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Variability regions of close-to-convex functions

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Abstract. M. Biernacki gave in 1936 concrete forms of the variability regions of $z/f(z)$ and $zf'(z)/f(z)$ of close-to-convex functions f for a fixed z with $|z| < 1$. The forms are, however, not necessarily convenient to determine the shape of the full variability region of $zf'(z)/f(z)$ over all close-to-convex functions f and all points z with $|z| < 1$. We propose a couple of other forms of the variability regions and see that the full variability region of $zf'(z)/f(z)$ is indeed the complex plane minus the origin. We also apply them to study the variability regions of $\log[z/f(z)]$ and $\log[zf'(z)/f(z)]$.

1. Introduction. Let $\mathcal{A}$ denote the class of analytic functions f on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}_0$ and $\mathcal{A}_1$ be its subclasses described by the conditions $f(0) = 1$ and $f(0) = f'(0) - 1 = 0$, respectively. Traditionally, the subclass of $\mathcal{A}_1$ consisting of univalent functions is denoted by $\mathcal{S}$. A function f in $\mathcal{A}_1$ is called *starlike* (resp. *convex*) if f is univalent and if $f(\mathbb{D})$ is starlike with respect to 0 (resp. convex). It is well known that $f \in \mathcal{A}_1$ is starlike (resp. convex) precisely when $\operatorname{Re}[zf'(z)/f(z)] > 0$ (resp. $\operatorname{Re}[1+zf''(z)/f'(z)] > 0$) in $|z| < 1$. The classes of starlike and convex functions in $\mathcal{A}_1$ will be denoted by $\mathcal{S}^*$ and $\mathcal{K}$ respectively.

A function $f \in \mathcal{A}_1$ is called *close-to-convex* if $\operatorname{Re}[e^{i\lambda} f'(z)/g'(z)] > 0$ in $|z| < 1$ for a convex function $g \in \mathcal{K}$ and a real constant λ. The set of close-to-convex functions in $\mathcal{A}_1$ will be denoted by $\mathcal{C}$. This class was first introduced and shown to be contained in $\mathcal{S}$ by Kaplan [7]. A domain is called *close-to-convex* if it is expressed as the image of $\mathbb{D}$ under the mapping $af + b$ for some $f \in \mathcal{C}$ and constants $a, b \in \mathbb{C}$ with $a \neq 0$. He also gave a geometric characterization of a close-to-convex domain in terms of turning of the boundary of the domain. We recommend the books [4], [5] and [13] as general references on these topics.

Prior to the work of Kaplan, Biernacki [2] introduced the notion of linearly accessible domains (in the strict sense). Here, a domain in $\mathbb{C}$ is called

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linearly accessible if its complement is a union of half-lines which do not cross each other. Lewandowski [11], [12] proved that the class of close-to-convex domains is identical with that of linearly accessible domains (see also [1] and [9] for simpler proofs of this fact). Therefore, the work of Biernacki on linearly accessible domains and their mapping functions can now be interpreted as concerning close-to-convex domains and functions.

For a non-vanishing function g in $\mathcal{A}_0$, unless otherwise stated, $\log g$ will mean the continuous branch of $\log g$ in $\mathbb{D}$ determined by $\log g(0) = 0$. For instance, $f(z)/z$ can be regarded as a non-vanishing function in $\mathcal{A}_0$ for $f \in \mathcal{S}$. Therefore, we can define $\log[f(z)/z]$ in the above sense. In the present note, we are interested in the following variability regions for a fixed $z \in \mathbb{D}$:

$$\begin{aligned} U_z &= \{z/f(z) : f \in \mathcal{C}\}, & LU_z &= \{\log(z/f(z)) : f \in \mathcal{C}\},\\ V_z &= \{f'(z) : f \in \mathcal{C}\}, & LV_z &= \{\log f'(z) : f \in \mathcal{C}\},\\ W_z &= \{zf'(z)/f(z) : f \in \mathcal{C}\}, & LW_z &= \{\log(zf'(z)/f(z)) : f \in \mathcal{C}\}. \end{aligned}$$

We collect basic properties of these sets.

Lemma 1.1.

(1) *X_z is a compact subset of $\mathbb{C}$ for $z \in \mathbb{D}$ and $X = U, V, W, LU, LV, LW$.*
(2) *$X_z = \exp(LX_z)$ for $z \in \mathbb{D}$ and $X = U, V, W$.*
(3) *$X_z = X_r$ for $|z| = r < 1$ and $X = U, V, W, LU, LV, LW$.*
(4) *$X_r \subset X_s$ for $0 \le r < s < 1$ and $X = U, V, W, LU, LV, LW$.*

Proof. It is enough to outline the proof since the reader can fill in the details easily. Assertion (1) follows from compactness of the family $\mathcal{C}$, whereas (2) is immediate by definition. To see (3) and (4), it is enough to show that $X_z \subset X_w$ for $|z| \le |w| < 1$. This follows from the fact that for $f \in \mathcal{C}$ and $a \in \mathbb{C}$ with $0 < |a| \le 1$ the function $f_a(z) = f(az)/a$ belongs to $\mathcal{C}$ again. ■

We remark that we can indeed show the stronger inclusion relation $X_r \subset \operatorname{Int} X_s$ for $0 \le r < s < 1$ by considering extremal functions corresponding to boundary points of X_r. Here, $\operatorname{Int} E$ means the set of interior points of a subset E of $\mathbb{C}$. However, we do not use this property in what follows.

Set $X_{1^-} = \bigcup_{0 \le r < 1} X_r$ for $X = U, V, W, LU, LV, LW$. Below, $\mathbb{D}(a, r)$ will stand for the open disk $|z - a| < r$ in $\mathbb{C}$ and $\overline{\mathbb{D}}(a, r)$ will stand for its closure, that is, the closed disk $|z - a| \le r$.

Biernacki [2] described U_z and W_z in his study on linearly accessible domains and their mapping functions. The results can be summarized as in the following.

Lemma 1.2 (Biernacki (1936)). *For $0 < r < 1$, the following hold:*

(1) $U_r = \{(1+s)^2/(1+(s+t)/2) : |s| \le r,\ |t| \le r\} = \{2u^2/(u+v) : |u-1| \le r, |v-1| \le r\}$.

(2) $W_r = (1-r^2)^{-2}U_r$.
(3) $U_{1^-} = \mathbb{D}(1,3) \setminus \{0\}$ *and* $LU_{1^-} \subset \{w \in \mathbb{C} : |\operatorname{Im} w| < 3\pi/2\}$.
(4) $LW_{1^-} \subset \{w \in \mathbb{C} : |\operatorname{Im} w| < 3\pi/2\}$.

