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Existence and multiplicity results for a nonlinear stationary Schrödinger equation

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Abstract. We revisit Kristály's result on the existence of weak solutions of the Schrödinger equation of the form

$$-\Delta u + a(x)u = \lambda b(x)f(u), \quad x \in \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N),$$

where λ is a positive parameter, a and b are positive functions, while $f : \mathbb{R} \to \mathbb{R}$ is sublinear at infinity and superlinear at the origin. In particular, by using Ricceri's recent three critical points theorem, we show that, under the same hypotheses, a much more precise conclusion can be obtained.

1. Introduction and statement of the main result. Sufficient conditions which ensure the multiplicity of weak solutions for nonlinear stationary Schrödinger-like equations have recently been proposed in the literature. In particular, Kristály [K] considers the Schrödinger equation of the form

$$(P_\lambda) \qquad -\Delta u + a(x)u = \lambda b(x)f(u), \quad x \in \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N),$$

with a positive parameter λ. He assumes that the potentials a and b satisfy the following conditions:

($\widetilde{a}$) $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, $\operatorname{ess\,inf}_{\mathbb{R}^N} a > 0$ and for any $M > 0$ and any $r > 0$,
$$\operatorname{mes}(\{x \in B_r(y) : a(x) \le M\}) \to 0 \quad \text{as } |y| \to +\infty$$
where "mes" stands for the Lebesgue measure and $B_r(y)$ denotes the open ball in $\mathbb{R}^N$ with center y and radius $r > 0$.

($\widetilde{b}$) $b \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $b \ge 0$ and
$$\sup_{R>0} \operatorname*{ess\,inf}_{|x|\le R} b(x) > 0.$$

($\widetilde{f_0}$) $f \in C^0(\mathbb{R})$ and there exist $\mathcal{C} > 0$ and $q \in\,]0,1[$ such that
$$|f(s)| \le \mathcal{C}|s|^q \quad \text{for each } s \in \mathbb{R}.$$

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($\widetilde{f_1}$) $f(s) = o(|s|)$ as $s \to 0$.
($\widetilde{f_2}$) $\sup_{s\in\mathbb{R}} F(s) > 0$ where $F(s) = \int_0^s f(t)\,dt$.

Following the suggestions of Bartsch and Wang ([BW]), due to ($\widetilde{a}$), Kristály defines the Hilbert space

$$E = \Big\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x)u^2\,dx < +\infty\Big\}$$

endowed with the inner product

$$(u,v)_E = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + a(x)uv)\,dx \qquad \text{for } u, v, \in E$$

and consequently with the induced norm which we denote by $\|\cdot\|$. The condition ($\widetilde{a}$) implies that the space E can be continuously embedded into $L^\ell(\mathbb{R}^N)$ whenever $2 \le \ell \le 2^*$ and the embedding is compact when $2 \le \ell < 2^*$ (see [Ba]). Here, 2^* denotes the critical Sobolev exponent, i.e., $2^* = 2N/(N-2)$ for $N \ge 3$ and $2^* = +\infty$ for $N = 1, 2$. By applying a result established by Bonanno [B], Kristály has proved in [K] that (P_λ) admits at least two solutions in E, provided that λ belongs to a suitable open interval. The aim of the present paper is to significantly improve Kristály's result, showing that, essentially under the same hypotheses, a more exact conclusion can be reached. Denoting by $\mathcal{A}$ the class of all Carathéodory functions $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that the functional

$$\mathcal{G}(u) = \int_{\mathbb{R}^N} \Big(\int_0^{u(x)} g(x,t)\,dt\Big)\,dx$$

belongs to $C^1(E)$ and has compact derivative, our main result reads as follows:

THEOREM 1.1. *Assume ($\widetilde{a}$), ($\widetilde{b}$), ($\widetilde{f_0}$), ($\widetilde{f_1}$), and ($\widetilde{f_2}$). Then, setting*

$$\gamma = \frac{1}{2}\inf\Big\{\frac{\|u\|^2}{\int_{\mathbb{R}^N} b(x)F(u(x))\,dx} : u \in E, \int_{\mathbb{R}^N} b(x)F(u(x))\,dx > 0\Big\},$$

for each compact interval $[c,d] \subset\,]\gamma, +\infty[$ there exists a number $r > 0$ with the following property: for every $\lambda \in [c,d]$ and every $g \in \mathcal{A}$ there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the problem

$$(P_{\lambda,\mu}) \qquad -\Delta u + a(x)u = \lambda b(x)f(u) + \mu g(x,u), \qquad x \in \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N),$$

has at least three weak solutions whose norms in E are less than r.

