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A note on generalized projections in c_0

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Abstract. Let $V \subset Z$ be two subspaces of a Banach space X. We define the set of generalized projections by

$$\mathcal{P}_V(X,Z) := \{P \in \mathcal{L}(X,Z) : P|_V = \mathrm{id}\}.$$

Now let $X = c_0$ or l_∞^m, $Z := \ker f$ for some $f \in X^*$ and $V := Z \cap l_\infty^n$ $(n < m)$. The main goal of this paper is to discuss existence, uniqueness and strong uniqueness of a minimal generalized projection in this case. Also formulas for the relative generalized projection constant and the strong uniqueness constant will be given (cf. J. Blatter and E. W. Cheney [Ann. Mat. Pura Appl. 101 (1974), 215–227] and G. Lewicki and A. Micek [J. Approx. Theory 162 (2010), 2278–2289] where the case of projections has been considered). We discuss both the real and complex cases.

1. Introduction. Let X be a Banach space and V be a closed linear subspace of X. Then we denote by $\mathcal{P}(X,V)$ the set of all linear projections continuous with respect to the operator norm. Recall that an operator $P : X \to V$ is called a *projection* if $P|_V = \mathrm{id}_V$. A projection $P_0 \in \mathcal{P}(X,V)$ is called *minimal* if

$$\|P_0\| = \lambda(V,X) := \inf\{\|P\| : P \in \mathcal{P}(X,V)\}. \tag{1.1}$$

Minimal projections in the context of functional analysis and approximation theory were extensively studied by many authors (see e.g., [3], [5]–[12], [14]–[17], [20], [21], [25], [27], [29]–[31]). Mainly the problems of existence of minimal projections, their uniqueness, finding concrete formulas for minimal projections and estimates of the constant $\lambda(V,X)$ were considered.

Projections play an important role in numerical analysis, as the error of approximation of an element x by Px can be estimated by means of the elementary inequality

$$\|x - Px\| \le \|\mathrm{id} - P\| \cdot \mathrm{dist}(x,V) \le (1 + \|P\|) \cdot \mathrm{dist}(x,V), \tag{1.2}$$

where $\mathrm{dist}(x,V) := \inf\{\|x - v\| : v \in V\}$.

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Let us introduce a generalization of projections (compare with *quasi-projection*, [13]), for which the above inequality also holds.

DEFINITION 1.1. Let $V \subset Z$ be two subspaces of a Banach space X. Then

$$\mathcal{P}_V(X, Z) := \{P \in \mathcal{L}(X, Z) : P|_V = \mathrm{id}\}. \tag{1.3}$$

An element $P_0 \in \mathcal{P}_V(X, Z)$ is called a *minimal generalized projection* (MGP) if

$$\|P_0\| = \lambda_Z(V, X) := \inf\{\|P\| : P \in \mathcal{P}_V(X, Z)\} \tag{1.4}$$

Notice that $\lambda_Z(V, X) \le \lambda(V, X)$ for any $V \subset Z \subset X$. In general $\lambda(V, X)$ and $\lambda_Z(V, X)$ are not equal (see Example 2.5). It is worth mentioning that some classical operators like Bernstein operators, Fejér operators and de La Vallée Poussin operators are generalized projections.

In this paper we discuss existence, uniqueness and strong uniqueness of minimal generalized projections in the case $X = c_0$ or $X = l_\infty^m$, $Z = \ker f$ for some $f \in X^* \setminus \{0\}$ and $V = Z \cap l_\infty^n$ $(n < m)$. Also formulas for the relative generalized projection constant $\lambda_Z(V, X)$ and the strong uniqueness constant will be given. This generalizes some results of J. Blatter and E. W. Cheney [3] and G. Lewicki and A. Micek [19]. Our results seem interesting because cases in which exact values of the above constants can be given are rare.

The notion of strong uniqueness was introduced by Newman and Shapiro [26]. Let X be a normed space and let $Y \subset X$ be a nonempty subset. An element $y \in Y$ is called a *strongly unique best approximation* (SUBA) to $x \in X$ if there exists $r > 0$ such that for every $v \in Y$,

$$\|x - v\| \ge \|x - y\| + r\|v - y\|. \tag{1.5}$$

The largest such r is called the *strong uniqueness constant.* The significance of this notion can be illustrated by its two main applications. The error estimate of the Remez algorithm is based on an iteration process for finding the constant r satisfying (1.5). The strong uniqueness of best approximation yields the Lipschitz continuity of the best approximation mapping (see e.g. [4]).

In the case of operators, the notion of strong uniqueness reduces to the following definition:

DEFINITION 1.2. Let $T_0 \in \mathcal{T} \subset \mathcal{L}(X, Z)$. Then T_0 is called a *strongly unique minimal operator* in $\mathcal{T}$ if there exists $r > 0$ such that for any $T \in \mathcal{T}$,

$$\|T\| \ge \|T_0\| + r\|T - T_0\|. \tag{1.6}$$

The largest such r, is called the *strong uniqueness operator constant* in $\mathcal{T}$.

For results concerning strong uniqueness in general and in the context of minimal projections see e.g. [1], [2], [18], [19], [23], [24], [26], [28], [32].

The main tool to study strong uniqueness is a Kolmogorov type criterion [17, Theorem 1.2.5]. The following theorem is a special case of this criterion.