The above expressions of U_r and W_r are simple but somewhat implicit. For instance, the parametrization of the boundary curve cannot be obtained immediately and the shape of the limit W_{1^-} is not clear (as we will see below, this set is equal to $\mathbb{C} \setminus \{0\}$). Therefore, it would be nice to have more explicit or more convenient expressions of U_r and W_r. We propose two such expressions in the present note.

THEOREM 1.3. *For* $0 < r < 1$, $U_r = F(\overline{\mathbb{D}}(0,r))$, *where*

$$F(z) = \frac{(3+\bar{z})(1+z)^3}{3+3z+\bar{z}+z^2}, \qquad z \in \mathbb{D}.$$

We will prove the theorem by describing explicitly the envelope of the family of circles $M_s(\partial\mathbb{D}(0,r))$ for $s = re^{i\theta}$, $0 \le \theta < 2\pi$, where M_s is the Möbius transformation $t \mapsto (1+s)^2/(1+(s+t)/2)$. Lewandowski [12, p. 45] used the envelope to prove that the inclusion $U_r \subset \{w \in \mathbb{C} : \operatorname{Re} w \ge 0\}$ (equivalently, $W_r \subset \{w \in \mathbb{C} : \operatorname{Re} w \ge 0\}$) is valid precisely when $r \le 4\sqrt{2}-5$. (This implies that the radius of starlikeness of close-to-convex functions is $4\sqrt{2}-5$.) However, no explicit form of the envelope is given in [12] because it is not necessary for the results there.

We note that $F(e^{i\theta}) = 1 + 3e^{i\theta}$ for $\theta \in \mathbb{R}$, which agrees with Lemma 1.2(3). This does not, however, give enough information to determine the boundary curve of the domain LU_{1^-}. It turns out that LU_{1^-} has a relatively simple description though LU_r does not. We indeed derive the following result by making use of Theorem 1.3.

THEOREM 1.4. *The variability region* LU_{1^-} *is an unbounded Jordan domain with the boundary curve* $\gamma(t)$, $-2\pi < t < 2\pi$, *given by*

$$\gamma(t) = \begin{cases} \operatorname{Log}(1+3e^{it}) & \text{if } |t| < \pi, \\ \operatorname{Log}(1-e^{it}) + \dfrac{t}{|t|}\pi i & \text{if } \pi \le |t| < 2\pi. \end{cases}$$

Here and hereafter, $\operatorname{Log} w = \log|w| + i \operatorname{Arg} w$ denotes the principal branch of $\log w$ with $-\pi < \operatorname{Im} \operatorname{Log} w = \operatorname{Arg} w \le \pi$.

As we will see in the next section, the function $\log F$ is univalent in $\mathbb{D}$. Therefore, the last theorem tells us that $F : \mathbb{D} \to U_{1^-}$ covers the disk $\mathbb{D}(-1,1)$ bivalently whereas it covers $\mathbb{D}(1,3) \setminus (\mathbb{D}(-1,1) \cup \{0\})$ univalently. See Figure 1 for the mapping behavior of F and $G = \log F$.

The following expression of W_r is not very explicit but useful in some situations.

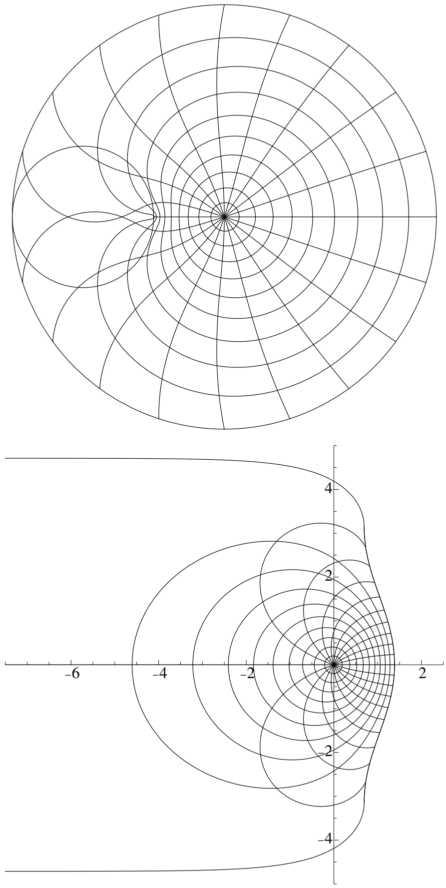

Fig. 1. U_{1^-} and LU_{1^-} as the images of $\mathbb{D}$ under F and G

THEOREM 1.5. *For* $0 < r < 1$,

$$W_r = \left\{ \frac{2u}{v(u+v)} : |u-1| \le r,\ |v-1| \le r \right\}.$$

Indeed, as an application of the last theorem, we can show the following result.

THEOREM 1.6. $LW_{1^-} = \{w \in \mathbb{C} : |\mathrm{Im}\, w| < 3\pi/2\}$.

Since $W_{1^-} = \exp(LW_{1^-})$, we obtain the following corollary, which was used in [8].

COROLLARY 1.7. *The full variability region* $\{zf'(z)/f(z) : z \in \mathbb{D}, f \in \mathcal{C}\}$ *is equal to* $\mathbb{C} \setminus \{0\}$.

The corollary means that $W_{1^-} = \mathbb{C} \setminus \{0\}$. We note here that this does not seem to follow immediately from Lemma 1.2.

Krzyż [10] showed that LV_r is convex and determined its shape for $0 < r < 1$.

Proposition 1.8 (Krzyż). *For $0 < r < 1$, the variability region LV_r is convex and its boundary is described by the curve $\sigma_r(t) = \log[(1 - re^{i\theta_2(t)})/(1 - re^{i\theta_1(t)})^3]$, $-\pi \le t \le \pi$. Here,*

$$\theta_1(t) = t - \arcsin(r \sin t), \qquad \theta_2(t) = \pi + t + \arcsin(r \sin t).$$

He also proved that LV_r is contained in the domain $|\mathrm{Im}\, w| < 4 \arcsin r$ for each $0 < r < 1$ and that this bound is sharp. (See also [5, Chap. 11].) In particular, $LV_{1^-} \subset \{w : |\mathrm{Im}\, w| < 2\pi\}$. Since $\mathrm{Re}\, \sigma_r(t) \to +\infty$ as $r \to 1^-$ for $|t| < \pi/2$ and $\mathrm{Re}\, \sigma_r(t) \to -\infty$ as $r \to 1^-$ for $\pi/2 < |t| \le \pi$, it is not easy to determine the limiting shape of LV_r as $r \to 1^-$ from the above proposition. We thus complement Krzyż' results by showing the following.

Theorem 1.9.

$$V_r = \{(1+s)/(1+t)^3 : |s| \le r,\ |t| \le r\} = \{u/v^3 : |u-1| \le r,\ |v-1| \le r\}$$

for $0 < r < 1$. Moreover, $LV_{1^-} = \{w : |\mathrm{Im}\, w| < 2\pi\}$ and $V_{1^-} = \mathbb{C} \setminus \{0\}$.

