ANNALES
POLONICI MATHEMATICI
111.1 (2014)

On the behavior of algebraic polynomials in regions with piecewise smooth boundary without cusps

by F. G. Abdullayev and C. D. Gün (Mersin)

Abstract. We continue studying the estimation of Bernstein–Walsh type for algebraic polynomials in regions with piecewise smooth boundary.

1. Introduction and main results. Let $G \subset \mathbb{C}$ be a finite region, with $0 \in G$, bounded by a Jordan curve $L := \partial G$, $\Delta(t,R) := \{w : |w-t| > R\}$, $\Delta := \Delta(0,1)$, $\Omega := \operatorname{ext} \overline{G} = \overline{\mathbb{C}} \setminus \overline{G}$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $w = \Phi(z)$ be the univalent conformal mapping of Ω onto Δ normalized by $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$, and $\Psi := \Phi^{-1}$.

Let $\wp_n$, $n \in \mathbb{N}$, denote the class of all algebraic polynomials $P_n(z)$ with $\deg P_n \le n$. Let $h(z)$ be a weight function defined in G. Denote by $A(G)$ the class of functions f which are analytic in G.

For any $p > 0$ we define

$$A_p(h,G) := \Big\{ f \in A(G) : \|f\|^p_{A_p(h,G)} := \iint\limits_G h(z)|f(z)|^p \, d\sigma_z < \infty \Big\},$$

where σ_z is two dimensional Lebesgue measure; we write $A_p(1,G) \equiv A_p(G)$.

When L is rectifiable, for any $p > 0$, let

$$\mathcal{L}_p(L) := \Big\{ f : \|f\|^p_{\mathcal{L}_p(h,L)} := \int\limits_L h(z)|f(z)|^p \, |dz| < \infty \Big\},$$

and $\mathcal{L}_p(1,L) \equiv \mathcal{L}_p(L)$.

For $R > 1$, set $L_R := \{z : |\Phi(z)| = R\}$, $G_R := \operatorname{int} L_R$, $\Omega_R := \operatorname{ext} L_R$. The well known Bernstein–Walsh Lemma [13] says that

$$\|P_n\|_{C(\overline{G}_R)} \le R^n \|P_n\|_{C(\overline{G})}. \tag{1.1}$$

Hence, setting $R = 1 + 1/n$, we see that the C-norms of a polynomial

2010 *Mathematics Subject Classification*: Primary 30A10, 30C10; Secondary 41A17.
Key words and phrases: algebraic polynomials, conformal mapping, smooth curve, quasiconformal curve.

DOI: 10.4064/ap111-1-4

$P_n(z)$ in $\overline{G}_R$ and $\overline{G}$ are identical, i.e. the norm $\|P_n\|_{C(\overline{G})}$ increases up to multiplication by a constant in $\overline{G}_R$.

A similar estimate to (1.1) in the space $\mathcal{L}_p(L)$ was obtained in [9]:

$$(1.2)\qquad \|P_n\|_{\mathcal{L}_p(L_R)} \le R^{n+1/p}\|P_n\|_{\mathcal{L}_p(L)}, \qquad p > 0.$$

To give a similar estimate for the $A_p(G)$-norm, we first give some definitions and notations.

DEFINITION 1.1 ([10, p. 97], [11]). The Jordan arc (or curve) L is called *K-quasiconformal* ($K \ge 1$) if there is a K-quasiconformal mapping f of a region $D \supset L$ such that $f(L)$ is a line segment (or circle).

Let $F(L)$ denote the set of all sense preserving plane homeomorphisms f of $D \supset L$ such that $f(L)$ is a line segment (or circle), and let

$$K_L := \inf\{K(f) : f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of f. Then L is a quasiconformal curve if $K_L < \infty$, and L is a K-quasiconformal curve if $K_L \le K$.

We note that the region D in Definition 1.1 can be $\mathbb{C}$ or a proper subset of $\mathbb{C}$. The case $D \equiv \mathbb{C}$ gives the global definition of a K-quasiconformal arc or curve. If $D \supset L$ is a neighborhood of the curve L, Definition 1.1 is called local. This local definition has an advantage for determining the coefficients of quasiconformality for some simple arcs or curves.

Let $z = z(s)$, $s \in [0, \operatorname{mes} L]$, denote the natural representation of L.

DEFINITION 1.2. We say that $L \in C_\theta$ if L has a continuous tangent $\theta(z) := \theta(z(s))$ at every point $z(s)$. We write $G \in C_\theta$ if $\partial G \in C_\theta$.

According to [11], we have the following facts:

COROLLARY 1.3. *If $L \in C_\theta$, then $L = \partial G$ is $(1+\varepsilon)$-quasiconformal for all $\varepsilon > 0$.*

COROLLARY 1.4. *If L is an analytic curve or arc, then L is 1-quasiconformal.*

It is known that there exist quasiconformal curves which are not rectifiable [10, p. 104].

Let $\{z_j\}_{j=1}^m$ be a fixed system of distinct points which are ordered in the positive direction on the curve L. Consider a *generalized Jacobi weight function* $h(z)$ defined as follows:

$$(1.3)\qquad h(z) := \prod_{j=1}^{m} |z - z_j|^{\gamma_j}, \qquad z \in G_R,$$

where $\gamma_j > -2$ for every $j = 1, \ldots, m$.

The Bernstein–Walsh type estimate for a region G with quasiconformal boundary and weight function $h(z)$ as in (1.3) in the space $A_p(h,G),\ p>0$, was given in [2]. In particular, for $h(z)\equiv 1$,

$$\|P_n\|_{A_p(G_R)} \le c_2 {R^*}^{n+1/p} \|P_n\|_{A_p(G)}, \quad p>0, \tag{1.4}$$

where $R^* := 1 + c_3(R-1)$. Therefore, if we choose $R = 1 + c_1/n$, then (1.4) shows that the A_p-norms of the polynomial $P_n(z)$ in G_R and in G are identical.

N. Stylianopoulos [12] replaced the norm $\|P_n\|_{C(\overline{G})}$ with $\|P_n\|_{A_2(G)}$ on the right-hand side of (1.1) and found a new version of the Bernstein–Walsh Lemma:

Lemma A ([12]). *Assume that L is quasiconformal and rectifiable. Then there exists a constant $c = c(L) > 0$ depending only on L such that*

$$|P_n(z)| \le c \frac{\sqrt{n}}{d(z,L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega, \tag{1.5}$$

for every $P_n \in \wp_n$, where $d(z,L) := \inf\{|\zeta - z| : \zeta \in L\}$.

On the other hand, using the mean value theorem, for an arbitrary Jordan region G, $P_n \in \wp_n$, and any $p>0$, we find

$$|P_n(z)| \le \left(\frac{1}{\sqrt{\pi}\, d(z,L)} \right)^{2/p} \|P_n\|_{A_p(G)}, \quad z \in G. \tag{1.6}$$

Hence, according to Corollary 1.3, from (1.5) and (1.6), we obtain an estimate of $|P_n(z)|$ for any $P_n \in \wp_n$ and $G \in C_\theta$, in the whole complex plane:

$$|P_n(z)| \le \frac{c_3}{d(z,L)} \|P_n\|_{A_2(G)} \begin{cases} 1, & z \in G, \\ \sqrt{n}\, |\Phi(z)|^{n+1}, & z \in \Omega. \end{cases} \tag{1.7}$$

To estimate $|P_n(z)|$ on the closed domain $\overline{G}$, we give the following theorem:

Theorem 1.5. *Let $p > 1$, let G be a region bounded by a K-quasiconformal curve $L := \partial G$, and let $h(z)$ be a weight function as in* (1.3). *Then, for any $P_n \in \wp_n,\ n \in \mathbb{N}$, and every $z_j \in L,\ j = 1, \dots, m$,*

$$|P_n(z_j)| \le c_4 n^{(2+\gamma_j)s/p} \|P_n\|_{A_p(h,G)}, \tag{1.8}$$

and consequently

$$\|P_n\|_{C(\overline{G})} \le c_4 n^{(2+\widehat{\gamma})s/p} \|P_n\|_{A_p(h,G)}, \tag{1.9}$$

where $c_4 = c_4(G,\ p) > 0$, $s := \min\{2, K^2\}$, $\widehat{\gamma} := \max\{0, \gamma_j : j = 1, \dots, m\}$.

Therefore, if $G \in C_\theta$, then for any $P_n \in \wp_n,\ n \in \mathbb{N}$, and $p > 0$, from Corollary 1.3, we have

$$|P_n(z)| \le c_4 n^{2/p+\varepsilon} \|P_n\|_{A_p(G)}, \quad \forall z \in \overline{G}, \tag{1.10}$$

for an arbitrarily small $\varepsilon > 0$, and consequently, from (1.5) and (1.10),

$$(1.11) \qquad |P_n(z)| \leq c_5 \|P_n\|_{A_2(G)} \begin{cases} n^{1+\varepsilon}, & z \in \overline{G}, \\ \dfrac{\sqrt{n}}{d(z,L)} \, |\Phi(z)|^{n+1}, & z \in \Omega. \end{cases}$$

(1.11) gives an estimate of $|P_n(z)|$ in the whole complex plane in the case of $h(z) \equiv 1,\ p = 2$ for $G \in C_\theta$.

In this work, we study similar problems for regions with piecewise smooth boundary (without cusps) and a generalized Jacobi weight function $h(z)$, as defined in (1.3), in $A_p(h,G)$, $p > 1$.