THEOREM 1.3. *Let $Z \subset X$ be finite-dimensional spaces and let $\mathcal{T}$ be an affine subspace of $\mathcal{L}(X, Z)$. Then T_0 is a strongly unique minimal operator in $\mathcal{T}$ with constant $r > 0$ iff for every $T \in \mathcal{T}$ there exists $z^* \in \operatorname{crit}^*(T_0)$ such that*

$$\inf\{\operatorname{Re}(z^*((T - T_0)x)) : x \in A_{z^*}(T_0)\} \le -r\|T - T_0\|, \tag{1.7}$$

where $\operatorname{crit}^*(T_0) := \{z^* \in \operatorname{ext}(S_{X^*}) : \|z^* \circ T_0\| = \|T_0\|\}$ *and* $A_{z^*}(T_0) := \{x \in S_X : z^*(T_0 x) = \|T_0\|\}$.

2. Results and applications. In this section, unless otherwise stated, we consider both real and complex cases. Let $n \in \mathbb{N}$. For every $f \in l_1$ we define $f^{(n)} := (f_1, \ldots, f_n)$. We denote by Q_n the operator given by

$$Q_n : c_0 \ni (x_1, x_2, \ldots) \mapsto (x_1, \ldots, x_n) \in l_\infty^n. \tag{2.1}$$

LEMMA 2.1. *Let $X := c_0$, $Z := \ker f$ with $f \in l_1$ such that $f^{(n)} \neq 0$, and $V = Z \cap l_\infty^n$. If $P \in \mathcal{P}_V(X, Z)$ then $P \circ Q_n \in \mathcal{P}_V(X, Z)$. Moreover there exists $w \in X$ such that $f(w) = 1$ and*

$$P \circ Q_n(x) = Q_n(x) - f(Q_n(x))w \quad \text{for all } x \in X. \tag{2.2}$$

Proof. It is obvious that $P \circ Q_n \in \mathcal{P}_V(X, Z)$. Now let $\{y_k\}_{i=1}^n$ be a basis of l_∞^n such that $f(y_n) = 1$ and $f(y_k) = 0$ for all $k \in \{1, \ldots, n-1\}$. Fix $x \in X$. Since $Q_n(x) \in l_\infty^n$, there exist $v \in V$ and $\alpha \in \mathbb{K}$ such that $Q_n(x) = v + \alpha y_n$. Hence

$$\begin{aligned} P(Q_n(x)) &= P(v) + \alpha P(y_n) = v + \alpha y_n - \alpha y_n + \alpha P(y_n) \\ &= Q_n(x) - \alpha(y_n - P(y_n)), \\ f(Q_n(x)) &= f(v) + \alpha f(y_n) = \alpha, \\ f(y_n - P(y_n)) &= f(y_n) - f(P(y_n)) = 1, \end{aligned}$$

as required. ■

Since $\|Q_n\| = 1$, we can state

COROLLARY 2.2. *Let X, Z, V be as in Lemma 2.1. Then*

$$\lambda_Z(V, X) = \inf\{\|P\| : P \in \widetilde{\mathcal{P}}_V(X, Z)\}, \tag{2.3}$$

where $\widetilde{\mathcal{P}}_V(X, Z) := \{Q_n - f \circ Q_n(\cdot)w : w \in X,\ f(w) = 1\}$.

LEMMA 2.3. *Let X, Z, V be as in Lemma 2.1. Fix $w \in X$ such that $f(w) = 1$ and let*

$$P(x) := Q_n(x) - f(Q_n(x))w \quad \text{for all } x \in X. \tag{2.4}$$

Then

(2.5)
$$\|P\| = \max\Big\{\max_{k\le n}\{|1-f_kw_k| + (\|f^{(n)}\|_1 - |f_k|)|w_k|\}, \max_{k>n}\{\|f^{(n)}\|_1|w_k|\}\Big\}.$$

Proof. Let $M := \max\{\|f^{(n)}\|_1|w_k| : k > n\}$. Observe that

$$\begin{aligned}\|P\| &= \sup_{\|x\|=1}\Big\{\max\Big\{\max_{k\le n}\Big\{\Big|x_k - \sum_{j=1}^n f_jx_jw_k\Big|\Big\}, \max_{k>n}\Big\{\Big|\sum_{j=1}^n f_jx_j\Big|\,|w_k|\Big\}\Big\}\Big\}\\ &= \max\Big\{\sup_{\|x\|=1}\max_{k\le n}\Big\{\Big|x_k(1-f_kw_k) - \sum_{j=1}^{k-1} f_jx_jw_k - \sum_{j=k+1}^{n} f_jx_jw_k\Big|\Big\}, M\Big\}\\ &= \max\Big\{\max_{k\le n}\Big\{|1-f_kw_k| + \Big(\sum_{j=1}^n |f_j| - |f_k|\Big)|w_k|\Big\}, M\Big\}\\ &= \max\Big\{\max_{k\le n}\{|1-f_kw_k| + (\|f^{(n)}\|_1 - |f_k|)|w_k|\}, \max_{k>n}\{\|f^{(n)}\|_1|w_k|\}\Big\},\end{aligned}$$

as required. ■

THEOREM 2.4. *Let $X := c_0$, $Z := \ker f$ with $f \in S_{l_1}$ such that $f^{(n)} \neq 0$, and $V = Z \cap l_\infty^n$. If $\|f^{(n)}\|_1 < 1/2$ or $\|f^{(n)}\|_1 \le 2\|f^{(n)}\|_\infty$ then there exists a minimal generalized projection in $\mathcal{P}_V(X,Z)$ and $\lambda_Z(V,X) = 1$.*