One might expect that LU_r and LW_r would also be convex for each $0 < r < 1$. This, however, is not true.

Theorem 1.10. *The variability regions LU_r and LW_r are closed Jordan domains for each $0 < r < 1$. Moreover, there exists a number $0 < r_0 < 1$ such that both LU_r and LW_r are non-convex for every r with $r_0 < r < 1$.*

We prove the above results in Section 3. Section 2 will be devoted to the study of mapping properties of the function $G = \log F$ that are necessary to show our results.

2. Univalence of the function $G = \log F$. In order to analyze the shape of LU_r or LW_r, we need to investigate mapping properties of the functions F and $G = \log F$, where F is given in Theorem 1.3. Therefore, before showing the main results from the Introduction, we exhibit the basic properties of the functions F and G. Here, we remark that F can be expressed in the form

$$F(z) = \frac{(1+z)^3}{1 + z(3+z)/(3+\bar{z})}.$$

Therefore, the continuous branch G of $\log F$ with $G(0) = 0$ is represented by

$$G(z) = 3\,\mathrm{Log}(1+z) - \mathrm{Log}(1 + ze^{2i\phi}), \qquad \phi = \mathrm{Arg}(3+z). \tag{2.1}$$

The goal in this section is to prove the following:

Theorem 2.1. *The function $G = \log F$ is a homeomorphism of the unit disk $\mathbb{D}$ onto a domain in $\mathbb{C}$.*

For $r \in (0,1)$ and $x \in (0,\pi)$, we set

$$\Phi_r(x) = \operatorname{Arg}(1 + re^{ix}).$$

We will use the following elementary properties of the function Φ_r.

LEMMA 2.2. *Let $r \in (0,1)$. Then*

$$\Phi_r'(x) = \frac{r(r+\cos x)}{1+2r\cos x + r^2}, \qquad x \in (0,\pi).$$

In particular, Φ_r is increasing in $(0, x_r)$ and decreasing in (x_r, π), where $x_r = \pi - \arccos r$. Furthermore, Φ_r' is decreasing in $(0,\pi)$ and therefore Φ_r is concave in $(0,\pi)$.

We also need the following information.

LEMMA 2.3. *Let $0 < r < 1$. Then the inequalities $0 < \theta + 2\phi < \pi$ hold for $0 < \theta < \pi$ and $\phi = \operatorname{Arg}(3 + re^{i\theta})$.*

Proof. Set $h_r(\theta) = \theta + 2\phi = \theta + 2\operatorname{Arg}(3 + re^{i\theta})$, $0 \le \theta \le \pi$. Then

$$\begin{aligned} h_r'(\theta) &= 1 + 2\frac{\partial \phi}{\partial \theta} = \frac{3(3 + 4r\cos\theta + r^2)}{9 + 6r\cos\theta + r^2} \\ &\ge \frac{3(3 - 4r + r^2)}{9 + 6r\cos\theta + r^2} = \frac{3(1-r)(3-r)}{9 + 6r\cos\theta + r^2} > 0. \end{aligned}$$

Therefore, h_r is increasing in $(0,\pi)$, which implies that $0 = h_r(0) < h_r(\theta) < h_r(\pi) = \pi$ for $0 < \theta < \pi$. ∎

As for the function G, we first show its local univalence.

LEMMA 2.4. *The function G is orientation-preserving and locally univalent in $\mathbb{D}$.*

Proof. The partial derivatives of G are given by

$$G_z(z) = \frac{6 + 4z + 3\bar{z} + z^2}{(1+z)(3 + 3z + \bar{z} + z^2)}, \tag{2.2}$$

$$G_{\bar{z}}(z) = \frac{z(3+z)}{(3+\bar{z})(3 + 3z + \bar{z} + z^2)}. \tag{2.3}$$

It suffices to show that the Jacobian $J_G = |G_z|^2 - |G_{\bar{z}}|^2$ is positive in $\mathbb{D}$, which is equivalent to the condition $|6 + 4z + 3\bar{z} + z^2| > |z(1+z)|$ in $|z| < 1$. If we write $z = re^{i\theta}$, then

$$\begin{aligned} &|6 + 4z + 3\bar{z} + z^2|^2 - |z(1+z)|^2 \\ &\qquad = 6(6 + 14r\cos\theta + 4r^2 + 6r^2\cos 2\theta + r^3\cos\theta + r^3\cos 3\theta) \\ &\qquad = 12(1 + r\cos\theta)\big(2(1 + r\cos\theta)^2 + 1 - r^2\big) > 0. \ ∎ \end{aligned}$$

LEMMA 2.5. *For a fixed $0 < r < 1$, the real part of $G(re^{i\theta})$ is a decreasing function of θ in $[0,\pi]$.*

Proof. Letting $z = re^{i\theta}$, we first have the expression

$$\frac{\partial}{\partial\theta}G(re^{i\theta}) = izG_z(z) - i\bar{z}G_{\bar{z}}(z) = \frac{3iz(6+4z+4\bar{z}+z^2+\bar{z}^2)}{P(z)}.$$

Here, we have put

$$P(z) = (1+z)(3+\bar{z})(3+3z+\bar{z}+z^2)$$

and used (2.2) and (2.3). Thus,

$$\frac{\partial}{\partial\theta}\operatorname{Re}G(re^{i\theta}) = -6(3+4r\cos\theta + r^2\cos 2\theta)\operatorname{Im}\frac{z}{P(z)}.$$

Since

$$\begin{aligned}\operatorname{Im}\frac{z}{P(z)} &= \frac{1}{2i}\left(\frac{z}{P(z)} - \frac{\bar{z}}{P(\bar{z})}\right)\\ &= \frac{(z-\bar{z})(9+12\operatorname{Re}z + 2\operatorname{Re}z^2 - 2|z|^2(1+2\operatorname{Re}z) - |z|^4)}{2i|P(z)|^2},\end{aligned}$$

we obtain the expression

$$\frac{\partial}{\partial\theta}\operatorname{Re}G(re^{i\theta}) = -\frac{6r\sin\theta(3+4r\cos\theta+r^2\cos 2\theta)H(r,\theta)}{|P(re^{i\theta})|^2},$$

where

$$H(r,\theta) = 9 + 12r\cos\theta - 4r^2\sin^2\theta - 4r^3\cos\theta - r^4.$$

Firstly, we note that

$$3+4r\cos\theta + r^2\cos 2\theta = 1 - r^2 + 2(1+r\cos\theta)^2 > 0.$$

Secondly, we compute

$$\frac{\partial H(r,\theta)}{\partial\theta} = -4r(3+2r\cos\theta - r^2)\sin\theta.$$

In particular, $H(r,\theta)$ is decreasing as a function of $\theta \in (0,\pi)$ for each fixed $0 < r < 1$. Hence,

$$H(r,\theta) \geq H(r,\pi) = (1-r)(3-r)(3-r^2) > 0.$$

Summarizing, we conclude that $\partial(\operatorname{Re}G)/\partial\theta < 0$ for $0 < \theta < \pi$. ∎

We next prove the following:

LEMMA 2.6. $\operatorname{Im}G(z) > 0$ *for* $z \in \mathbb{D}$ *with* $\operatorname{Im}z > 0$.