Let us give the corresponding definitions and some notations that will be used later.

Definition 1.6. We say that a Jordan region G is in $C_\theta(\lambda_1, \ldots, \lambda_m)$, $0 < \lambda_j < 2,\ j = 1, \ldots, m$, if $L = \partial G$ is the union of finitely many smooth arcs $\{L_j\}_{j=1}^m$, such that they have exterior angles $\lambda_j \pi,\ 0 < \lambda_j < 2$, (with respect to $\overline{G}$) at the corner points $\{z_j\}_{j=1}^m \in L$, where two arcs meet.

According to the "three-point" criterion [6, p. 100], every piecewise smooth curve (without cusps) is quasiconformal.

Now we can state our new results.

For $0 < \delta_j < \delta_0 := \frac{1}{4} \min\{|z_i - z_j| : i,j = 1, \ldots, m,\ i \neq j\}$, let $\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$, $\delta := \min_{1 \leq j \leq m} \delta_j$, $\Omega(\delta) := \bigcup_{j=1}^m \Omega(z_j, \delta)$, $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$.

We first consider the case when there is only one singular point on the curve L, i.e. $m = 1$ and for simplicity assume that $\lambda_1 =: \lambda$, $\gamma_1 =: \gamma$.

Theorem 1.7. *Let $p > 1$, let $G \in C_\theta(\lambda)$, $0 < \lambda < 2$, and let $h(z)$ be as defined in* (1.3) *for $m = 1$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and $R_1 = 1 + 1/n$,*

$$(1.12) \qquad |P_n(z)| \leq c_6 \frac{G_{n,1}}{d(z, L_{R_1})} \|P_n\|_{A_p(h,G)} |\Phi(z)|^{n+1}, \qquad z \in \Omega_{R_1},$$

where $c_6 = c_6(G, p, \varepsilon) > 0$,

(1.13)

$$G_{n,1} = \begin{cases} n^{1/p + \varepsilon_p} & \text{if } p \geq 2,\ 0 < \lambda < 2,\ -2 < \gamma < 1/\lambda + (p-2), \\ & \quad \text{or } p < 2,\ 1 \leq \lambda < 2,\ -2 < \gamma < 1/\lambda - (2-p), \\ & \quad \text{or } p < 2,\ 0 < \lambda < 1,\ -2 < \gamma < (p-1)/\lambda; \\ n^{\gamma\lambda/p + (2/p - 1)\lambda + \varepsilon} & \text{if } p \geq 2,\ 0 < \lambda < 2,\ \gamma \geq 1/\lambda + (p-2), \\ & \quad \text{or } p < 2,\ 1 \leq \lambda < 2,\ \gamma \geq 1/\lambda - (2-p), \\ n^{\gamma\lambda/p + (2/p - 1) + \varepsilon} & \text{if } p < 2,\ 0 < \lambda < 1,\ \gamma \geq (p-1)/\lambda, \end{cases}$$

and $\varepsilon_p = \varepsilon$ if $p \neq 2$ while $\varepsilon_p = 0$ if $p = 2$.

In particular, in the case of $p = 2$, we obtain:

Corollary 1.8. *Let $p = 2$, let $G \in C_\theta(\lambda)$, $0 < \lambda < 2$, and let $h(z)$ be as in* (1.3) *for $m = 1$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and $R_1 = 1 + 1/n$,*

$$|P_n(z)| \le c_7 \frac{G_{n,2}}{d(z, L_{R_1})} \|P_n\|_{A_2(h,G)} |\Phi(z)|^{n+1}, \qquad z \in \Omega_{R_1}, \tag{1.14}$$

where $c_7 = c_7(G) > 0$ and

$$G_{n,2} < \begin{cases} n^{1/2}, & -2 < \gamma < 1/\lambda,\ 0 < \lambda < 2, \\ n^{\gamma\lambda/2+\varepsilon}, & \gamma \ge 1/\lambda,\ 0 < \lambda < 2. \end{cases} \tag{1.15}$$

Now, we will give an estimate similar to (1.10) for $G \in C_\theta(\lambda_1, \ldots, \lambda_m)$, $0 < \lambda_j < 2$, $j = 1, \ldots, m$, $m \ge 2$, in $A_p(h, G)$.

Theorem 1.9. *Let $p > 1$, let $G \in C_\theta(\lambda_1, \ldots, \lambda_m)$, $0 < \lambda_j < 2$, $j = 1, \ldots, m$, and let $h(z)$ be as defined in* (1.3)*. Then, for any $j = 1, \ldots, m$ and $P_n \in \wp_n$, $n \in \mathbb{N}$,*

$$|P_n(z_j)| \le c_8 n^{(2+\gamma_j)\lambda_j/p+\varepsilon} \|P_n\|_{A_p(h,G)}, \tag{1.16}$$

for an arbitrarily small $\varepsilon > 0$, where $c_8 = c_8(G, p, \lambda_j, \varepsilon) > 0$.

Combining (1.12) and (1.16), we obtain:

Corollary 1.10. *Let $p > 1$, let $G \in C_\theta(\lambda)$, $0 < \lambda < 2$, and let $h(z)$ be as defined in* (1.3) *for $m = 1$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and $R_1 = 1 + 1/n$,*

$$|P_n(z)| \le c_9 \|P_n\|_{A_p(h,G)} \begin{cases} n^{(2+\widehat{\gamma})\widehat{\lambda}/p+\varepsilon},\ \forall \varepsilon > 0, & z \in \overline{G}_{R_1}, \\ \dfrac{G_{n,1}}{d(z, L_{R_1})} |\Phi(z)|^{n+1}, & z \in \Omega_{R_1}, \end{cases} \tag{1.17}$$

where $c_9 = c_9(G, p, \lambda, \varepsilon) > 0$, $\widehat{\gamma} := \max\{0, \gamma\}$,

$$\widehat{\lambda} = \begin{cases} \lambda & \text{if } z \in \Omega(z_1, \delta_1), \\ 1 & \text{if } z \in \Omega \setminus \Omega(z_1, \delta_1), \end{cases}$$

and $G_{n,1}$ is defined in (1.13)*.*

Corollary 1.11. *Let $p = 2$, let $G \in C_\theta(\lambda)$, $0 < \lambda < 2$, and let $h(z)$ be as defined in* (1.3) *for $m = 1$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and $R_1 = 1 + 1/n$,*

$$|P_n(z)| \le c_{10} \|P_n\|_{A_p(h,G)} \begin{cases} n^{(1+\widehat{\gamma}/2)\widehat{\lambda}+\varepsilon},\ \forall \varepsilon > 0, & z \in \overline{G}_{R_1}, \\ \dfrac{G_{n,2}}{d(z, L_{R_1})} |\Phi(z)|^{n+1}, & z \in \Omega_{R_1}, \end{cases} \tag{1.18}$$

where $c_{10} = c_{10}(G, \lambda, \varepsilon) > 0$, $\widehat{\gamma} := \max\{0, \gamma\}$, $\widehat{\lambda}$ is as in Corollary 1.10*, and $G_{n,2}$ is defined in* (1.15)*.*

1.1. The general case. In this section, we consider the general case, i.e. $m \geq 2$. Let us first introduce some notations.

Let $\{z_j\}_{j=1}^m$ be points on the curve L ordered in the positive direction. Set $\lambda_k^* := \max\{\lambda_j : j = 1, \ldots, k,\ k \leq m\}$, $\lambda_{k*} := \min\{\lambda_j : j = 1, \ldots, k, k \leq m\}$, $\lambda^* := \lambda_m^*,\ \lambda_* := \lambda_{m*}$,

$$\widetilde{\lambda} := \begin{cases} \lambda_* & \text{if } p \geq 2, \\ \lambda^* & \text{if } p < 2, \end{cases} \qquad \widetilde{\lambda}_k := \begin{cases} \lambda_{k*} & \text{if } p \geq 2, \\ \lambda_k^* & \text{if } p < 2. \end{cases}$$

For any $j = 1, \ldots, m$, let $\mu_j := 1/\lambda_j + (p-2)$, $\eta_j := 1/\lambda_j - (2-p)$, $\omega_j := (p-1)/\lambda_j$, $\gamma_k^* := \max\{\gamma_j : j = 1, \ldots, k,\ k \leq m\}$, $\gamma^* := \gamma_m^*$, $\mathit{\Gamma} := \{\gamma_j : j = 1, \ldots, m\}$, $\mathit{\Gamma}_{j,k} := \{\gamma_j \in \mathit{\Gamma} : \gamma_j \leq \mu_k,\ k, j = 1, \ldots, m\}$, $\widetilde{\mathit{\Gamma}}_{j,k} := \mathit{\Gamma} \setminus \mathit{\Gamma}_{j,k}$. Let $w_j := \Phi(z_j)$.

