beta c(A_n \setminus D_1)^{\alpha}))} < \infty.$$

Thus the complement of D_2 is thin at 0 by virtue of Lemma 4.1. The proof is complete.

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