Proof. Fix $r \in (0,1)$. Let $\theta \in (0,\pi)$ and let ϕ be given in (2.1). Note that $0 < \phi < \theta$. We need to show that

$$g_r(\theta) = \operatorname{Im}G(re^{i\theta}) = 3\Phi_r(\theta) - \Phi_r(\theta+2\phi) \tag{2.4}$$

is positive. Note that $0 < \theta < \theta + 2\phi < \pi$ by Lemma 2.3. Lemma 2.2 implies that Φ_r takes its maximum value $\Phi_r(x_r) = \arcsin r$ in $(0,\pi)$. In particular, we have $\Phi_r(\theta+2\phi) \leq \arcsin r$. Therefore, $g_r(\theta) > 0$ for $x_r^- < \theta < x_r^+$, where x_r^- and x_r^+ are the solutions to the equation $3\Phi_r(x) = \arcsin r$ with $0 < x_r^- < x_r < x_r^+ < \pi$.

We next assume that $x_r^+ \le \theta < \pi$. Since Φ_r is decreasing in (x_r^+, π) by Lemma 2.2, we see that $g_r(\theta) > \Phi_r(\theta) - \Phi_r(\theta + 2\phi) > 0$. Finally, we assume that $0 < \theta \le x_r^-$. In view of concavity of Φ_r (see Lemma 2.2) together with $\Phi_r(0) = 0$, we have $\Phi_r(x_r^-) = \Phi_r(x_r)/3 \le \Phi_r(x_r/3)$. Hence, $x_r^- \le x_r/3$. In particular, $\theta + 2\phi \le 3\theta \le 3x_r^- \le x_r$. Since Φ_r is increasing and concave in $(0, x_r)$, the inequalities $\Phi_r(\theta + 2\phi) < \Phi_r(3\theta) \le 3\Phi_r(\theta)$ follow. Thus we have shown that $g_r(\theta) > 0$ in this case, too. ∎

The following property will be used for the proof of Theorem 1.4.

Lemma 2.7. *The function g_r defined in* (2.4) *satisfies $g_r'(\theta) > 0$ for $0 < \theta < x_r = \pi - \arccos r$.*

Proof. By definition, we have

$$g_r'(\theta) = 3\Phi_r'(\theta) - \left(1 + 2\frac{\partial\phi}{\partial\theta}\right)\Phi_r'(\theta + 2\phi).$$

Since $1 + 2\partial\phi/\partial\theta > 0$ (see the proof of Lemma 2.3), Lemma 2.2 implies

$$g_r'(\theta) \ge 3\Phi_r'(\theta) - \left(1 + 2\frac{\partial\phi}{\partial\theta}\right)\Phi_r'(\theta) = 2\left(1 - \frac{\partial\phi}{\partial\theta}\right)\Phi_r'(\theta).$$

We note here that

$$1 - \frac{\partial\phi}{\partial\theta} = \frac{3(3 + r\cos\theta)}{9 + 6r\cos\theta + r^2} > 0.$$

Since $\Phi_r'(\theta) > 0$ for $0 < \theta < x_r$ by Lemma 2.2, the required assertion follows. ∎

We are now ready to prove the theorem.

Proof of Theorem 2.1. Since G is orientation-preserving and locally univalent by Lemma 2.4, it is enough to show that G is injective on the circle $|z| = r$ for each $r \in (0, 1)$. We note here that G is symmetric in the real axis, in other words, $G(\bar{z}) = \overline{G(z)}$ for $z \in \mathbb{D}$. By Lemmas 2.5 and 2.6, G maps the upper half of the circle $|z| = r$ univalently onto a Jordan arc in the upper half-plane. Taking into account symmetry, we have shown that G maps the circle $|z| = r$ univalently onto a Jordan curve which is symmetric in the real axis. Thus the proof is complete. ∎

3. Proof of main results. In this section, we prove the main results presented in Section 1. In the proofs, we will make use of a weakened version of Lemma 5.1 in Greiner and Roth [6], which is an outcome of the duality methods developed by Ruscheweyh and Sheil–Small.

For complex numbers a, b with $|a| \le 1$, $|b| \le 1$, define a function $f_{a,b} \in \mathcal{A}_1$ by

$$f_{a,b}(z) = z\frac{1+(a+b)z/2}{(1+bz)^2}.$$

It is easy to see that $f_{a,b}$ belongs to the class $\mathcal{C}$ of close-to-convex functions. The linear space $\mathcal{A}$ is naturally equipped with the topology of uniform convergence on compact subsets in $\mathbb{D}$.

LEMMA 3.1 ([6, Lemma 5.1]). *Let λ_1 and λ_2 be continuous linear functionals on $\mathcal{A}$ such that λ_2 does not vanish on $\mathcal{C}$. Then for every $f \in \mathcal{C}$ there exist complex numbers a, b with $|a|, |b| \le 1$ such that*

$$\frac{\lambda_1(f)}{\lambda_2(f)} = \frac{\lambda_1(f_{a,b})}{\lambda_2(f_{a,b})}.$$

As an application of this lemma, one can give a simple proof of Theorem 1.5.

Proof of Theorem 1.5. Fix $0 < r < 1$. Let $f \in \mathcal{C}$. We now apply Lemma 3.1 to the choice $\lambda_1(f) = rf'(r)$ and $\lambda_2(f) = f(r)$ to see that

$$\frac{rf'(r)}{f(r)} = \frac{\lambda_1(f)}{\lambda_2(f)} = \frac{\lambda_1(f_{a,b})}{\lambda_2(f_{a,b})} = \frac{2(1+ar)}{(1+br)(2+(a+b)r)}$$

for some $a, b \in \overline{\mathbb{D}}$. The proof is completed by letting $u = 1 + ar$ and $v = 1 + br$. ∎

We need a topological observation for the proof of Theorem 1.3. The following result is a basis of it.