Proof. First assume that $\|f^{(n)}\|_1 \le 2\|f^{(n)}\|_\infty$. Then there exists $k \in \{1,\dots,n\}$ such that $|f_k| \ge \frac{1}{2}\|f^{(n)}\|_1$. Now let $w := (1/f_k)e_k$ (where $\{e_j\}_{j=1}^\infty$ is the canonical basis of c_0) and

$$P(x) := Q_n(x) - f(Q_n(x))w \quad \text{for } x \in X. \tag{2.6}$$

It is easy to see that $f(w) = 1$ and $P \in \mathcal{P}_V(X,Z)$. According to Lemma 2.3,

$$\|P\| = \max\{1, \|f^{(n)}\|_1/|f_k| - 1\}. \tag{2.7}$$

By assumption $\|f^{(n)}\|_1/|f_k| - 1 \le 2\|f^{(n)}\|_1/\|f^{(n)}\|_1 - 1 = 1$. Hence P is a MGP and $\|P\| = \lambda_Z(V,X) = 1$.

Now assume that $\|f^{(n)}\|_1 < 1/2$. We know that $\|f\|_1 = 1$, so there exists $M \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{M} |f_k| > \frac{1}{2}. \tag{2.8}$$

Define

$$\varphi : [0,1]^{M-n} \ni (\alpha_{n+1},\dots,\alpha_M) \mapsto \sum_{k=n+1}^{M} \frac{\alpha_k}{\|f^{(n)}\|_1}|f_k| \in \mathbb{R}. \tag{2.9}$$

The function φ is continuous, $\varphi(0) = 0$ and $\varphi(1,\dots,1) > 1$, hence there exists $\beta \in [0,1]^{M-n}$ such that $\varphi(\beta) = 1$. Now let $w := (w_1,\dots,w_M,0,\dots)$

where $w_k = 0$ for $k = 1, \dots, n$ and

$$w_k = \frac{\beta_k}{\|f^{(n)}\|_1} \frac{\bar{f}_k}{|f_k|} \quad \text{for } k = n+1, \dots, M.$$

Define the generalized projection $P \in \mathcal{P}_V(X, Z)$ by

$$P(x) := Q_n(x) - f(Q_n(x))w \quad \text{for } x \in X. \tag{2.10}$$

By Lemma 2.3, $\|P\| = \max\{1, \max\{\beta_j : j \in \{n+1, \dots, M\}\}\} = 1$. ∎

EXAMPLE 2.5. Let $f := \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{5}{16}, \frac{5}{32}, \dots\right) \in l_1$, and let $X := c_0$, $Z := \ker f$ and $V := \ker f \cap l_\infty^3$.

Notice that the assumptions of the above theorem are satisfied. Hence e.g. $P_0 := Q_3 - \frac{32}{15} f \circ Q_3(\cdot)(e_4 + e_5)$ is a MGP in $\mathcal{P}_V(X, Z)$ and $\|P_0\| = \lambda_Z(V, X) = 1$. By [3], $P_1 := \mathrm{id} - \frac{8}{3} f \circ Q_3(\cdot)(e_1 + e_2 + e_3)$ is a minimal projection in $\mathcal{P}(X, V)$ and $\|P_1\| = \lambda(V, X) = \frac{4}{3}$, but there does not exist a minimal projection in $\mathcal{P}(X, Z)$, and $\lambda(Z, X) \approx 1.58$.

REMARK 2.6. Let the assumptions of the previous theorem hold. Then there exist more than one MGP in $\widetilde{\mathcal{P}}_V(X, Z)$.

Proof. First assume that there exists $k > n$ such that $f_k = 0$. Let P be a MGP defined as in Theorem 2.4 (P is given by (2.6) when $\|f^{(n)}\|_1 \le 2\|f^{(n)}\|_\infty$ and by (2.10) when $\|f^{(n)}\|_1 < 1/2$). Then $w_k = 0$. Hence $Q := Q_n - f \circ Q_n(\cdot)(w + e_k)$ is also a MGP in $\widetilde{\mathcal{P}}_V(X, Z)$.

Now assume that $f_k \neq 0$ for every $k > n$ and consider two cases.

(i) $\|f^{(n)}\|_1 \le 2\|f^{(n)}\|_\infty$. Let $k \in \{1, \dots, n\}$ be such that $|f_k| = \|f^{(n)}\|_\infty$. Now for every $\alpha \in (0, 1)$ with $\alpha\|f^{(n)}\|_1 \le |f_{n+1}|$ we define

$$Q_\alpha := Q_n - f \circ Q_n(\cdot) y \quad \text{where} \quad y := \frac{1-\alpha}{f_k} e_k + \frac{\alpha}{f_{n+1}} e_{n+1}.$$

Then

$$\|Q_\alpha\| \overset{(2.5)}{=} \max\left\{1, 2\alpha - 1 + (1-\alpha)\frac{\|f^{(n)}\|_1}{|f_k|}, \frac{\alpha\|f^{(n)}\|_1}{|f_{n+1}|}\right\} = 1.$$

Hence for any $\alpha \in (0, 1)$, Q_α is a MGP.