Now, we can give new results for the general case:

THEOREM 1.12. *Let $p > 1$, let $G \in C_\theta(\lambda_1, \ldots, \lambda_m)$, $0 < \lambda_j < 2$, $j = 1, \ldots, m$, and let $h(z)$ be as defined in (1.3). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, $R_1 = 1 + 1/n$, and all sufficiently small $\varepsilon > 0$,*

$$|P_n(z)| \leq c_{11} \frac{D_{n,1}}{d(z, L_{R_1})} \|P_n\|_{A_p(h,G)} |\Phi(z)|^{n+1}, \qquad z \in \Omega_{R_1}, \tag{1.19}$$

where $c_{11} = c_{11}(G, p) > 0$,

$$D_{n,1} = \begin{cases} n^{1/p+\varepsilon_p} & \text{if } p \geq 2,\ 0 < \lambda_j < 2,\ -2 < \gamma_j < 1/\lambda_j + (p-2),\ \forall j, \\ & \text{or } p < 2,\ 1 \leq \lambda_j < 2,\ -2 < \gamma_j < 1/\lambda_j - (2-p),\ \forall j, \\ & \text{or } p < 2,\ 0 < \lambda_j < 1,\ -2 < \gamma_j < (p-1)/\lambda_j,\ \forall j; \\ \sum_{j=1}^m n^{\gamma_j \lambda_j / p + (2/p - 1)\lambda_j + \varepsilon} & \text{if } p \geq 2,\ 0 < \lambda_j < 2,\ \gamma_j \geq 1/\lambda_j + (p-2),\ \forall j, \\ & \text{or } p < 2,\ 1 \leq \lambda_j < 2,\ \gamma_j \geq 1/\lambda_j - (2-p),\ \forall j; \\ \sum_{j=1}^m n^{\gamma_j \lambda_j / p + (2/p - 1) + \varepsilon} & \text{if } p < 2,\ 0 < \lambda_j < 1,\ \gamma_j \geq (p-1)/\lambda_j,\ \forall j; \end{cases}$$

and $\varepsilon_p = \varepsilon$ if $p \neq 2$, while $\varepsilon_p = 0$ if $p = 2$.

Theorem 1.12 is local, that is, each term in the sum that gives $D_{n,1}$ shows the growth of $|P_n(z)|$, depending on the behavior of the weight function $h(z)$ and the neighborhood of the point z_j for any $j = 1, \ldots, m$ outside the corner λ_j.

Comparing the terms in the sum for each point z_j, $j = 1, \ldots, m$, and using the above notation, we can obtain the following global result:

THEOREM 1.13. *Let $p > 1$, $G \in C_\theta(\lambda_1, \ldots, \lambda_m)$, $0 < \lambda_j < 2$, $j = 1, \ldots, m$, and let $h(z)$ be as defined in* (1.3)*. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, and $R_1 = 1 + 1/n$, we have*

$$|P_n(z)| \le c_{12} \frac{D_{n,2}}{d(z, L_{R_1})} \|P_n\|_{A_p(h,G)} |\Phi(z)|^{n+1}, \quad z \in \Omega_{R_1}, \tag{1.20}$$

where $c_{12} = c_{12}(G, p, m) > 0$,

$$D_{n,2} = \begin{cases} n^{1/p+\varepsilon_p} & \text{if } p \ge 2,\ 0 < \lambda_j < 2, \text{ and } -2 < \gamma_j < \mu_1, \forall j, \\ & \quad \text{or } p < 2,\ 1 \le \lambda_j < 2, \text{ and } -2 < \gamma_j < \eta_1, \forall j, \\ & \quad \text{or } p < 2,\ 0 < \lambda_j < 1, \text{ and } -2 < \gamma_j < \omega_1, \forall j; \\ n^{\gamma^*\lambda^*/p+(2/p-1)\widetilde{\lambda}+\varepsilon} & \text{if } p \ge 2,\ 0 < \lambda_j < 2,\ \gamma_j \ge \mu_m, \forall j, \\ & \quad \text{or } p < 2,\ 1 \le \lambda_j < 2,\ \gamma_j \ge \eta_m, \forall j; \\ n^{\gamma_k^*\lambda_k^*/p+(2/p-1)\widetilde{\lambda}_k+\varepsilon} & \text{if } p \ge 2,\ 0 < \lambda_j < 2,\ \mu_k \le \gamma_j < \mu_{k+1}, \forall j, k, \\ & \quad \text{or } p < 2,\ 1 \le \lambda_j < 2,\ \eta_k \le \gamma_j < \eta_{k+1}, \forall j, k; \\ n^{\gamma^*\lambda^*/p+(2/p-1)+\varepsilon} & \text{if } p < 2,\ 0 < \lambda_j < 1,\ \gamma_j \ge \omega_m, \forall j; \\ n^{\gamma_k^*\lambda_k^*/p+(2/p-1)+\varepsilon} & \text{if } p < 2,\ 0 < \lambda_j < 1,\ \omega_k \le \gamma_j < \omega_{k+1}, \forall j, k, \end{cases}$$

$(k = 1, \ldots, m-1)$ and $\varepsilon_p = \varepsilon$ if $p \ne 2$, while $\varepsilon_p = 0$ if $p = 2$.

1.2. Sharpness. The sharpness of (1.12)–(1.20) can be seen from the following:

REMARK 1.14. (a) For any $n \in \mathbb{N}$ there exists a polynomial $Q_n^* \in \wp_n$, a region $G_1^* \subset \mathbb{C}$, and a constant $c_{13} = c_{13}(G_1^*, \varepsilon) > 0$ such that for all $z \in \overline{G_1^*}$,

$$|Q_n^*(z)| \ge c_{13} n \|Q_n^*\|_{A_2(G_1^*)}.$$

(b) For any $n \in \mathbb{N}$ there exists a polynomial $P_n^* \in \wp_n$, a region $G_2^* \subset \mathbb{C}$, a compact set $F \Subset \Omega^* := \overline{\mathbb{C}} \setminus \overline{G_2^*}$, and a constant $c_{14} = c_{15}(G_2^*, F^*) > 0$ such that

$$|P_n^*(z)| \ge c_{14} \frac{\sqrt{n}}{d(z, L)} \|P_n^*\|_{A_2(G_2^*)} |\Phi(z)|^{n+1}, \quad z \in F \Subset \Omega^*.$$

2. Some auxiliary results. Let $G \subset \mathbb{C}$ be a finite region bounded by a Jordan curve L, let $B(\zeta, r) := \{z : |z - \zeta| < r\}$, and let $w = \varphi(z)$ be the univalent conformal mapping of G onto $B := B(0, 1)$, normalized by $\varphi(0) = 0$, $\varphi'(0) > 0$, and $\psi := \varphi^{-1}$.

The interior or exterior level curve can be defined for $t > 0$ as

$$L_t := \{z : |\varphi(z)| = t \text{ if } t < 1,\ |\Phi(z)| = t \text{ if } t > 1\},\ L_1 \equiv L,$$

and let $G_t := \operatorname{int} L_t$, $\Omega_t := \operatorname{ext} L_t$.

Throughout this paper, $c, c_0, c_1, c_2, \ldots$ are positive constants (in general, different in different relations), which depend on G in general. For nonnegative functions $a > 0$ and $b > 0$, we shall use the notations "$a \prec b$" if $a \le cb$, and "$a \asymp b$" if $c_1 a \le b \le c_2 a$, for some constants c, c_1, c_2 (independent of a and b), respectively.

Lemma 2.1 ([3]). *Let L be a K-quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \prec d(z_1, L_{r_0})\}$, $w_j = \Phi(z_j)$ (or $z_2, z_3 \in G \cap \{z : |z - z_1| \prec d(z_1, L_{R_0})\}$, $w_j = \varphi(z_j)$), $j = 1, 2, 3$. Then*

(a) *The statements $|z_1 - z_2| \prec |z_1 - z_3|$ and $|w_1 - w_2| \prec |w_1 - w_3|$ are equivalent. So are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.*

(b) *If $|z_1 - z_2| \prec |z_1 - z_3|$, then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}},$$

where $\varepsilon < 1$, $c > 1$, $0 < r_0 < 1$, $R_0 := r_0^{-1}$ are constants, depending on G.

Corollary 2.2. *Under the assumptions of Lemma 2.1, if $z_3 \in L_{r_0}$ (or $z_3 \in L_{Rr_0}$), then*

$$|w_1 - w_2|^{K^2} \prec |z_1 - z_2| \prec |w_1 - w_2|^{K^{-2}}.$$

Corollary 2.3. *If $L \in C_\theta$, then*

$$|w_1 - w_2|^{1+\varepsilon} \prec |z_1 - z_2| \prec |w_1 - w_2|^{1-\varepsilon}$$

for all $\varepsilon > 0$.

Recall that for $0 < \delta_j < \delta_0 := \frac{1}{4}\min\{|z_i - z_j| : i, j = 1, \ldots, m,\ i \ne j\}$, we put $\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \le \delta_j\}$; $\delta := \min_{1 \le j \le m} \delta_j$, $\Omega(\delta) := \bigcup_{j=1}^m \Omega(z_j, \delta)$, $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega(z_j, \delta))$, $\Delta(\delta) := \bigcup_{j=1}^m \Phi(\Omega(z_j, \delta))$, $\widehat{\Delta}(\delta) := \Delta \setminus \Delta(\delta)$. Let $w_j := \Phi(z_j)$. For $\varphi_j := \arg w_j$, $j = 1, \ldots, m$, we put

$$\Delta_j' := \left\{ t = \mathrm{Re}^{i\theta} : R > 1,\ \frac{\varphi_{j-1} + \varphi_j}{2} \le \theta < \frac{\varphi_j + \varphi_{j+1}}{2} \right\},$$

where $\varphi_0 \equiv \varphi_m$, $\varphi_1 \equiv \varphi_{m+1}$, $\Omega_j := \Psi(\Delta_j')$, and $L_{R_1}^j := L_{R_1} \cap \Omega_j$. Clearly, $\Omega = \bigcup_{j=1}^m \Omega_j$.