LEMMA 3.2. *Let D be a domain in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Suppose that the boundary ∂E of a compact set E in $\widehat{\mathbb{C}}$ is contained in ∂D. Then either $E \cap D = \emptyset$ or $D \subset E$.*

Proof. Suppose that the open set $D \setminus E$ is non-empty. Let D_0 be a connected component of $D \setminus E$. Then $\partial D_0 \subset \partial D \cup \partial E = \partial D$ by assumption. This means that D_0 is open and closed in D. Since D is connected, D_0 must be equal to D; in other words, $D \cap E = \emptyset$. ∎

Let J be a Jordan curve in $\widehat{\mathbb{C}}$. We recall that the Jordan curve theorem implies that $\widehat{\mathbb{C}} \setminus J$ consists of exactly two connected components, say D_1 and D_2, and that $J = \partial D_j$ for $j = 1, 2$ (see, for instance, [13, Theorem 1.10]). As a corollary of the above lemma, we have the following.

LEMMA 3.3. *Let J be a Jordan curve in $\widehat{\mathbb{C}}$ and let D_1 and D_2 be the connected components of $\widehat{\mathbb{C}} \setminus J$. Suppose that the boundary ∂E of a compact set E in $\widehat{\mathbb{C}}$ is a non-empty subset of J. Then one of the following three cases occurs:* (1) $E \subset J$, (2) $E = J \cup D_1$, (3) $E = J \cup D_2$.

Proof. We apply Lemma 3.2 to the domain D_1 to obtain either $E \subset \widehat{\mathbb{C}} \setminus D_1 = J \cup D_2$ or $D_1 \subset E$, which implies $J \cup D_1 = \overline{D_1} \subset E$. Similarly, we have either $E \subset J \cup D_1$ or $J \cup D_2 \subset E$. Since the case when $\widehat{\mathbb{C}} = J \cup D_1 \cup D_2 \subset E$ is excluded by assumption, the required assertion follows. ∎

We prepare one more lemma for the proof of Theorem 1.3.

Let $c(\alpha)$ and $\rho(\alpha)$ be continuously differentiable functions of the real variable α with values in $\mathbb{C}$ and $\mathbb{R}_+ = (0, +\infty)$, respectively. Denote by Δ_α and Γ_α the disk $|w - c(\alpha)| < \rho(\alpha)$ and its boundary, respectively. If $|\rho'(\alpha_0)| < |c'(\alpha_0)|$ for an $\alpha_0 \in \mathbb{R}$, then we can find a $\delta > 0$ so that

$$|\rho(\alpha)^2 - \rho(\alpha_0)^2 - |c(\alpha) - c(\alpha_0)|^2| < 2\rho(\alpha_0)|c(\alpha) - c(\alpha_0)| \tag{3.1}$$

whenever $0 < |\alpha - \alpha_0| < \delta$. This can be shown by observing that

$$\frac{|\rho(\alpha)^2 - \rho(\alpha_0)^2 - |c(\alpha) - c(\alpha_0)|^2|}{|c(\alpha) - c(\alpha_0)|} \to \frac{2\rho(\alpha_0)|\rho'(\alpha_0)|}{|c'(\alpha_0)|} < 2\rho(\alpha_0)$$

as $\alpha \to \alpha_0$. In this situation, the possible boundary points of the union $\bigcup_{|\alpha - \alpha_0| < \delta} \Delta_\alpha$ lying on Γ_{α_0} are described by the next lemma. We recall here that $\theta = \arccos x$ is a decreasing function in $-1 \le x \le 1$ with $0 = \arccos 1 \le \theta \le \pi = \arccos(-1)$.

LEMMA 3.4. *Suppose that $|\rho'(\alpha_0)| < |c'(\alpha_0)|$ for an $\alpha_0 \in \mathbb{R}$ and fix a δ as above. Then, for $\alpha \in \mathbb{R}$ with $0 < |\alpha - \alpha_0| < \delta$, Γ_α intersects Γ_{α_0} exactly in the two points $\zeta_+(\alpha, \alpha_0)$ and $\zeta_-(\alpha, \alpha_0)$ given by*

$$\zeta_\pm(\alpha, \alpha_0) = c(\alpha_0) + \rho(\alpha_0)\frac{c(\alpha) - c(\alpha_0)}{|c(\alpha) - c(\alpha_0)|}e^{\pm i\varepsilon\theta(\alpha,\alpha_0)},$$

where

$$\theta(\alpha, \alpha_0) = \arccos\left[-\frac{\rho(\alpha)^2 - \rho(\alpha_0)^2 - |c(\alpha) - c(\alpha_0)|^2}{2\rho(\alpha_0)|c(\alpha) - c(\alpha_0)|}\right],$$

and

$$\varepsilon = \operatorname{sgn}(\alpha - \alpha_0) = \frac{\alpha - \alpha_0}{|\alpha - \alpha_0|}.$$

Moreover,

$$\lim_{\alpha \to \alpha_0} \zeta_\pm(\alpha, \alpha_0) = c(\alpha_0) + \rho(\alpha_0)\frac{c'(\alpha_0)}{|c'(\alpha_0)|}e^{\pm i\theta(\alpha_0,\alpha_0)} =: \zeta_\pm(\alpha_0, \alpha_0),$$

where

$$\theta(\alpha_0, \alpha_0) = \arccos\left[-\frac{\rho'(\alpha_0)}{|c'(\alpha_0)|}\right],$$

and for any $\zeta \in \Gamma_{\alpha_0} \setminus \{\zeta_+(\alpha_0, \alpha_0), \zeta_-(\alpha_0, \alpha_0)\}$, there exists an $\alpha \in \mathbb{R}$ with $0 < |\alpha - \alpha_0| < \delta$ such that $\zeta \in \Delta_\alpha$.

Proof. It is elementary to see that the inequality (3.1) is equivalent to the double inequality

$$|\rho(\alpha)-\rho(\alpha_0)|<|c(\alpha)-c(\alpha_0)|<\rho(\alpha)+\rho(\alpha_0), \tag{3.2}$$

which means that the two circles Γ_α and Γ_{α_0} intersect in exactly two points. It is easily verified that the two intersection points are $\zeta_\pm(\alpha,\alpha_0)$, which are defined above. Noting that

$$\lim_{\alpha\to\alpha_0-}\theta(\alpha,\alpha_0)=\arccos\left[\frac{\rho'(\alpha_0)}{|c'(\alpha_0)|}\right]=\pi-\lim_{\alpha\to\alpha_0+}\theta(\alpha,\alpha_0),$$

one can show that the limit of $\zeta_\pm(\alpha,\alpha_0)$ as $\alpha\to\alpha_0$ is as given in the above formula.