(ii) $\|f^{(n)}\|_1 < 1/2$. It is easy to see that the function φ given by (2.9) is equal to 1 at more than one point. Each such point can be used to define a MGP (cf. (2.10)). Since $f_k \neq 0$ for all $k > n$, these projections are different. ∎

THEOREM 2.7. *Let X, Z, V be as in Theorem 2.4. Assume additionally that $\|f^{(n)}\|_1 \ge 1/2$ and $\|f^{(n)}\|_\infty < \|f^{(n)}\|_1/2$. Then*

$$\lambda_Z(V, X) = \frac{1 + \sum_{k=1}^n \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|}}{\frac{1 - \|f^{(n)}\|_1}{\|f^{(n)}\|_1} + \sum_{k=1}^n \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|}}. \tag{2.11}$$

Proof. Since $\|f^{(n)}\|_\infty < \|f^{(n)}\|_1/2$, we have $\|f^{(n)}\|_1 - 2|f_k| > 0$ for $k \in \{1, \dots, n\}$. For $m > n$ ($m \in \mathbb{N}$) put

$$\lambda_m := \Big(1 + \sum_{k=1}^{n} \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|}\Big)\Big(\frac{\sum_{k=n+1}^{m}|f_k|}{\|f^{(n)}\|_1} + \sum_{k=1}^{n} \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|}\Big)^{-1},$$

and set

$$\lambda := \frac{1 + \sum_{k=1}^{n} \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|}}{\frac{1-\|f^{(n)}\|_1}{\|f^{(n)}\|_1} + \sum_{k=1}^{n} \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|}}.$$

We will construct a sequence $\{P_m\}_{m>n}$ of generalized projections such that $\|P_m\| = \lambda_m$. To do this, fix $m > n$ and define $w \in c_0$ as follows:

$$w_k := \begin{cases} \dfrac{\bar{f}_k}{|f_k|}\,\dfrac{\lambda_m - 1}{\|f^{(n)}\|_1 - 2|f_k|} & \text{for } k = 1, \dots, n,\\[2ex] \dfrac{\bar{f}_k}{|f_k|}\,\dfrac{\lambda_m}{\|f^{(n)}\|_1} & \text{for } k = n+1, \dots m,\\[2ex] 0 & \text{for } k > m. \end{cases}$$

Then

$$\begin{aligned} f(w) &= \sum_{k=1}^{n} |f_k| \frac{\lambda_m - 1}{\|f^{(n)}\|_1 - 2|f_k|} + \sum_{k=n+1}^{m} |f_k| \frac{\lambda_m}{\|f^{(n)}\|_1} \\ &= (\lambda_m - 1)\sum_{k=1}^{n} \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|} + \frac{\lambda_m}{\|f^{(n)}\|_1} \sum_{k=n+1}^{m} |f_k| \\ &= \lambda_m\Big(\sum_{k=1}^{n} \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|} + \frac{\sum_{k=n+1}^{m}|f_k|}{\|f^{(n)}\|_1}\Big) - \sum_{k=1}^{n} \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|} \\ &= 1. \end{aligned}$$

Therefore, the operator

$$P_m(x) := Q_n(x) - f(Q_n(x))w \tag{2.12}$$

is a generalized projection. Since $f_k w_k \geq 0$ and $\sum_{k=1}^{\infty} f_k w_k = 1$, we have $f_k w_k \leq 1$ for all $k \in \mathbb{N}$. Using this observation and Lemma 2.3 we get

$$\begin{aligned} \|P_m\| &= \max\Big\{\max_{k\leq n}\{1 - f_k w_k + \|f^{(n)}\|_1 |w_k| - f_k w_k\}, \lambda_m\Big\} \\ &= \max\Big\{\max_{k\leq n}\{1 - 2 f_k w_k + \|f^{(n)}\|_1 |w_k|\}, \lambda_m\Big\} \\ &= \max\Big\{\max_{k\leq n}\{1 + |w_k|(\|f^{(n)}\|_1 - 2|f_k|)\}, \lambda_m\Big\} = \lambda_m. \end{aligned}$$

It is easy to see that $\lambda_m \to \lambda$ ($m \to \infty$), which shows that $\lambda_Z(V, X) \leq \lambda$. To prove the opposite inequality, suppose that there exists a generalized projection P such that $\|P\| < \lambda$. According to Corollary 2.2 we may assume that P is given by $P(x) = Q_n(x) - f(Q_n(x))y$ for some $y \in c_0$ such that $f(y) = 1$. Using Lemma 2.3, we obtain

$$|1 - f_k y_k| + |y_k|(\|f^{(n)}\|_1 - |f_k|) \le \|P\| \quad \text{for } k \in \{1, \dots, n\},$$

which implies

$$|y_k|(\|f^{(n)}\|_1 - 2|f_k|) \le \|P\| - 1 \quad \text{for } k \in \{1, \dots, n\}.$$