The following lemma is a consequence of the results given in [8], [14].

Lemma 2.4. *Let $G \in C_\theta(\lambda_1, \ldots, \lambda_m)$, $0 < \lambda_j < 2$, $j = 1, \ldots, m$. Then*

(i) *for any $w \in \Delta_j$,*

$$|w - w_j|^{\lambda_j + \varepsilon} \prec |\Psi(w) - \Psi(w_j)| \prec |w - w_j|^{\lambda_j - \varepsilon},$$
$$|w - w_j|^{\lambda_j - 1 + \varepsilon} \prec |\Psi'(w)| \prec |w - w_j|^{\lambda_j - 1 - \varepsilon},$$

(ii) *for any* $w \in \overline{\Delta} \setminus \Delta_j$,

$$(|w|-1)^{1+\varepsilon} \prec d(\Psi(w), L)| \prec (|w|-1)^{1-\varepsilon},$$
$$(|w|-1)^{\varepsilon} \prec |\Psi'(w)| \prec (|w|-1)^{-\varepsilon}.$$

Let $\{z_j\}_{j=1}^m$ be a fixed system of distinct points on the curve L ordered in the positive direction and let $h(z)$ be a weight function as defined in (1.3).

Lemma 2.5 ([5]). *Let L be a K-quasiconformal curve and $R = 1 + c/n$. Then for any fixed $\varepsilon \in (0,1)$ there exists a level curve $L_{1+\varepsilon(R-1)}$ such that for any polynomial $P_n(z) \in \wp_n$, $n \in \mathbb{N}$,*

$$\|P_n\|_{\mathcal{L}_p(h/\Phi',\ L_{1+\varepsilon(R-1)})} \prec n^{1/p} \|P_n\|_{A_p(h,\ G)}, \qquad p > 0. \tag{2.1}$$

Lemma 2.6 ([2]). *Let L be a K-quasiconformal curve and let $h(z)$ be as in* (1.3). *Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$, and $n = 1, 2, \ldots$,*

$$\|P_n\|_{A_p(h,G_R)} \prec \widetilde{R}^{n+1/p} \|P_n\|_{A_p(h,G)}, \qquad p > 0, \tag{2.2}$$

where $\widetilde{R} = 1 + c(R-1)$ and c is independent of n and R.

3. Proofs

Proof of Theorem 1.12. For $z \in \Omega$ let

$$T_n(z) := P_n(z)/\Phi^{n+1}(z). \tag{3.1}$$

For any $R > 1$ and $R_1 := 1 + (R-1)/2$, the Cauchy integral representation for Ω_{R_1} gives

$$T_n(z) = -\frac{1}{2\pi i} \int\limits_{L_{R_1}} T_n(\zeta) \frac{d\zeta}{\zeta - z}, \qquad z \in \Omega_{R_1}.$$

Since $|\Phi(\zeta)| > 1$ for $\zeta \in L_{R_1}$, we have

$$|P_n(z)| = \frac{|\Phi(z)|^{n+1}}{2\pi} \int\limits_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|} \le \frac{|\Phi(z)|^{n+1}}{2\pi d(z, L_{R_1})} \int\limits_{L_{R_1}} |P_n(\zeta)|\,|d\zeta|. \tag{3.2}$$

Let

$$\begin{aligned} A_n &:= \int\limits_{L_{R_1}} |P_n(\zeta)|\,|d\zeta| = \sum_{i=1}^m \int\limits_{L^i_{R_1}} |P_n(\zeta)|\,|d\zeta| \\ &= \sum_{i=1}^m \int\limits_{F^i_{R_1}} |P_n(\Psi(\tau))|\,|\Psi'(\tau)|\,|d\tau|, \end{aligned} \tag{3.3}$$

where $F^i_{R_1} := \Phi(L^i_{R_1}) = \Delta'_i \cap \{\tau : |\tau| = R_1\}$, $i = 1, \ldots, m$. Changing variable $\tau = \Phi(\zeta)$ and multiplying the numerator and denominator of the integrand by $\prod_{j=1}^{m}|\Psi(\tau)-\Psi(w_j)|^{\gamma_j/p}|\Psi'(\tau)|^{2/p}$ and applying the Hölder inequality, we obtain

(3.4)

$$\begin{aligned}
A_n &= \sum_{i=1}^{m}\int_{F_{R_1}^i}\frac{\prod_{j=1}^{m}|\Psi(\tau)-\Psi(w_j)|^{\gamma_j/p}|P_n(\Psi(\tau))(\Psi'(\tau))^{2/p}|\,|\Psi'(\tau)|^{1-2/p}}{\prod_{j=1}^{m}|\Psi(\tau)-\Psi(w_j)|^{\gamma_j/p}}\,|d\tau| \\
&\le \sum_{i=1}^{m}\Big(\int_{F_{R_1}^i}\prod_{j=1}^{m}|\Psi(\tau)-\Psi(w_j)|^{\gamma_j}|P_n(\Psi(\tau))|^p|\Psi'(\tau)|^2\,|d\tau|\Big)^{1/p} \\
&\quad\times\Big(\int_{F_{R_1}^i}\Big(\frac{|\Psi'(\tau)|^{1-2/p}}{\prod_{j=1}^{m}|\Psi(\tau)-\Psi(w_j)|^{\gamma_j/p}}\Big)^q|d\tau|\Big)^{1/q} \\
&\le \sum_{i=1}^{m}A_n^i,
\end{aligned}$$

where

$$\begin{aligned}
A_n^i &:= \Big(\int_{F_{R_1}^i}|f_{n,p}(\tau)|^p\,|d\tau|\Big)^{1/p}\Big(\int_{F_{R_1}^i}\frac{|\Psi'(\tau)|^{2-q}}{\prod_{j=1}^{m}|\Psi(\tau)-\Psi(w_j)|^{\gamma_j(q-1)}}\,|d\tau|\Big)^{1/q} \\
&=: J_{n,1}^i\cdot J_{n,2}^i,
\end{aligned}$$

with

$$f_{n,p}(\tau) := \prod_{j=1}^{m}(\Psi(\tau)-\Psi(w_j))^{\gamma_j/p}P_n(\Psi(\tau))(\Psi'(\tau))^{2/p},\qquad |\tau|=R_1.$$

Applying Lemma 2.5, we get

$$J_{n,1}^i \prec n^{1/p}\|P_n\|_{A_p(h,G)},\qquad i=1,\dots,m. \tag{3.5}$$

Moreover

$$\begin{aligned}
(J_{n,2}^i)^q &= \int_{F_{R_1}^i}\frac{|\Psi'(\tau)|^{2-q}}{\prod_{j=1}^{m}|\Psi(\tau)-\Psi(w_j)|^{\gamma_j(q-1)}}\,|d\tau| \\
&\asymp \int_{F_{R_1}^i}\frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau)-\Psi(w_i)|^{\gamma_i(q-1)}}\,|d\tau|,
\end{aligned} \tag{3.6}$$

since the points $w_j := \Phi(z_j)$ are distinct.

For simplicity, we take $i=1$, $J_{n,1}^i =: J_1$, $J_{n,2}^1 =: J_2$. Let $w_1 := \Phi(z_1)$ and

$$\begin{aligned}
E_{R_1}^{11} &:= \{\tau\in F_{R_1}^1 : |\tau-w_1| < c_1(R_1-1)\}, \\
E_{R_1}^{12} &:= \{\tau\in F_{R_1}^1 : c_1(R_1-1)\le|\tau-w_1| < c_2\}, \\
E_{R_1}^{13} &:= \{\tau\in F_{R_1}^1 : |\tau-w_1|\ge c_2\}.
\end{aligned}$$

Then

$$F^1_{R_1} = \bigcup_{k=1}^{3} E^{1k}_{R_1}.$$

In these notations, (3.6) can be written as

$$J_2 = J_2(E^{11}_{R_1}) + J_2(E^{12}_{R_1}) + J_2(E^{13}_{R_1}) =: J^1_2 + J^2_2 + J^3_2, \tag{3.7}$$

and consequently

$$A^1_n =: J_1 \cdot (J^1_2 + J^2_2 + J^3_2) =: A^1_{n,1} + A^1_{n,2} + A^1_{n,3}, \tag{3.8}$$

where

$$A^1_{n,k} := n^{1/p} \|P_n\|_{A_p(h,G)} \int\limits_{E^{1k}_{R_1}} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1(q-1)}} \, |d\tau|, \qquad k = 1, 2, 3. \tag{3.9}$$

Given the possible values q ($q > 2$ and $q < 2$), λ_1 ($0 < \lambda_1 < 1$ and $1 < \lambda_1 < 2$), and γ_1 ($-2 < \gamma_1 < 0$ and $\gamma_1 \geq 0$), we will consider several cases separately.