Finally, we fix α with $0<|\alpha-\alpha_0|<\delta$. Let $\omega=c(\alpha_0)+\rho(\alpha_0)(c(\alpha)-c(\alpha_0))/|c(\alpha)-c(\alpha_0)|$. Then,

$$|\omega-c(\alpha)|=\big||c(\alpha_0)-c(\alpha)|-\rho(\alpha_0)\big|,$$

which is smaller than $\rho(\alpha)$ by (3.2). We have thus obtained $\omega\in\Delta_\alpha$. Hence, the connected component of $\Gamma_{\alpha_0}\setminus\{\zeta_+(\alpha,\alpha_0),\zeta_-(\alpha,\alpha_0)\}$ containing ω is contained in Δ_α. In other words, the component in the direction of the vector $c(\alpha)-c(\alpha_0)$ is contained in Δ_α. Noting that $c(\alpha)-c(\alpha_0)=(\alpha-\alpha_0)\cdot\{c'(\alpha_0)+o(1)\}$, the direction is chosen according to the sign of $\alpha-\alpha_0$. In view of the continuity of $\zeta_\pm(\alpha,\alpha_0)$ in α, we obtain the final assertion. ■

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Fix $0<r<1$. We will prove the stronger assertion that $LU_r=G(\overline{\mathbb{D}}(0,r))$. Here, we recall that $G=\log F$, which is univalent on $\mathbb{D}$ by Theorem 2.1. In particular, $\partial G(\mathbb{D}(0,r))=G(\partial\mathbb{D}(0,r))$ is a bounded Jordan curve. Let $H(s,t)=(1+(s+t)/2)/(1+s)^2$ and $h(s,t)=2\operatorname{Log}(1+s)-\operatorname{Log}(1+(s+t)/2)=-\log H(s,t)$.

By Lemma 1.2(1), we have the relation $LU_r=h(\overline{\mathbb{D}}(0,r)\times\overline{\mathbb{D}}(0,r))$. Since LU_r is a compact subset of $\mathbb{C}$ with non-empty interior, it is enough to show the inclusion $\partial LU_r\subset G(\partial\overline{\mathbb{D}}(0,r))$ by Lemma 3.3.

Let z_0 be a boundary point of LU_r. Then $z_0=h(s_0,t_0)$ for some $s_0,t_0\in\overline{\mathbb{D}}(0,r)$. Since $s\mapsto h(s,t_0)$ is an open mapping, s_0 must be on the boundary; in other words, $|s_0|=r$. Similarly, we have $|t_0|=r$. We note that $\zeta_0=e^{-z_0}=H(s_0,t_0)$. Let $\alpha_0=\arg s_0$.

Let Γ_α denote the boundary circle of the disk $\Delta_\alpha=H(re^{i\alpha},\mathbb{D}(0,r))$. Then ζ_0 is a boundary point of the union of Δ_α over a neighborhood of α_0. (Note that ζ_0 might not necessarily appear in the boundary of the union of Δ_α over all $\alpha\in\mathbb{R}$.) Observe that the center $c(\alpha)$ and the radius $\rho(\alpha)$ of Γ_α are given respectively by

$$c(\alpha)=H(s,0)=\frac{1+s/2}{(1+s)^2}\quad\text{and}\quad\rho(\alpha)=\frac{r}{2|1+s|^2}$$

for $s = re^{i\alpha}$. We compute

$$c'(\alpha) = -\frac{is(3+s)}{2(1+s)^3} \quad \text{and} \quad \rho'(\alpha) = \frac{r^2 \sin\alpha}{|1+s|^4}. \tag{3.3}$$

We now see that

$$-\frac{\rho'(\alpha)}{|c'(\alpha)|} = \frac{-2r\sin\alpha}{|1+s|\,|3+s|} \in (-1,1)$$

for any $\alpha \in \mathbb{R}$, because

$$1 - \frac{\rho'(\alpha)^2}{|c'(\alpha)|^2} = \frac{(3+4r\cos\alpha + r^2)^2}{|1+s|^2|3+s|^2} > 0.$$

Hence, by Lemma 3.4, the boundary point ζ_0 has to be of the form $\zeta_\pm(\alpha_0,\alpha_0)$.

We now find a concrete form of $\zeta_\pm(\alpha) := \zeta_\pm(\alpha,\alpha)$. Recall that $\zeta_\pm(\alpha) = c(\alpha) + \rho(\alpha)e^{\pm i\theta(\alpha,\alpha)}c'(\alpha)/|c'(\alpha)|$, where, by the above computations,

$$e^{\pm i\theta(\alpha,\alpha)} = \frac{-2r\sin\alpha}{|1+s|\,|3+s|} \pm i\frac{3+4r\cos\alpha + r^2}{|1+s|\,|3+s|}.$$

Hence,

$$e^{i\theta(\alpha,\alpha)} = i\frac{(1+s)(3+\bar{s})}{|1+s|\,|3+s|} \quad \text{and} \quad e^{-i\theta(\alpha,\alpha)} = -i\frac{(1+\bar{s})(3+s)}{|1+s|\,|3+s|}.$$

Since

$$\frac{c'(\alpha)}{|c'(\alpha)|} = -\frac{is(3+s)}{(1+s)^3}\cdot\frac{|1+s|^3}{r|3+s|},$$

we finally obtain

$$\begin{aligned}
\zeta_+(\alpha) &= \frac{1+s/2}{(1+s)^2} + \frac{r}{2|1+s|^2}\cdot\frac{s|1+s|^2}{r(1+s)^2} = \frac{1}{1+s},\\
\zeta_-(\alpha) &= \frac{1+s/2}{(1+s)^2} - \frac{r}{2|1+s|^2}\cdot\frac{s(3+s)(1+\bar{s})^2}{r(3+\bar{s})(1+s)^2}\\
&= \frac{3+3s+\bar{s}+s^2}{(3+\bar{s})(1+s)^3} = \frac{1}{F(s)}.
\end{aligned}$$

We now show that the point $\zeta_+(\alpha)$ is contained in the interior of U_r. To this end, it is enough to show that the function $\varphi(z) = 1/(1+z)$ is subordinate to $\psi(z) = (1+z/2)/(1+z)^2 = H(z,0)$; in other words, there is an analytic function ω with $|\omega(z)| \le |z|$ on the unit disk $|z| < 1$ such that $\varphi = \psi\circ\omega$. Indeed, this would imply that

$$\zeta_+(\alpha) \in \varphi(\overline{\mathbb{D}}(0,r)) \subset \psi(\overline{\mathbb{D}}(0,r)) = H(\overline{\mathbb{D}}(0,r),0) \subset \operatorname{Int} U_r.$$

To see the above subordination, it suffices to show $\varphi(\mathbb{D}) \subset \psi(\mathbb{D})$ because ψ is univalent on $\mathbb{D}$ as we can verify directly. We note here that $\varphi(\mathbb{D})$ is the half-plane $\operatorname{Re} w > 1/2$. On the other hand,

$$\operatorname{Re}\psi(e^{i\theta}) = \frac{\cos\theta + 1/2}{4\cos^2(\theta/2)} = \frac{1}{2} - \frac{1}{8\cos^2(\theta/2)} < \frac{1}{2}.$$

Though ψ has a singularity at $z = -1$, it is easily checked that $\operatorname{Re}\psi(z) \to -\infty$ as $z \to -1$ in $\mathbb{D}$. Hence, we conclude that $\psi(\mathbb{D})$ contains $\varphi(\mathbb{D}) = \{w : \operatorname{Re} w > 1/2\}$ and therefore that φ is subordinate to ψ.