Since $|f_k| < \|f^{(n)}\|_1/2$, we have $|y_k| \le \frac{\|P\|-1}{\|f^{(n)}\|_1 - 2|f_k|}$ for $k \in \{1, \dots, n\}$. Analogously, $|y_k| \le \|P\|/\|f^{(n)}\|_1$ for $k > n$. By the above estimates we get

$$\begin{aligned}
f(y) &\le \sum_{k=1}^{n} |f_k y_k| + \sum_{k=n+1}^{\infty} |f_k y_k| \\
&\le (\|P\| - 1) \sum_{k=1}^{n} \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|} + \frac{\|P\|}{\|f^{(n)}\|_1} \sum_{k=n+1}^{\infty} |f_k| \\
&= \|P\| \bigg(\sum_{k=1}^{n} \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|} + \frac{1 - \|f^{(n)}\|_1}{\|f^{(n)}\|_1} \bigg) - \sum_{k=1}^{n} \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|} \\
&< \lambda \bigg(\sum_{k=1}^{n} \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|} + \frac{1 - \|f^{(n)}\|_1}{\|f^{(n)}\|_1} \bigg) - \sum_{k=1}^{n} \frac{|f_k|}{\|f^{(n)}\|_1 - 2|f_k|} \\
&< 1,
\end{aligned}$$

a contradiction. ∎

Corollary 2.8. *Let the assumptions of the previous theorem hold. Then there exists a MGP in $\mathcal{P}_V(X, Z)$ iff $f \in c_{00}$.*

Proof. By the proof of Theorem 2.7 it is easy to see that if $f \in c_{00}$ (i.e. there exists $m_0 \ge n$ such that $f_k = 0$ for all $k > m_0$) then P_{m_0} given by (2.12) is a MGP.

Now assume conversely that $f \notin c_{00}$ and $P \in \mathcal{P}_V(X, Z)$ is a MGP. By Lemma 2.1 and Corollary 2.2 we can assume that P is given by

$$P(x) = Q_n(x) - f(Q_n(x))w \quad \text{for } x \in X,$$

for some $w \in c_0$ such that $f(w) = 1$. Then equality holds in the last inequality of the proof of Theorem 2.7. This is possible only if $|w_k| = \lambda/\|f^{(n)}\|_1$ for all $k > n$ such that $f_k \ne 0$, which implies that $w \notin c_0$, a contradiction. ∎

Theorem 2.9. *Let $X := l_\infty^m$ over $\mathbb{R}$, $Z := \ker f$ with $f \in S_{l_1^m}$, and $V := Z \cap l_\infty^n$ for fixed $n \le m$. Let $\|f^{(n)}\|_1 \ge 1/2$ and $\|f^{(n)}\|_\infty < \|f^{(n)}\|_1/2$. Assume additionally that $f_j \ne 0$ for all $j \in \{1, \dots, m\}$. Then*

(a) *The MGP given by* (2.12) *is strongly unique in* $\widetilde{\mathcal{P}}_V(X, Z)$ (*see Corollary* 2.2).

(b) *Let*

$$A(f) := \sum_{k=1}^{n} \frac{f_k}{\|f^{(n)}\|_1 - 2|f_k|} + \frac{1 - \|f^{(n)}\|_1}{\|f^{(n)}\|_1}.$$

Then the strong uniqueness constant is given by

$$r = \frac{1}{\|f^{(n)}\|_1} \min\left\{ \frac{|f_{j_0}|}{A(f) - \frac{|f_{j_0}|}{\|f^{(n)}\|_1 - 2|f_{j_0}|}}, \frac{|f_{k_0}|}{A(f) - \frac{|f_{k_0}|}{\|f^{(n)}\|_1}} \right\}, \tag{2.13}$$

where $|f_{j_0}| = \min\{|f_j| : j \in \{1, \dots, n\}\}$ *and* $|f_{k_0}| = \min\{|f_k| : k \in \{n+1, \dots, m\}\}$.

Proof. (a) Since

$$\|P_m\| = \|Q_n - |f| \circ Q_n(\cdot)\widetilde{w}\| \quad \text{where} \quad \widetilde{w} = \left(\frac{w_1 \bar{f}_1}{|f_1|}, \dots, \frac{w_m \bar{f}_m}{|f_m|} \right),$$

we can assume that $f_j > 0$ for all $j \in \{1, \dots, m\}$. By Theorem 1.3 it is enough to prove that there exists $r > 0$ such that for every $Q \in \widetilde{\mathcal{P}}_V(X, Z)$ there exists $k \in \{1, \dots, m\}$ with

$$M(Q, k) := \inf\{((Q - P_m)(x))_k : x \in S_X,\ (P_m x)_k = \|P_m\|\} \le -r\|Q - P_m\|.$$

By the proof of Lemma 2.3 it is easy to see that if $k \in \{1, \dots, n\}$ then

$$(P_m x)_k = \|P_m\| \quad \text{iff} \quad x_k = 1 \text{ and } x_j = -1 \text{ for } j \in \{1, \dots, k-1, k+1, \dots, n\} \tag{2.14}$$

and if $k \in \{n+1, \dots, m\}$ then

$$(P_m x)_k = \|P_m\| \quad \text{iff} \quad x_j = -1 \text{ for } j \in \{1, \dots, n\}. \tag{2.15}$$

We know that there exists $y \in X$ such that $f(y) = 1$ and

$$Q(x) = Q_n(x) - f(Q_n(x))y \quad \text{for all } x \in X.$$

Hence

$$M(Q, k) = \begin{cases} (2f_k - \|f^{(n)}\|_1)(w_k - y_k) & \text{for } k \in \{1, \dots, n\}, \\ -\|f^{(n)}\|_1(w_k - y_k) & \text{for } k \in \{n+1, \dots, m\}. \end{cases} \tag{2.16}$$