Case 1. Let $1 < q < 2$ ($p > 2$). Then

$$(J^1_2)^q \asymp \int\limits_{E^{11}_{R_1}} \frac{|\Psi'(\tau)|^{2-q}}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1(q-1)}} \, |d\tau|.$$

1.1. Let $1 \leq \lambda_1 < 2$.

1.1.1. If $\gamma_1 \geq 0$, applying Lemma 2.4 to (3.9), we get

$$\begin{aligned} (J^1_2)^q &\prec \int\limits_{E^{11}_{R_1}} \frac{|\tau - w_1|^{(\lambda_1 - 1 - \varepsilon)(2-q)}}{|\tau - w_1|^{\gamma_1(\lambda_1+\varepsilon)(q-1)}} \, |d\tau| \\ &\prec \left(\frac{1}{n}\right)^{(\lambda_1 - 1 - \varepsilon)(2-q)} \int\limits_{E^{11}_{R_1}} \frac{|d\tau|}{(|\tau| - 1)^{\gamma_1(\lambda_1+\varepsilon)(q-1)}} \\ &\prec n^{\gamma_1\lambda_1(q-1) - (\lambda_1 - 1)(2-q) - 1 + \varepsilon} \qquad \text{if } \gamma_1\lambda_1(q-1) > 1, \end{aligned}$$

so

$$J^1_2 \prec n^{\frac{\gamma_1\lambda_1(q-1) - (\lambda_1-1)(2-q) - 1}{q} + \varepsilon}, \ \forall \varepsilon > 0, \qquad \text{if } \gamma_1\lambda_1(q-1) \geq 1.$$

Moreover

$$\begin{aligned} (J^2_2)^q &\prec \int\limits_{E^{12}_{R_1}} \frac{|\tau - w_1|^{(\lambda_1 - 1 - \varepsilon)(2-q)}}{|\tau - w_1|^{\gamma_1(\lambda_1+\varepsilon)(q-1)}} \, |d\tau| \\ &\prec \left(\frac{1}{n}\right)^{(\lambda_1 - 1 - \varepsilon)(2-q)} \int\limits_{E^{12}_{R_1}} \frac{|d\tau|}{(|\tau| - 1)^{\gamma_1(\lambda_1+\varepsilon)(q-1)}} \\ &\prec n^{\gamma_1(\lambda_1+\varepsilon)(q-1) - (\lambda_1 - 1 - \varepsilon)(2-q) - 1 + \varepsilon} \qquad \text{if } \gamma_1\lambda_1(q-1) \geq 1, \end{aligned}$$

so

$$J_2^2 \prec n^{\frac{\gamma_1\lambda_1(q-1)-(\lambda_1-1)(2-q)-1}{q}+\varepsilon}, \ \forall\varepsilon>0, \quad \text{if } \gamma_1\lambda_1(q-1)\geq 1.$$

In this case, from (3.7) and (3.8), we obtain

$$\begin{aligned} A_{n,1}^1 &\prec n^{(\frac{2+\gamma_1}{p}-1)\lambda_1+\varepsilon}\|P_n\|_{A_p(h,G)}, \quad \forall\varepsilon>0,\\ A_{n,2}^1 &\prec n^{(\frac{2+\gamma_1}{p}-1)\lambda_1+\varepsilon}\|P_n\|_{A_p(h,G)}, \quad \forall\varepsilon>0, \end{aligned} \tag{3.10}$$

if $\gamma_1\lambda_1(q-1)\geq 1$.

1.1.2. If $\gamma_1<0$, we analogously have

$$\begin{aligned} (J_2^1)^q &\prec \int\limits_{E_{R_1}^{11}} \frac{|\tau-w_1|^{(\lambda_1-1-\varepsilon)(2-q)}}{|\tau-w_1|^{\gamma_1\lambda_1(q-1)}}\,|d\tau|\\ &\prec \int\limits_{E_{R_1}^{11}} (|\tau|-1)^{(\lambda_1-1-\varepsilon)(2-q)+(-\gamma_1)(\lambda_1-\varepsilon)(q-1)}\,|d\tau|\\ &\prec (1/n)^{(\lambda_1-1-\varepsilon)(2-q)+(-\gamma_1)(\lambda_1-\varepsilon)(q-1)}\cdot \operatorname{mes} E_{R_1}^{11}, \end{aligned}$$

so

$$J_2^1 \prec n^{\frac{\gamma_1\lambda_1(q-1)-(\lambda_1-1)(2-q)-1}{q}+\varepsilon}, \quad \forall\varepsilon>0.$$

Moreover

$$(J_2^2)^q \asymp \int\limits_{E_{R_1}^{12}} \frac{|\tau-w_1|^{(\lambda_1-1-\varepsilon)(2-q)}}{|\tau-w_1|^{\gamma_1(\lambda_1-\varepsilon)(q-1)}}\,|d\tau| \prec \int\limits_{E_{R_1}^{12}} |d\tau| \prec 1.$$

Also,

$$\begin{aligned} A_{n,1}^1 &\prec n^{(\frac{2+\gamma_1}{p}-1)\lambda_1+\varepsilon}\|P_n\|_{A_p(h,G)}, \quad \forall\varepsilon>0,\\ A_{n,2}^1 &\prec n^{1/p}\|P_n\|_{A_p(h,G)}. \end{aligned} \tag{3.11}$$

1.2. Let $0<\lambda_1<1$.

1.2.1. If $\gamma_1\geq 0$, applying Lemma 2.4 to (3.9) we get

$$(J_2^1)^q \prec \int\limits_{E_{R_1}^{12}} \frac{|\tau-w_1|^{(\lambda_1-1-\varepsilon)(2-q)}}{|\tau-w_1|^{\gamma_1(\lambda_1+\varepsilon)(q-1)}}\,|d\tau| \prec n^{\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)-1+\varepsilon}$$

if $\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)\geq 1$. Hence

$$J_2^1 \prec n^{\frac{\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)-1}{q}+\varepsilon}, \ \forall\varepsilon>0, \quad \text{if } \gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)\geq 1.$$

Moreover

$$(J_2^2)^q \prec \int\limits_{E_{R_1}^{12}} \frac{|\tau-w_1|^{(\lambda_1-1-\varepsilon)(2-q)}}{|\tau-w_1|^{\gamma_1(\lambda_1+\varepsilon)(q-1)}}\,|d\tau| \prec n^{\gamma_1(\lambda_1+\varepsilon)(q-1)+(1-\lambda_1+\varepsilon)(2-q)-1}$$

if $\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)\geq 1$, so that

$J_2^2 \prec n^{\frac{\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)-1}{q}+\varepsilon}$, $\forall\varepsilon>0$, if $\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)\geq 1$.

In this case, for $A_{n,1}^1$ and $A_{n,2}^1$ from (3.8) we obtain

$$\begin{aligned} &A_{n,1}^1 \prec n^{\gamma_1\lambda_1/p-(1-2/p)\lambda_1+\varepsilon}\|P_n\|_{A_p(h,G)}, \quad \forall\varepsilon>0,\\ &A_{n,2}^1 \prec n^{\gamma_1\lambda_1/p-(1-2/p)\lambda_1+\varepsilon}\|P_n\|_{A_p(h,G)}, \quad \forall\varepsilon>0, \end{aligned} \tag{3.12}$$

if $\gamma_1\lambda_1(q-1)+(1-\lambda_1)(2-q)\geq 1$.

1.2.2. If $\gamma_1<0$, we analogously have

$$\begin{aligned} (J_2^1)^q &\prec \int\limits_{E_{R_1}^{11}} \frac{|\tau-w_1|^{(-\gamma_1)(\lambda_1-\varepsilon)(q-1)}}{|\tau-w_1|^{(1-\lambda_1+\varepsilon)(2-q)}}\,|d\tau|\\ &\prec \left(\frac{1}{n}\right)^{(-\gamma_1)(\lambda_1-\varepsilon)(q-1)} \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau-w_1|^{(1-\lambda_1+\varepsilon)(2-q)}} \prec 1, \end{aligned}$$

and

$$(J_2^2)^q \asymp \int\limits_{E_{R_1}^{12}} \frac{|\tau-w_1|^{(-\gamma_1)(\lambda_1-\varepsilon)(q-1)}}{|\tau-w_1|^{(1-\lambda_1+\varepsilon)(2-q)}}\,|d\tau| \prec 1.$$

Also,

$$A_{n,1}^1 \prec n^{1/p}\|P_n\|_{A_p(h,G)}, \qquad A_{n,2}^1 \prec n^{1/p}\|P_n\|_{A_p(h,G)}. \tag{3.13}$$

Case 2. Let $q>2$ $(p<2)$. Then $2-q<0$ and so

$$(J_2^1)^q \asymp \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\zeta-z_1|^{\gamma_1(q-1)}}. \tag{3.14}$$

2.1. Let $1\leq\lambda_1<2$.

2.1.1. If $\gamma_1\geq 0$, applying Lemma 2.4 to (3.14), we obtain

$$\begin{aligned} (J_2^1)^q &\prec \int\limits_{E_{R_1}^{11}} \frac{|d\tau|}{|\tau-w_1|^{(\lambda_1-1+\varepsilon)(q-2)}|\tau-w_1|^{\gamma_1(\lambda_1+\varepsilon)(q-1)}}\\ &\prec n^{\gamma_1(\lambda_1+\varepsilon)(q-1)+(\lambda_1-1+\varepsilon)(q-2)-1} \end{aligned}$$

if $\gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)\geq 1$. Hence

$$J_2^1 \prec n^{\frac{\gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)-1}{q}+\varepsilon} \quad \text{if } \gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)\geq 1.$$

Moreover

$$\begin{aligned} (J_2^2)^q &\prec \int\limits_{E_{R_1}^{12}} \frac{|d\tau|}{|\tau-w_1|^{(\lambda_1-1+\varepsilon)(q-2)}\;|\tau-w_1|^{\gamma_1(\lambda_1+\varepsilon)(q-1)}}\\ &\prec n^{\gamma_1(\lambda_1+\varepsilon)(q-1)+(\lambda_1-1+\varepsilon)(q-2)-1}, \qquad \forall\varepsilon>0, \end{aligned}$$

if $\gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)\geq 1$, so that

$$J_2^2 \prec n^{\frac{\gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)-1}{q}+\varepsilon},\ \forall\varepsilon>0, \quad \text{if } \gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)\geq 1.$$

In this case, from (3.8), we have

$$(3.15)\qquad \begin{aligned} A_{n,1}^1 &\prec n^{\gamma_1\lambda_1/p+(2/p-1)\lambda_1+\varepsilon}\|P_n\|_{A_p(h,G)}, \quad \forall\varepsilon>0,\\ A_{n,2}^1 &\prec n^{\gamma_1\lambda_1/p+(2/p-1)\lambda_1+\varepsilon}\|P_n\|_{A_p(h,G)}, \quad \forall\varepsilon>0, \end{aligned}$$

if $\gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)\geq 1$.