What we have shown above tells us that $-\log\zeta_+(\alpha)$ does not lie on the boundary of LU_r. Hence, $z_0 \in \partial LU_r$ is of the form $G(re^{i\alpha_0}) = -\log\zeta_-(\alpha_0)$. We conclude that the boundary of LU_r is contained in the curve $\{G(re^{i\alpha}) : -\pi < \alpha \le \pi\}$. ∎

Remark 3.5. The curves $\zeta_+(\alpha)$ and $\zeta_-(\alpha)$ are the inner and outer envelopes of $\{\Gamma_\alpha\}_{\alpha\in\mathbb{R}}$ (see [3] as a general reference for the envelope of a family of curves). Figure 2 illustrates the curves Γ_α and the inner and outer envelopes in the case when $r = 3/4$. In view of the above proof, we can get precise information about the extremal functions for the functional $z/f(z)$ in the class of close-to-convex functions. Indeed, for a given complex number $s = re^{i\alpha}$ with $0 < r < 1$, a function $f \in \mathcal{C}$ satisfies $\log[r/f(r)] = G(re^{i\alpha}) \in \partial LU_r$ if and only if $f = f_{a,b}$ for

$$a = -\frac{s}{r}\cdot\frac{1+\bar{s}}{1+s}\cdot\frac{3+s}{3+\bar{s}} \quad\text{and}\quad b = \frac{s}{r} = e^{i\alpha}.$$

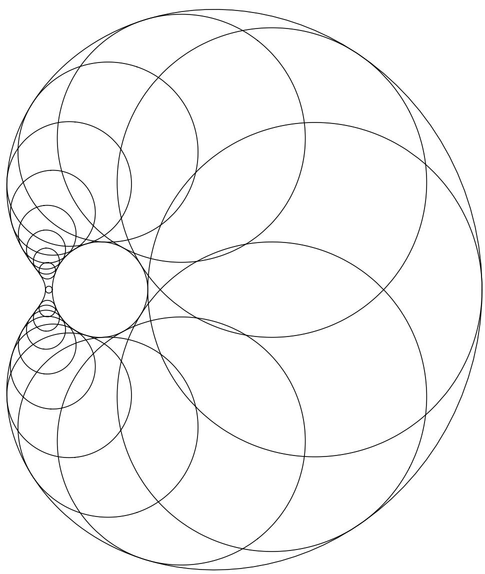

Fig. 2. Circles Γ_α and their inner and outer envelopes for $r = 3/4$

Proof of Theorem 1.4. By Theorem 2.1 and the proof of Theorem 1.3, we see that the mapping $G = \log F$ is a homeomorphism of $\mathbb{D}$ onto LU_{1^-}. We now observe that G extends continuously to $\overline{\mathbb{D}} \setminus \{-1\}$. Since $G(e^{it}) = \gamma(t)$ for $|t| < \pi$, the boundary of LU_{1^-} contains the arc $\gamma([-\pi, \pi])$.

We next investigate the limit points of $G(z)$ as $z \to -1$. Let $\alpha(\delta) = (a\delta)^{1/3}$ and put $z = (1-\delta)e^{i(\pi-\alpha(\delta))}$ for $0 < \delta < 1$ and a positive constant a. Then

$$\begin{aligned} z &= -(1-\delta)\left(1 - i\alpha(\delta) - \frac{\alpha(\delta)^2}{2} + \frac{i\alpha(\delta)^3}{6} + O(\delta^{4/3})\right) \\ &= -1 + i(a\delta)^{1/3} + \frac{(a\delta)^{2/3}}{2} + \left(1 - \frac{ia}{6}\right)\delta + O(\delta^{4/3}) \end{aligned}$$

as $\delta \to 0^+$. Therefore,

$$\frac{3+z}{3+\bar{z}} = 1 + i(a\delta)^{1/3} - \frac{(a\delta)^{2/3}}{2} - \frac{2ia\delta}{3} + O(\delta^{4/3})$$

and

$$1 + z\frac{3+z}{3+\bar{z}} = \left(1 + \frac{ia}{2}\right)\delta + O(\delta^{4/3}).$$

Since $(1+z)^3 = -ia\delta + O(\delta^{4/3})$, we have

$$\frac{(1+z)^3}{1 + z(3+z)/(3+\bar{z})} = -\left(1 + \frac{a+2i}{a-2i}\right) + O(\delta^{1/3})$$

as $\delta \to 0^+$. Thus

$$\lim_{\delta\to 0^+} G((1-\delta)e^{i(\pi-\alpha(\delta))}) = \pi i + \log\left(1 + \frac{a+2i}{a-2i}\right).$$

Since a is an arbitrary positive real number, the boundary of $LU_{1^-} = G(\mathbb{D})$ contains the curve $\gamma(t):\ \pi < t < 2\pi$. The same is true for $-2\pi < t < -\pi$ by the symmetry of the function G.

We have seen that the Jordan curve $J = \{\gamma(t) : |t| < 2\pi\} \cup \{\infty\}$ is contained in ∂LU_{1^-}. The remaining thing is to prove the converse implication $\partial LU_{1^-} \subset J$. We denote by D the domain bounded by J and containing the origin.

Note that LU_r is convex in the direction of the imaginary axis for each $0 < r < 1$ by Lemma 2.5. Therefore, the same is true for the limit LU_{1^-}. We observe also that D is convex in the direction of the imaginary axis.