Now define a function $\phi := S_X \to \mathbb{R}$ by

$$\phi(x) := \min\Big\{ \min_{k \le n}\{(2f_k - \|f^{(n)}\|_1)x_k\}, \min_{k > n}\{-\|f^{(n)}\|_1 x_k\} \Big\}.$$

Since $0 < f_k < \|f^{(n)}\|_1/2$ for $k = 1, \dots, m$, we have $\phi(x) < 0$ for every $x \in S_X \cap Z$. Because ϕ is continuous and $S_X \cap Z$ is a compact set, the number

$$\hat{r} := -\max\{\phi(x) : x \in S_X \cap Z\}\|f^{(n)}\|_1^{-1} \tag{2.17}$$

is positive. By (2.16) and (2.17) we can choose $k \in \{1, \dots, m\}$ such that

$$M(Q, k) = \phi\left(\frac{w - y}{\|w - y\|_\infty} \right) \|w - y\|_\infty \le -\hat{r}\|w - y\|_\infty \|f^{(n)}\|_1 = -\hat{r}\|Q - P_m\|.$$

(b) First we will show that the constant $\hat{r}$ given by (2.17) is the best possible. Take $r_1 > \hat{r}$. By (2.17) there exists $z \in S_X \cap Z$ such that $\phi(z) > -r_1\|f^{(n)}\|_1$. Now define $Q \in \widetilde{\mathcal{P}}_V(X,Z)$ by

$$Q(x) := P_m(x) + f(Q_n(x))z \quad \text{ for } x \in X.$$

Then

$$M(Q,k) \geq \phi(z) > -r_1\|f^{(n)}\|_1 = -r_1\|Q - P_m\|.$$

as required. Now we will show that $\hat{r} = r$ (where r is given by (2.13)). Let $x \in S_X \cap Z$ yield the maximum in (2.17) and consider four cases.

(i) There exists $i_0 \in \{1,\dots,n\}$ such that $x_{i_0} = 1$. Then

$$\begin{aligned}(2f_{i_0} - \|f^{(n)}\|_1)\bigg(A(f) - \frac{f_{j_0}}{\|f^{(n)}\|_1 - 2f_{j_0}}\bigg) &\leq (2f_{i_0} - \|f^{(n)}\|_1)\frac{f_{i_0}}{\|f^{(n)}\|_1 - 2f_{i_0}} \\ &= -f_{i_0} \leq -f_{j_0}.\end{aligned}$$

Hence

$$\|f^{(n)}\|_1\hat{r} = -\phi(x) \geq \frac{f_{j_0}}{A(f) - \frac{f_{j_0}}{\|f^{(n)}\|_1 - 2f_{j_0}}} \geq \|f^{(n)}\|_1 r.$$

(ii) There exists $i_0 \in \{n+1,\dots,m\}$ such that $x_{i_0} = 1$. Then

$$-\|f^{(n)}\|_1\bigg(A(f) - \frac{f_{k_0}}{\|f^{(n)}\|_1}\bigg) \leq -\|f^{(n)}\|_1\frac{f_{i_0}}{\|f^{(n)}\|_1} = -f_{i_0} \leq -f_{k_0}.$$

Hence

$$\|f^{(n)}\|_1\hat{r} = -\phi(x) \geq \frac{f_{k_0}}{A(f) - \frac{f_{k_0}}{\|f^{(n)}\|_1}} \geq \|f^{(n)}\|_1 r.$$

(iii) There exists $i_0 \in \{1,\dots,n\}$ such that $x_{i_0} = -1$. Assume that $-\phi(x) < r\|f^{(n)}\|_1$. Since $x \in \ker f$,

$$\begin{aligned} f_{i_0} &= \sum_{\substack{1\leq k\leq m\\ k\neq i_0}} f_k x_k \\ &< \frac{f_{j_0}}{A(f) - \frac{f_{j_0}}{\|f^{(n)}\|_1 - 2f_{j_0}}}\bigg(\sum_{\substack{1\leq k\leq n\\ k\neq i_0}} \frac{f_k}{\|f^{(n)}\|_1 - 2f_k} + \sum_{k=n+1}^{m} \frac{f_k}{\|f^{(n)}\|_1}\bigg) \\ &= \frac{f_{j_0}}{A(f) - \frac{f_{j_0}}{\|f^{(n)}\|_1 - 2f_{j_0}}}\bigg(A(f) - \frac{f_{i_0}}{\|f^{(n)}\|_1 - 2f_{i_0}}\bigg) \leq f_{j_0},\end{aligned}$$

a contradiction.

(iv) There exists $i_0 \in \{n+1,\dots,m\}$ such that $x_{i_0} = -1$. Assume that $-\phi(x) < r\|f^{(n)}\|_1$. Since $x \in \ker f$,

$$f_{i_0} = \sum_{\substack{1\le k\le m\\ k\neq i_0}} f_k x_k < \frac{f_{k_0}}{A(f) - \frac{f_{k_0}}{\|f^{(n)}\|_1}} \Big(\sum_{k=1}^{n} \frac{f_k}{\|f^{(n)}\|_1 - 2f_k} + \sum_{\substack{n+1\le k\le m\\ k\neq i_0}} \frac{f_k}{\|f^{(n)}\|_1} \Big)$$
$$= \frac{f_{k_0}}{A(f) - \frac{f_{k_0}}{\|f^{(n)}\|_1}} \Big(A(f) - \frac{f_{i_0}}{\|f^{(n)}\|_1} \Big) \le f_{k_0},$$

a contradiction.