REMARK. This result covers, as a particular case, the problem studied by Kristály [K]. Here, we prove it by a different method and we provide further information both on the size and location of the set containing the parameter λ and the location of the possible weak solutions of the problem at issue.

2. Proof of Theorem 1.1. First, we recall a theorem from [R] which is the basic tool in the proof of our result. In the following, if X is a real Banach space, the symbol $\mathcal{W}_X$ denotes the class of all functionals $I : X \to \mathbb{R}$ having the following property: if $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to+\infty} I(u_n) \le I(u)$, then $\{u_n\}$ has a subsequence converging strongly to u.

THEOREM 2.1 ([R, Theorem 2]). *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; and $J : X \to \mathbb{R}$ a C^1 functional with compact derivative. Assume that Φ has a strict local minimum at x_0 with $\Phi(x_0) = J(x_0) = 0$. Finally, setting*

$$\alpha = \max\left\{0, \limsup_{\|x\|\to+\infty} \frac{J(x)}{\Phi(x)}, \limsup_{x\to x_0} \frac{J(x)}{\Phi(x)}\right\},$$

$$\beta = \sup_{x\in\Phi^{-1}(]0,+\infty[)} \frac{J(x)}{\Phi(x)}$$

assume that $\alpha < \beta$. Then, for each compact interval $[c, d] \subset]1/\beta, 1/\alpha[$ (with the conventions $\frac{1}{0} = +\infty$, $\frac{1}{+\infty} = 0$) there exists $r > 0$ with the following property: for every $\lambda \in [c, d]$ and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(x) = \lambda J'(x) + \mu\Psi'(x)$$

has at least three solutions whose norms are less than r.

To use this theorem in our particular case, we begin by defining the functional $\mathcal{F} : E \to \mathbb{R}$ as

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} b(x)F(u(x))\,dx$$

for each $u \in E$. Standard arguments based on the hypothesis $(\widetilde{a})$ and on the fact that E is continuously embedded in $L^\ell(\mathbb{R}^N)$ when $2 \le \ell \le 2^*$ show that the functional $\mathcal{F}$ is well defined, it is of class C^1, and satisfies

$$\mathcal{F}'(u)(v) = \int_{\mathbb{R}^N} b(x)f(u(x))v(x)\,dx \quad \text{for all } u, v \in E.$$

Moreover, since the embedding $E \hookrightarrow L^\ell(\mathbb{R}^N)$ is compact for $2 \le \ell < 2^*$, $\mathcal{F}'$ is a compact operator. In the following, we denote by $\kappa_\ell > 0$ the Sobolev embedding constant for $E \hookrightarrow L^\ell(\mathbb{R}^N)$ where $\ell \in [2, 2^*]$. Finally, for any

$\lambda > 0$ and $\mu \geq 0$ we define the functional $\mathcal{H} : E \to \mathbb{R}$ by

$$\mathcal{H}(u) = \frac{1}{2}\|u\|^2 - \lambda\mathcal{F}(u) - \mu\mathcal{G}(u) \quad \text{for all } u \in E.$$