2.1.2. If $\gamma_1<0$, we analogously have

$$\begin{aligned}(J_2^1)^q &\asymp \int\limits_{E_{R_1}^{11}} \frac{|\tau-w_1|^{(-\gamma_1)(\lambda_1-\varepsilon)(q-1)}}{|\tau-w_1|^{(\lambda_1-1+\varepsilon)(q-2)}}\,|d\tau|\\ &\prec n^{(\lambda_1-1)(q-2)+\gamma_1\lambda_1(q-1)-1+\varepsilon} \quad \text{if } (\lambda_1-1)(q-2)\geq 1,\end{aligned}$$

so

$$J_2^1 \prec n^{\frac{(\lambda_1-1)(q-2)+\gamma_1\lambda_1(q-1)-1}{q}+\varepsilon},\ \forall\varepsilon>0, \quad \text{if } (\lambda_1-1)(q-2)\geq 1.$$

Moreover

$$\begin{aligned}(J_2^2)^q &\asymp \int\limits_{E_{R_1}^{12}} \frac{|\tau-w_1|^{(-\gamma_1)(\lambda_1-\varepsilon)(q-1)}}{|\tau-w_1|^{(\lambda_1-1+\varepsilon)(q-2)}}\,|d\tau|\\ &\prec n^{(\lambda_1-1)(q-2)-1+\varepsilon} \quad \text{if } (\lambda_1-1)(q-2)\geq 1,\end{aligned}$$

and hence

$$J_2^2 \prec n^{\frac{(\lambda_1-1)(q-2)-1}{q}+\varepsilon},\ \forall\varepsilon>0, \quad \text{if } (\lambda_1-1)(q-2)\geq 1.$$

So,

$$(3.16)\qquad \begin{aligned} A_{n,1}^1 &\prec n^{(2/p-1)\lambda_1+\gamma_1\lambda_1/p+\varepsilon}\|P_n\|_{A_p(h,G)}, \quad \forall\varepsilon>0,\\ A_{n,2}^1 &\prec n^{(2/p-1)\lambda_1+\varepsilon}\|P_n\|_{A_p(h,G)}, \quad \forall\varepsilon>0, \end{aligned}$$

if $(\lambda_1-1)(q-2)\geq 1$.

2.2. Let $0<\lambda_1<1$.

2.2.1. If $\gamma_1\geq 0$, applying Lemma 2.4 to (3.14), we obtain

$$\begin{aligned}(J_2^1)^q &\prec \int\limits_{E_{R_1}^{11}} \frac{|\tau-w_1|^{(1-\lambda_1-\varepsilon)(q-2)}}{|\tau-w_1|^{\gamma_1(\lambda_1+\varepsilon)(q-1)}}\,|d\tau|\\ &\prec n^{-(1-\lambda_1)(q-2)+\gamma_1\lambda_1(q-1)-1+\varepsilon} \quad \text{if } \gamma_1\lambda_1(q-1)\geq 1,\end{aligned}$$

so that

$$J_2^1 \prec n^{\frac{-(1-\lambda_1)(q-2)+\gamma_1\lambda_1(q-1)-1}{q}+\varepsilon},\ \forall\varepsilon>0, \quad \text{if } \gamma_1\lambda_1(q-1)\geq 1.$$

Moreover

$$(J_2^2)^q \prec \int\limits_{E_{R_1}^{12}} \frac{|\tau - w_1|^{(1-\lambda_1-\varepsilon)(q-2)}}{|\tau - w_1|^{\gamma_1(\lambda_1+\varepsilon)(q-1)}} |d\tau|$$
$$\prec n^{\gamma_1(\lambda_1+\varepsilon)(q-1)-1+\varepsilon} \quad \text{if } \gamma_1\lambda_1(q-1) \geq 1,$$

and hence

$$J_2^2 \prec n^{\frac{\gamma_1\lambda_1(q-1)-1}{q}+\varepsilon}, \forall\varepsilon > 0, \quad \text{if } \gamma_1\lambda_1(q-1) \geq 1.$$

In this case, from (3.8), we have

$$\begin{aligned} A_{n,1}^1 &\prec n^{(\frac{2+\gamma_1}{p}-1)\lambda_1+\varepsilon} \|P_n\|_{A_p(h,G)}, && \forall\varepsilon > 0, \\ A_{n,2}^1 &\prec n^{(2/p-1)+\gamma_1\lambda_1/p+\varepsilon} \|P_n\|_{A_p(h,G)}, && \forall\varepsilon > 0, \end{aligned} \tag{3.17}$$

if $\gamma_1\lambda_1(q-1) \geq 1$.

2.2.2. If $\gamma_1 < 0$, we analogously have

$$(J_2^1)^q \asymp \int\limits_{E_{R_1}^{11}} \frac{|\tau - w_1|^{(1-\lambda_1-\varepsilon)(q-2)}}{|\tau - w_1|^{\gamma_1(\lambda_1-\varepsilon)(q-1)}} |d\tau|$$
$$\prec \left(\frac{1}{n}\right)^{(1-\lambda_1-\varepsilon)(q-2)+(-\gamma_1)(\lambda_1-\varepsilon)(q-1)} \cdot \operatorname{mes} E_{R_1}^1 \prec 1,$$

and

$$(J_2^2)^q \asymp \int\limits_{E_{R_1}^{12}} \frac{|\tau - w_1|^{(\lambda_1-1-\varepsilon)(2-q)}}{|\tau - w_1|^{\gamma_1(\lambda_1-\varepsilon)(q-1)}} |d\tau| \prec \int\limits_{E_{R_1}^{12}} |d\tau| \prec 1.$$

Hence

$$A_{n,1}^1 \prec n^{1/p} \|P_n\|_{A_p(h,G)}, \quad A_{n,2}^1 \prec n^{1/p} \|P_n\|_{A_p(h,G)}. \tag{3.18}$$

To estimate $A_{n,3}^1$, in all cases, we note that $|\zeta - z_1| \asymp 1$ for each $\zeta \in E_{R_1}^{13}$, and so

$$(J_2^3)^q \asymp \int\limits_{E_{R_1}^{13}} \frac{|d\tau|}{(|\tau|-1)^{(2-q)\varepsilon}},$$

hence

$$J_2^3 \prec n^{\varepsilon}, \forall\varepsilon > 0, \text{ if } p \neq 2 \quad \text{and} \quad J_2^3 \prec 1 \text{ if } p = 2.$$

Consequently,

$$\begin{aligned} A_{n,3}^1 &\prec n^{1/p+\varepsilon} \|P_n\|_{A_p(h,G)}, \forall\varepsilon > 0 && \text{if } p \neq 2, \\ A_{n,3}^1 &\prec n^{1/2} \|P_n\|_{A_2(h,G)} && \text{if } p = 2. \end{aligned} \tag{3.19}$$

Therefore, combining (3.8)–(3.19), we get

$$A_n^1 = \sum_{k=1}^{3} A_{n,k}^1 \prec \|P_n\|_{A_p(h,G)}$$
$$\times \begin{cases} n^{(\frac{2+\gamma_1}{p}-1)\lambda_1+\varepsilon} + n^{(\frac{2+\gamma_1}{p}-1)\lambda_1+\varepsilon} + n^{1/p+\varepsilon}, \\ \qquad p>2,\ \lambda_1 \geq 1,\ \gamma_1\lambda_1(q-1)\geq 1; \\ n^{\gamma_1\lambda_1/p-(1-2/p)\lambda_1+\varepsilon} + n^{\gamma_1\lambda_1/p-(1-2/p)\lambda_1+\varepsilon} + n^{1/p+\varepsilon}, \\ \qquad p>2,\ \lambda_1<1,\ \gamma_1\lambda_1(q-1)+(\lambda_1-1)(q-2)\geq 1; \\ n^{\gamma_1\lambda_1/p+(2/p-1)\lambda_1+\varepsilon} + n^{\gamma_1\lambda_1/p+(2/p-1)\lambda_1+\varepsilon} + n^{1/p+\varepsilon}, \\ \qquad p<2,\ \lambda_1\geq 1,\ \gamma_1\lambda_1(q-1)\geq 1; \\ n^{(\frac{2+\gamma_1}{p}-1)\lambda_1+\varepsilon} + n^{(2/p-1)+\gamma_1\lambda_1/p+\varepsilon} + n^{1/p+\varepsilon}, \\ \qquad p<2,\ \lambda_1<1,\ \gamma_1\lambda_1(q-1)\geq 1, \end{cases}$$