By the above cases we have $\hat{r} \ge r$. Now define

$$x_k := \begin{cases} -1, & k = j_0,\\ \dfrac{f_{j_0}}{(\|f^{(n)}\|_1 - 2f_k)\big(A(f) - \frac{f_{j_0}}{\|f^{(n)}\|_1 - 2f_{j_0}}\big)}, & k \in \{1,\dots,j_0-1, j_0+1,\dots,n\},\\ \dfrac{f_{j_0}}{\|f^{(n)}\|_1\big(A(f) - \frac{f_{j_0}}{\|f^{(n)}\|_1 - 2f_{j_0}}\big)}, & k \in \{n+1,\dots,m\}, \end{cases}$$

and

$$y_k := \begin{cases} -1, & k = k_0,\\ \dfrac{f_{k_0}}{(\|f^{(n)}\|_1 - 2f_k)\big(A(f) - \frac{f_{j_0}}{\|f^{(n)}\|_1}\big)}, & k \in \{1,\dots,n\},\\ \dfrac{f_{k_0}}{\|f^{(n)}\|_1\big(A(f) - \frac{f_{j_0}}{\|f^{(n)}\|_1}\big)}, & k \in \{n+1,\dots,k_0-1,\dots,k_0+1,\dots m\}. \end{cases}$$

One can easily check that $x, y \in S_X \cap Z$ and

$$-\phi(x) = \frac{f_{j_0}}{A(f) - \frac{f_{j_0}}{\|f^{(n)}\|_1 - 2f_{j_0}}} \quad \text{and} \quad -\phi(y) = \frac{f_{k_0}}{A(f) - \frac{f_{k_0}}{\|f^{(n)}\|_1}},$$

which implies the converse inequality. ∎

Now we consider the complex case.

THEOREM 2.10. *Let $X := l_\infty^m$ over $\mathbb{C}$, $Z := \ker f$ with $f \in S_{l_1^m}$, and $V := Z \cap l_\infty^m$. Let $\|f^{(n)}\|_1 \ge 1/2$ and $\|f^{(n)}\|_\infty < \|f^{(n)}\|_1/2$. Assume additionally that $f_j \neq 0$ for all $j \in \{1,\dots,m\}$. Then the operator given by (2.12) is the only MGP in $\widetilde{\mathcal{P}}_V(X,Z)$ but it is not strongly unique.*

Proof. Without loss of generality we can assume that $f_1,\dots,f_m > 0$. In the complex case we define

$$M(Q,k) := \inf\{\operatorname{Re}((Q - P_m)(x))_k : x \in S_X,\ (P_m x)_k = \|P_m\|\},$$

where $P_m = Q_n + f \circ Q_n(\cdot)w$ is given by (2.12). As in the proof of Theorem 2.9, we can show that

$$M(Q,k) = \begin{cases} \mathrm{Re}((2f_k - \|f^{(n)}\|_1)(w_k - y_k)) & \text{for } k \in \{1,\dots,n\}, \\ -\|f^{(n)}\|_1 \mathrm{Re}(w_k - y_k) & \text{for } k \in \{n+1,\dots,m\}. \end{cases}$$

To prove that P_m is not a strongly unique MGP in $\widetilde{\mathcal{P}}_V(X,Z)$ it is enough to find $Q \in \widetilde{\mathcal{P}}_V(X,Z)$ such that $M(Q,k) = 0$ for every $k \in \{1,\dots,m\}$. Let $y := w + (i/f_1, -i/f_2, 0, \dots, 0)$ and $Q := Q_n + f \circ Q_n(\cdot)y$. Then one can easily check that $M(Q,k) = 0$.

Now we will show that P_m is the unique MGP in $\widetilde{\mathcal{P}}_V(X,Z)$. Let $Q \in \widetilde{\mathcal{P}}_V(X,Z)$ be such that $\|Q\| = \|P_m\|$. Then $Q(x) = Q_n(x) - f(Q_n(x))y$ for some $y \in l_\infty^m$ such that $f(y) = 1$. Observe that $\widetilde{Q} := Q_n - f \circ Q_n(\cdot)\,\mathrm{Re}(y)$ is also an element of $\widetilde{\mathcal{P}}_V(X,Z)$ and $\|\widetilde{Q}\| = \|Q\|$. Indeed, $1 = f(y) = \mathrm{Re}(f(y)) = f(\mathrm{Re}(y))$ and

$$\begin{aligned} \|\widetilde{Q}\| &= \max\Big\{\max_{k\le n}\{|\mathrm{Re}(1 - f_k y_k)| + (\|f^{(n)}\|_1 - |f_k|)|\mathrm{Re}(y_k)|\}, \\ &\qquad\qquad \max_{k>n}\{\|f^{(n)}\|_1|\mathrm{Re}(y_k)|\}\Big\} \\ &\le \max\Big\{\max_{k\le n}\{|1 - f_k y_k| + (\|f^{(n)}\|_1 - |f_k|)|y_k|\}, \max_{k>n}\{\|f^{(n)}\|_1|y_k|\}\Big\} \\ &= \|Q\|. \end{aligned}$$