Obviously, the weak solutions of the problem $(P_{\lambda,\mu})$ are the critical points of $\mathcal{H}$.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We apply Theorem 2.1 for $X = E$, $\Phi(u) = \frac{1}{2}\|u\|^2$ and $J = \mathcal{F}$. Note that Φ is a coercive, sequentially weakly lower semicontinuous C^1 functional which belongs to $\mathcal{W}_E$. The latter assertion is a classical result, since the space E is uniformly convex and $\Phi(u) = h(\|u\|)$ with $h(t) = \frac{1}{2}t^2 : [0, +\infty[\to \mathbb{R}$, which is a continuous and strictly increasing function. Because Φ is continuous, it is bounded on each bounded subset of E, its derivative is a homeomorphism between E and its dual (see [Z, Theorem 26. A]), and the hypotheses on J of Theorem 2.1 are satisfied as well. Putting $u_0 = \theta_E$, where θ_E is the zero element of E, observe that Φ has at u_0 the only global minimum. Moreover, if $u \neq \theta_E$ then $\Phi(u) > 0$ by $(\widetilde{a})$ and $\Phi(u_0) = J(u_0) = 0$. Now, we fix a number $\epsilon > 0$; in view of $(\widetilde{f_0})$ and $(\widetilde{f_1})$ there exist ρ_1, ρ_2 with $0 < \rho_1 < \rho_2$ such that

$$b(x)F(s) < \epsilon a(x)|s|^2 \tag{2.1}$$

for a.e. $x \in \mathbb{R}^N$ and all $s \in \mathbb{R} \setminus ([-\rho_2, -\rho_1] \cup [\rho_1, \rho_2])$. Then, as F is bounded on $[-\rho_2, -\rho_1] \cup [\rho_1, \rho_2]$, we can choose $\mathcal{D} > 0$ and $2 < q < 2^*$ in such a way that

$$b(x)F(s) < \epsilon a(x)|s|^2 + \mathcal{D}|s|^q$$

for a.e. $x \in \mathbb{R}^N$ and all $s \in \mathbb{R}$. Thus, by continuous embedding,

$$\mathcal{F}(u) \leq \epsilon\|u\|^2 + \mathcal{D}\kappa_q^q\|u\|^q$$

for all $u \in E$. Hence,

$$\limsup_{u \to 0} \frac{2\mathcal{F}(u)}{\|u\|^2} \leq 2\epsilon. \tag{2.2}$$

Further, by (2.1) again, for each $u \in E \setminus \{\theta_E\}$, we obtain

$$\begin{aligned}\frac{\mathcal{F}(u)}{\|u\|^2} &= \frac{\int_{(|u| \leq \rho_2)} b(x)F(u(x))\,dx}{\|u\|^2} + \frac{\int_{(|u| > \rho_2)} b(x)F(u(x))\,dx}{\|u\|^2} \\ &\leq \frac{\sup_{[-\rho_2, \rho_2]} F \int_{\mathbb{R}^N} b(x)\,dx}{\|u\|^2} + \epsilon.\end{aligned}$$

So, we get

$$\limsup_{\|u\| \to +\infty} \frac{2\mathcal{F}(u)}{\|u\|^2} \leq 2\epsilon. \tag{2.3}$$

Since ϵ is arbitrary, from (2.2) and (2.3) it follows that

$$\max\left\{\limsup_{\|u\|\to+\infty}\frac{2\mathcal{F}(u)}{\|u\|^2},\limsup_{u\to 0}\frac{2\mathcal{F}(u)}{\|u\|^2}\right\}\le 0.$$

Thus, by using the notation of Theorem 2.1, we have $\alpha = 0$ and by our assumption $0 < \beta \le +\infty$. Therefore, for $\gamma = 1/\beta$, the conclusion follows from Theorem 2.1 with $\Psi = \mathcal{G}$. ∎

Example 2.2. Let κ, h and ξ be arbitrary real positive. We choose $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(s) = \begin{cases} s|s|[4m|s| + 3n], & |s| \le \xi, \\ \kappa s e^{-h|s|}, & |s| \ge \xi, \end{cases}$$

where

$$m = m(\kappa, h, \xi) = -\frac{\kappa e^{-h\xi}(h\xi + 1)}{4\xi^2}, \qquad n = n(\kappa, h, \xi) = \frac{\kappa e^{-h\xi}(2 + h\xi)}{3\xi}.$$

Then, we take as potentials $a(x) = |x|^2 + \ell$ with ℓ a positive constant and $b(x) = e^{-|x|^2}$, $x \in \mathbb{R}^N$. It follows easily that the assumptions $(\widetilde{a})$, $(\widetilde{b})$, $(\widetilde{f_0})$, $(\widetilde{f_1})$ and $(\widetilde{f_2})$ of Theorem 1.1 hold.

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