if $\gamma_1 \geq 0$, and

$$A_n^1 = \sum_{k=1}^{3} A_{n,k}^1 \prec \|P_n\|_{A_p(h,G)}$$
$$\times \begin{cases} n^{(\frac{2+\gamma_1}{p}-1)\lambda_1+\varepsilon} + n^{1/p} + n^{1/p+\varepsilon}, & p>2,\ \lambda_1\geq 1; \\ n^{1/p} + n^{1/p} + n^{1/p+\varepsilon}, & p>2,\ \lambda_1<1; \\ n^{(2/p-1)\lambda_1+\gamma_1\lambda_1/p+\varepsilon} + n^{(2/p-1)\lambda_1+\varepsilon} + n^{1/p+\varepsilon}, & p<2,\ \lambda_1\geq 1, \\ & (\lambda_1-1)(q-2)\geq 1; \\ n^{(2/p-1)+\gamma_1\lambda_1/p+\varepsilon} + n^{1/p} + n^{1/p+\varepsilon}, & p<2,\ \lambda_1<1, \end{cases}$$

if $\gamma_1 < 0$, for any sufficiently small $\varepsilon > 0$.

Hence,

$$A_n^1 \prec \|P_n\|_{A_p(h,G)} \tag{3.20}$$
$$\times \begin{cases} n^{1/p+\varepsilon}, & p>2,\ \lambda_1\geq 1,\ 0\leq\gamma_1\lambda_1<1+\lambda_1(p-2); \\ n^{\gamma_1\lambda_1/p-(1-2/p)\lambda_1+\varepsilon}, & p>2,\ \lambda_1\geq 1,\ \gamma_1\lambda_1\geq 1+\lambda_1(p-2); \\ n^{1/p+\varepsilon}, & p>2,\ \lambda_1<1,\ 0\leq\gamma_1\lambda_1<1+\lambda_1(p-2); \\ n^{\gamma_1\lambda_1/p-(1-2/p)\lambda_1+\varepsilon}, & p>2,\ \lambda_1<1,\ \gamma_1\lambda_1\geq 1+\lambda_1(p-2); \\ n^{1/p+\varepsilon}, & p<2,\ \lambda_1>1,\ 0\leq\gamma_1\lambda_1<1-\lambda_1(2-p); \\ n^{\gamma_1\lambda_1/p+(2/p-1)\lambda_1+\varepsilon}, & p<2,\ \lambda_1>1,\ \gamma_1\lambda_1\geq 1-\lambda_1(2-p); \\ n^{1/p+\varepsilon}, & p<2,\ \lambda_1<1,\ 0\leq\gamma_1\lambda_1<p-1; \\ n^{\gamma_1\lambda_1/p+(2/p-1)+\varepsilon}, & p<2,\ \lambda_1<1,\ \gamma_1\lambda_1\geq p-1, \end{cases}$$

if $\gamma_1 \geq 0$, and

$$(3.21)\qquad A_n \prec \|P_n\|_{A_p(h,G)} \begin{cases} n^{1/p+\varepsilon}, & p>2,\ \lambda_1 \geq 1,\ \gamma_1<0; \\ n^{1/p+\varepsilon}, & p>2,\ \lambda_1 < 1,\ \gamma_1<0; \\ n^{1/p+\varepsilon}, & p<2,\ \lambda_1 \geq 1,\ \gamma_1<0; \\ n^{1/p+\varepsilon}, & p<2,\ \lambda_1 < 1,\ \gamma_1<0, \end{cases}$$

if $\gamma_1 < 0$, for any sufficiently small $\varepsilon > 0$. Therefore, taking into account also the case $p = 2$, and summing over all $j = 1, \ldots, m$, from (3.8) and (3.9), we get

$$A_n \leq \sum_{j=1}^{m} A_n^j \prec \|P_n\|_{A_p(h,G)}$$

$$\times \begin{cases} n^{1/p+\varepsilon_p}, & p \geq 2,\ 0<\lambda_j<2,\ -2<\gamma_j<1/\lambda_j+(p-2); \\ \sum_{j=1}^{m} n^{\gamma_j\lambda_j/p-(1-2/p)\lambda_j+\varepsilon_p}, & p \geq 2,\ 0<\lambda_j<2,\ \gamma_j \geq 1/\lambda_j+(p-2); \\ n^{1/p+\varepsilon}, & p<2,\ 1 \leq \lambda_j<2,\ -2<\gamma_j<1/\lambda_j-(2-p); \\ \sum_{j=1}^{m} n^{\gamma_j\lambda_j/p+(2/p-1)\lambda_j+\varepsilon}, & p<2,\ 1 \leq \lambda_j<2,\ \gamma_j \geq 1/\lambda_j-(2-p); \\ n^{1/p+\varepsilon}, & p<2,\ 0<\lambda_j<1,\ -2<\gamma_j<(p-1)/\lambda_j; \\ \sum_{j=1}^{m} n^{\gamma_j\lambda_j/p+(2/p-1)+\varepsilon}, & p<2,\ 0<\lambda_j<1,\ \gamma_j \geq (p-1)/\lambda_j, \end{cases}$$

for any sufficiently small $\varepsilon > 0$, where $\varepsilon_p = \varepsilon$ if $p > 2$, and $\varepsilon_p = 0$ if $p = 2$.

Also,

$$A_n \prec \|P_n\|_{A_p(h,G)}$$

$$\times \begin{cases} n^{1/p+\varepsilon_p}, & p \geq 2,\ 0<\lambda_j<2,\ -2<\gamma_j<1/\lambda_1+(p-2),\ \forall j; \\ n^{\gamma^*\lambda^*/p-(1-2/p)\lambda_*+\varepsilon}, & p \geq 2,\ 0<\lambda_j<2,\ \gamma_j \geq 1/\lambda_m+(p-2),\ \forall j; \\ n^{\gamma_k^*\lambda_k^*/p-(1-2/p)\lambda_{k*}+\varepsilon}, & p \geq 2,\ 0<\lambda_j<2,\ \mu_k \leq \gamma_j < \mu_{k+1},\ \forall j; \\ n^{1/p+\varepsilon}, & p<2,\ 1 \leq \lambda_j<2,\ -2<\gamma_j<1/\lambda_1-(2-p),\ \forall j; \\ n^{\gamma^*\lambda^*/p+(2/p-1)\lambda^*+\varepsilon}, & p<2,\ 1 \leq \lambda_j<2,\ \gamma_j \geq 1/\lambda m-(2-p),\ \forall j; \\ n^{\gamma_k^*\lambda_k^*/p+(2/p-1)\lambda_k^*+\varepsilon}, & p<2,\ 1 \leq \lambda_j<2,\ \eta_k \leq \gamma_j < \eta_{k+1},\ \forall j; \\ n^{1/p+\varepsilon}, & p<2,\ 0<\lambda_j<1,\ -2<\gamma_j<(p-1)/\lambda_1,\ \forall j; \\ n^{\gamma^*\lambda^*/p+(2/p-1)+\varepsilon}, & p<2,\ 0<\lambda_j<1,\ \gamma_j \geq (p-1)/\lambda_m,\ \forall j; \\ n^{\gamma_k^*\lambda_k^*/p+(2/p-1)+\varepsilon}, & p<2,\ 0<\lambda_j<1,\ \omega_k \leq \gamma_j < \omega_{k+1},\ \forall j, \end{cases}$$

(with $k = 1, \ldots, m-1$) for any sufficiently small $\varepsilon > 0$, where $\varepsilon_p = \varepsilon$ if $p > 2$, and $\varepsilon_p = 0$ if $p = 2$. Combining the formulas (3.2), (3.3), (3.5), (3.8), (3.20), and (3.21) we complete the proof of Theorem 1.12. ■

Proof of Corollary 1.8. Let $p = 2,\ m = 1$. First of all we note that from (3.5), (3.8) and (3.9), for $\gamma_1 = 0$, we can easily obtain

$$(3.22)\qquad A_n \prec n^{1/2}\|P_n\|_{A_2(G)}.$$

1. Let $\gamma_1 > 0$. Then, for any $0 < \lambda_1 < 2$, we obtain

$$(J_2^1)^2 \prec \int\limits_{E^1_{R_1}} \frac{|d\tau|}{|\tau - w_1|^{\gamma_1(\lambda_1+\varepsilon)}} \prec n^{\gamma_1(\lambda_1+\varepsilon)-1}, \forall\varepsilon > 0, \quad \text{if } \gamma_1\lambda_1 \geq 1;$$

$$(J_2^2)^2 \prec \int\limits_{E^2_{R_1}} \frac{|d\tau|}{|\tau - w_1|^{\gamma_1(\lambda_1+\varepsilon)}} \prec n^{\gamma_1(\lambda_1+\varepsilon)-1}, \forall\varepsilon > 0, \quad \text{if } \gamma_1\lambda_1 \geq 1.$$