When we consider l_∞^m over $\mathbb{R}$, then $\widetilde{Q}$ is also a MGP. Indeed, by Lemma 2.3, $\lambda(V, l_\infty^m(\mathbb{R})) \le \lambda(V, l_\infty^m(\mathbb{C})) = \|\widetilde{Q}\|_{\mathbb{C}} = \|\widetilde{Q}\|_{\mathbb{R}}$. Hence by Theorem 2.9, $\mathrm{Re}(y) = w$ and by the proof of Theorem 2.7, for all $k \in \{1,\dots,n\}$,

$$\begin{aligned} \lambda_Z(V,X) &= |\mathrm{Re}(1 - f_k y_k)| + (\|f^{(n)}\|_1 - |f_k|)|\mathrm{Re}(y_k)| \\ &= |1 - f_k y_k| + (\|f^{(n)}\|_1 - |f_k|)|y_k|, \end{aligned}$$

and for $k \in \{n+1,\dots,m\}$,

$$\lambda_Z(V,X) = \|f^{(n)}\|_1|\mathrm{Re}(y_k)| = \|f^{(n)}\|_1|y_k|.$$

This implies that $y = \mathrm{Re}(y) = w$, as required. ∎

REMARK 2.11. If $\|P_m\| > 1$ then in Theorems 2.9 and 2.10 the assumption that $f_j \ne 0$ for $j \in \{1,\dots,m\}$ is necessary.

Proof. Let $k \in \{1,\dots,m\}$ be such that $f_k = 0$. Let $P_m = Q_n - f \circ Q_n(\cdot)w$ be as in Theorem 2.7. Now define $Q_\varepsilon = Q_n - f \circ Q_n(\cdot)(w + \varepsilon e_k)$. If $k \le n$ then

$$\|Q_\varepsilon\| = \max\{\|P_m\|, 1 + \varepsilon\|f^{(n)}\|_1\} = \|P_m\| \quad \text{for } 0 < \varepsilon \le \frac{\|P_m\| - 1}{\|f^{(n)}\|_1},$$

and if $k > n$ then

$$\|Q_\varepsilon\| = \max\{\|P_m\|, \varepsilon\|f^{(n)}\|_1\} = \|P_m\| \quad \text{for } 0 < \varepsilon \le \frac{\|P_m\|}{\|f^{(n)}\|_1}.$$

Corollary 2.12. *Let $X := l_\infty^m$, $Z := \ker f$ with $f \in S_{l_1^m}$, and $V := Z \cap l_\infty^m$. Let $\|f^{(n)}\|_1 \ge 1/2$ and $\|f^{(n)}\|_\infty < \|f^{(n)}\|_1/2$. Assume additionally that $f_j \neq 0$ for all $j \in \{1, \dots, m\}$. Then the MGP given by* (2.12) *is unique in $\mathcal{P}_V(X, Z)$.*

Proof. We know that P_m given by (2.12) is a MGP. Let $Q \in \mathcal{P}_V(X, Z)$ be such that $\|Q\| = \|P_m\|$. By Theorems 2.9 and 2.10, $Q \circ Q_n = P_m$. Now let $\{e_k\}_{k=1}^m$ be the canonical basis of l_∞^m. By the proof of Theorem 2.7 we know that for every $k \in \{1, \dots, m\}$ there exists $x^k \in S_{l_\infty^n}$ such that $(P_m(x^k))_k = \|P_m\|$. Now fix $j \in \{n+1, \dots, m\}$. Then $\|x^k + \alpha e_j\| = 1$ for some $\alpha \in \mathbb{K}$ such that $|\alpha| = 1$, and

$$\|Q\| \ge \max\{(Q(x^k + \alpha e_j))_k\} = \|P_m\| + |Q(e_j)_k| \quad \text{for all } k \in \{1, \dots, m\}.$$

Hence $Q(e_j) = 0$, as required. ∎

Now we present an application of Theorem 2.7 and Corollary 2.8 .

Reasoning as in [12] or [14], we first prove the following.

Theorem 2.13. *Let $V \subset Z$ be two subspaces of a Banach space X. Let Z be a dual space. If $\mathcal{P}_V(X, Z) \neq \emptyset$ then there exists a minimal generalized projection.*

Proof. Let

$$\mathcal{P}_\varepsilon := \{P \in \mathcal{P}_V(X, Z) : \|P\| \le \lambda_Z(V, X) + \varepsilon\}.$$

Observe that $\mathcal{P}_\varepsilon$ is closed in $Z^X = \prod_{x \in X} Z$, where in Z^X we take the product topology with respect to the weak* topology in Z. By the Banach–Alaoglu theorem and the Tychonoff theorem, $\mathcal{P}_\varepsilon$ is compact. Hence $\mathcal{P}_0 = \bigcap_{n \in \mathbb{N}} \mathcal{P}_{1/n}$ is nonempty as the intersection of a decreasing family of nonempty compact sets. Notice that any $P \in \mathcal{P}_0$ is a MGP in $\mathcal{P}_V(X, Z)$. ∎

Observe that the assumption that Z is a dual space cannot be replaced by the assumption that V is dual, as the following example shows.

Example 2.14. Let $f := \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right) \in l_1$ and let $X := c_0$, $Z := \ker f$, $V := \ker f \cap l_\infty^3$.

Since V is finite-dimensional, it is a dual space. Applying Theorem 2.7 and Corollary 2.8 one can easily check that $\mathcal{P}_V(X, Z) \neq \emptyset$ but no MGP exists in $\mathcal{P}_V(X, Z)$.

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