Suppose that there is a boundary point p_0 of LU_{1^-} with $p_0 \notin J$. We may assume that $\operatorname{Im} p_0 > 0$. Let p_1 be the point in J with $\operatorname{Im} p_1 > 0$ and $\operatorname{Re} p_1 = \operatorname{Re} p_0$. Then the convexity of LU_{1^-} in the direction of the imaginary axis implies that the segment $[p_0, p_1]$ is contained in ∂LU_{1^-}. We can choose p_0 so that the segment is maximal. Since the family of smooth Jordan domains $LU_r,\ 0 < r < 1$, exhausts the domain LU_{1^-}, for a small enough $\delta > 0$ there exist three points $z_1^-(\delta), z_0(\delta), z_1^+(\delta)$ on the circle $|z| = 1-\delta$ with $0 < \operatorname{Arg} z_1^-(\delta) < \operatorname{Arg} z_0(\delta) < \operatorname{Arg} z_1^+(\delta)$ such that $G(z_1^-(\delta)) \to p_1$, $G(z_0(\delta)) \to p_0$, $G(z_1^+(\delta)) \to p_1$ as $\delta \to 0^+$. In particular, $\operatorname{Im} G(z_0(\delta)) < \operatorname{Im} G(z_1^{\pm}(\delta))$ for sufficiently small $\delta > 0$. Hence, $g_{1-\delta}(\theta) = \operatorname{Im} G((1-\delta)e^{i\theta})$ takes a local minimum at a point θ_0 with $\operatorname{Arg} z_1^-(\delta) < \theta_0 < \operatorname{Arg} z_1^+(\delta)$. In particular, $g'_{1-\delta}(\theta_0) = 0$. Note here that

$$\operatorname{Re} G((1-\delta)e^{i\theta_0}) \to \operatorname{Re} p_0 \quad (\delta \to 0^+).$$

We write $\theta_0 = \pi - \beta(\delta)$. Then, by Lemma 2.7, we see that $\theta_0 \geq x_{1-\delta}$, equivalently, $\beta(\delta) \leq \arccos(1-\delta)$. This implies that $\beta(\delta) = O(\delta^{1/2})$ as $\delta \to 0^+$. Therefore, $z = (1-\delta)e^{i(\pi-\beta(\delta))} = -1+i\beta(\delta)+O(\delta)$, $(3+z)/(3+\bar{z}) = 1+i\beta(\delta)+O(\delta)$ and thus $1+z(3+z)/(3+\bar{z}) = O(\delta)$ as $\delta \to 0^+$. In particular,

$$\operatorname{Re} G((1-\delta)e^{i\theta_0}) \to -\infty \quad (\delta \to 0^+),$$

which is a contradiction.

We now conclude that $\partial LU_{1^-} = J$. ∎

Proof of Theorem 1.6. Let $\Omega = \{(r,s,t) \in \mathbb{R}^3 : 0 < s < 2,\ 0 < rs^2 < 2,\ -\pi/2 < t < \pi/2\}$. Then $u = rs^2 e^{it}\cos^2 t$ and $v = se^{-it}\cos t$ satisfy $|u-1| < 1$ and $|v-1| < 1$, whence the point

$$w(r,s,t) = \log \frac{2u}{v(u+v)} = \log(2r) + 3it - \log(1 + rse^{2it}\cos t)$$

belongs to the region LW_{1^-} for $(r,s,t) \in \Omega$ by Theorem 1.5.

For a given point $z_0 = x_0 + iy_0$ with $|y_0| < 3\pi/2$, we now look for $(r,s,t) \in \Omega$ such that $w(r,s,t) = z_0$. Let $r_0 = e^{x_0}/2$ and take $0 < s_0 < 1$ small enough that $r_0 s_0 < 1/2$. Then $r_0 s_0^2 < s_0 < 2$ and $x_0 \pm 3\pi i/2$ are the endpoints of the curve $\alpha(t) = w(r_0, s_0, t),\ -\pi/2 < t < \pi/2$. We now take a $t_0 \in (-\pi/2, \pi/2)$ such that $\operatorname{Im}\alpha(t_0) = y_0$ and let $x_1 = x_0 - \operatorname{Re}\alpha(t_0)$. Since the function $-\log(1-x)$ is convex, we have the inequality $-\log(1-x) \leq 2x\log 2$ for $0 \leq x < 1/2$. We now estimate $-x_1$ in the following way:

$$-x_1 = -\log|1 + r_0 s_0 e^{2it_0}\cos t_0| \leq -\log(1 - r_0 s_0) \leq 2r_0 s_0 \log 2,$$

which implies

$$r_0 s_0^2 e^{-x_1} < s_0 e^{-x_1} \leq s_0 e^{2r_0 s_0 \log 2} < s_0 e^{\log 2} = 2s_0 < 2.$$

Therefore $(r_0 e^{x_1}, s_0 e^{-x_1}, t_0) \in \Omega$ and

$$w(r_0 e^{x_1}, s_0 e^{-x_1}, t_0) = x_1 + w(r_0, s_0, t_0) = x_0 + iy_0 = z_0$$

as desired. ∎

Proof of Theorem 1.9. For a fixed $0 < r < 1$, we consider the continuous linear functionals λ_1 and λ_2 on $\mathcal{A}$ defined by $\lambda_1(f) = f'(r)$ and $\lambda_2(f) = f'(0)$

for $f \in \mathcal{A}$. Then Lemma 3.1 implies that for any $f \in \mathcal{C}$,

$$f'(r) = \frac{\lambda_1(f)}{\lambda_2(f)} = \frac{\lambda_1(f_{a,b})}{\lambda_2(f_{a,b})} = \frac{1+ar}{(1+br)^3}$$

for some $a, b \in \overline{\mathbb{D}}$. Thus the first part of the theorem has been proved.

By the first part, we have $LV_{1^-} = \{\mathrm{Log}(1+z) - 3\,\mathrm{Log}(1+w) : z, w \in \mathbb{D}\}$. Let a and b be real numbers with $|b| < \pi/2$. We shall show that $a + 4bi \in LV_{1^-}$. It is easy to observe that the domain $\{\mathrm{Log}(1+z) : z \in \mathbb{D}\}$ is convex and its boundary curve

$$\tau(t) = \mathrm{Log}(1 + e^{it}) = \log(2\cos(t/2)) + ti/2 \qquad (-\pi < t < \pi)$$

satisfies $\mathrm{Re}\,\tau(t) \to -\infty$ and $\mathrm{Im}\,\tau(t) \to \pm\pi/2$ as $t \to \pm\pi^{\mp}$. Therefore, there are $z, w \in \mathbb{D}$ such that $a - 3c + bi = \mathrm{Log}(1+z)$ and $-c - bi = \mathrm{Log}(1+w)$ for a sufficiently large $c > 0$. In particular, $a + 4bi = \mathrm{Log}(1+z) - 3\,\mathrm{Log}(1+w) \in LV_{1^-}$. ∎

Proof of Theorem 1.10. Since $LU_r = LW_r + 2\log(1-r^2)$ by Lemma 1.2(2), it suffices to prove the assertion for LU_r. If there is no r_0 as in the assertion, then the limiting domain LU_{1^-} must be convex. Note that LU_{1^-} is convex if and only if $\frac{d}{dt}\arg\gamma'(t) \geq 0$, where γ is given in Theorem 1.4. A simple computation gives us

$$\frac{d}{dt}\arg\gamma'(t) = \mathrm{Im}\,\frac{d}{dt}\log\gamma'(t) = \mathrm{Re}\,\frac{1}{1+3e^{it}} = \frac{1+3\cos t}{|1+3e^{it}|^2}$$

for $|t| < \pi$. This is negative when $\cos t < -1/3$ and thus we get a contradiction. ∎

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