Consequently,

$$\begin{aligned} A^1_{n,1} &\prec n^{1/2+(\gamma_1\lambda_1-1)/2+\varepsilon}\|P_n\|_{A_2(h,G)} \\ &\prec \|P_n\|_{A_2(h,G)} \begin{cases} n^{\gamma_1\lambda_1/2+\varepsilon}, \ \forall\varepsilon > 0, & \text{if } \gamma_1\lambda_1 \geq 1, \\ n^{1/2}, & \text{if } \gamma_1\lambda_1 < 1; \end{cases} \\ A^1_{n,2} &\prec \|P_n\|_{A_2(h,G)} \begin{cases} n^{\gamma_1\lambda_1/2+\varepsilon}, \ \forall\varepsilon > 0, & \text{if } \gamma_1\lambda_1 \geq 1, \\ n^{1/2}, & \text{if } \gamma_1\lambda_1 < 1. \end{cases} \end{aligned} \tag{3.23}$$

2. Let $-2 < \gamma_1 < 0$. In this case, for any $0 < \lambda_1 < 2$, we also obtain

$$(J_2^1)^2 \prec \int\limits_{E^1_{R_1}} \frac{|d\tau|}{|\tau - w_1|^{\gamma_1(\lambda_1-\varepsilon)}} \prec n^{\gamma_1(\lambda_1+\varepsilon)-1}, \quad \forall\varepsilon > 0,$$

$$(J_2^2)^2 \prec \int\limits_{E^2_{R_1}} \frac{|d\tau|}{|\tau - w_1|^{\gamma_1(\lambda_1-\varepsilon)}} \prec 1;$$

hence

$$\begin{aligned} A^1_{n,1} &\prec n^{\gamma_1\lambda_1/2+\varepsilon}\|P_n\|_{A_2(h,G)}, \quad \forall\varepsilon > 0, \\ A^1_{n,2} &\prec n^{1/2}\|P_n\|_{A_2(h,G)}. \end{aligned} \tag{3.24}$$

To estimate $A^1_{n,3}$ in all cases, we note that $|\zeta - z_1| \asymp 1$ for each $\zeta \in E^{13}_{R_1}$. Therefore, $J_2^k \prec 1, \ k = 1, 2, 3$, and we get

$$A^1_{n,3} \prec n^{1/2}\|P_n\|_{A_2(h,G)}. \tag{3.25}$$

Therefore, combining (3.2)–(3.5), (3.8), (3.22)–(3.25), for any $0 < \lambda_1 < 2$, we get

$$A_n \prec \|P_n\|_{A_2(h,G)} \begin{cases} n^{1/2}, & \text{if } -2 < \gamma_1 < 1/\lambda_1, \\ n^{\gamma_1\lambda_1/2+\varepsilon}, \ \forall\varepsilon > 0, & \text{if } \gamma_1 \geq 1/\lambda_1. \end{cases}$$

The proof is completed. ■

Proof of Theorem 1.5. For an arbitrary polynomial $P_n \in \wp_n$ and the Bergman polynomials (i.e. polynomials $K_n(z)$ orthonormal over the region, $\|K_n\|_{A_2(h,G)} = 1$), the following theorem was proved in [4, Ths. 2.1 and 5.1].

THEOREM. *Let G be a region bounded by a k-quasidisk for some $0 \leq k < 1$, and let $h(z)$ be a weight function as defined in* (1.3). *Then, for any*

$P_n \in \wp_n$, $n \in \mathbb{N}$, and every point $z_j \in L$, $j = 1, \ldots, m$,

$$|P_n(z_j)| \prec n^{(2+\gamma_j)(1+k)/p} \|P_n\|_{A_p(h,G)}.$$

Proceeding as in the proof of that theorem we can obtain the following estimate [4, (4.11)] for regions with quasiconformal boundary:

$$|K_n(z_j)| \prec d(z_j, L_R)^{-(2+\gamma)/p}.$$

Repeating the proof of this formula for the polynomial $P_n(z)$, we get

$$|P_n(z_j)| \prec \frac{1}{d(z_j, L_R)^{(2+\gamma)/p}} \|P_n\|_{A_p(h,G)}.$$

Since L is K-quasiconformal, $d(z_j, L_R) \succ (R-1)^s$, by Corollary 2.2, where $s := \min\{2, K^2\}$. According to Lemma 2.4, we get

$$|P_n(z_j)| \prec n^{(2+\gamma)s/p} \|P_n\|_{A_p(h,G)}. \quad \blacksquare \tag{3.26}$$

Proof of Remark 1.14. (a) Let $Q_n^*(z) := \sum_{j=0}^n (j+1) z^j$, $G_1^* = B$, and $p = 2$. In this case,

$$\|Q_n^*\|_{C(\overline{G_1^*})} = (n+1)(n+2)/2, \qquad \|Q_n^*\|_{A_2(G_1^*)} = \sqrt{\pi(n+1)(n+2)/2}.$$

Thus, we have

$$\|Q_n^*\|_{C(\overline{G_1^*})} \geq \frac{1}{\sqrt{2\pi}} n \|Q_n^*\|_{A_2(G_1^*)}.$$

(b) Let $G_2^* \subset \mathbb{C}$ be a region bounded by a smooth curve $L = \partial G_2^* \in C_\theta$. According to the "three-point" criterion [6, p. 100], the curve L is quasiconformal. Let $\{K_n(z)\}$, $\deg K_n = n$, $n = 0, 1, 2, \ldots$, denote the system of Bergman polynomials for the region G_2^*, i.e. $K_n(z) := \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_0$, $\alpha_n > 0$, and

$$\iint_{G_2^*} K_n(z) \overline{K_m(z)}\, d\sigma_z = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta. Let $\overline{G_2^*}$ be the closure of the convex hull of the G_2^* and $F := \overline{\mathbb{C}} \setminus \overline{G_2^*}$. We know from [7, p. 245] that all zeros of the Bergman polynomials $K_n(z)$ are contained in $\overline{G_2^*}$. According to [1], for arbitrary quasidiscs, we have:

$$K_n(z) = \alpha_n \rho^{n+1} \Phi^n(z) \Phi'(z) A_n(z), \qquad z \in F \Subset \Omega,$$

where

$$\sqrt{\frac{n+1}{\pi}} \leq \alpha_n \rho^{n+1} \leq c_1 \sqrt{\frac{n+1}{\pi}}$$

for some $c_1 = c_1(G_2^*) > 1$ and

$$c_2 \leq |A_n(z)| \leq 1 + \frac{c_3}{\sqrt{|\Phi(z)| - 1}},$$

for some $c_i = c_i(G_2^*) > 0$, $i = 2, 3$. Therefore, since $\|K_n\|_{A_2(G_2^*)} = 1$, we have

$$
\begin{aligned}
|K_n(z)| &\geq c_2\sqrt{\frac{n+1}{\pi}}|\Phi(z)|^n\frac{|\Phi(z)|-1}{d(z,L)} \\
&\geq c_3\frac{\sqrt{n}}{d(z,L)}|\Phi(z)|^{n+1}(1-1/|\Phi(z)|) \\
&\geq c_4\frac{\sqrt{n}}{d(z,L)}|\Phi(z)|^{n+1}\|K_n\|_{A_2(G_2^*)}.
\end{aligned}
$$

The proof is complete. ∎

References

[1] F. G. Abdullayev, Ph.D. Dissertation, Donetsk, 1986, 120 pp.

[2] F. G. Abdullayev, *On some properties of orthogonal polynomials over the region of the complex plane (Part III)*, Ukrain. Math. J. 53 (2001), 1934–1948.

[3] F. G. Abdullayev and V. V. Andrievskii, *On the orthogonal polynomials in the domains with K-quasiconformal boundary*, Izv. Akad. Nauk Azerbaĭjan. SSR Ser. Fiz.-Tekh. Mat. Nauk 4 (1983), no 1, 7–11 (in Russian).

[4] F. G. Abdullayev and U. Deger, *On the orthogonal polynomials with weight having singularities on the boundary of regions in the complex plane*, Bull. Belg. Math. Soc. Simon Stevin 16 (2009), 235–250.

[5] F. G. Abdullayev and P. Özkartepe, *On the behavior of the algebraic polynomial in unbounded regions of complex plane*, 2012 (to appear).

[6] L. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, Princeton, NJ, 1966.

[7] P. J. Davis, *Interpolation and Approximation*, Blaisdell, 1963.

[8] D. Gaier, *On the convergence of the Bieberbach polynomials in regions with corners*, Constr. Approx. 4 (1988), 289–305.

[9] E. Hille, G. Szegö and J. D. Tamarkin, *On some generalization of a theorem of A. Markoff*, Duke Math. J. 3 (1937), 729–739.

[10] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer, Berlin, 1973.

[11] S. Rickman, *Characterisation of quasiconformal arcs*, Ann. Acad. Sci. Fenn. Ser. A Math. 395 (1966), 30 pp.

[12] N. Stylianopoulos, *Fine asymptotics for Bergman orthogonal polynomials over domains with corners*, CMFT 2009, Ankara, 2009.

[13] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, Amer. Math. Soc., 1960.

[14] S. E. Warschawski, *Über das Randverhalten der Ableitung der Abbildungsfunktion bei konformer Abbildung*, Math. Z. 35 (1932), 321–456.

F. G. Abdullayev, C. D. Gün
Department of Mathematics
Faculty of Arts and Science
Mersin University
33343 Mersin, Turkey
E-mail: fabdul@mersin.edu.tr
cevahirdoganaygun@gmail.com

Received 22.3.2013
and in final form 22.4.2013 (3070)