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Bounded Toeplitz and Hankel products on weighted Bergman spaces of the unit ball

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Abstract. We prove a sufficient condition for products of Toeplitz operators $T_f T_{\bar{g}}$, where f, g are square integrable holomorphic functions in the unit ball in $\mathbb{C}^n$, to be bounded on the weighted Bergman space. This condition slightly improves the result obtained by K. Stroethoff and D. Zheng. The analogous condition for boundedness of products of Hankel operators $H_f H_g^*$ is also given.

1. Introduction. Let $dv(z)$ denote the Lebesgue measure on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ normalized so that $\int_{\mathbb{B}} dv = 1$. For $\alpha > -1$ let

$$dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dv(z),$$

where $c_\alpha = \Gamma(n+1+\alpha)/(n!\Gamma(\alpha+1))$, denote the weighted Lebesgue measure on the unit ball. The *Bergman space* A^2_α is the Hilbert space consisting of holomorphic functions on $\mathbb{B}$ for which

$$\|f\| = \|f\|_\alpha = \Big(\int_{\mathbb{B}} |f(z)|^2\, dv_\alpha(z)\Big)^{1/2} < \infty.$$

Let P_α denote the orthogonal projection from $L^2(\mathbb{B}, dv_\alpha)$ onto A^2_α. For $f \in L^2(\mathbb{B}, dv_\alpha)$, the *Toeplitz operator* T_f and the *Hankel operator* H_f with symbol f are defined densely on the space A^2_α by $T_f(h) = P_\alpha(fh)$ and $H_f(h) = fh - P_\alpha(fh)$, respectively. The Bergman space A^2_α has the reproducing kernel $K_w^{(\alpha)}$ given by

$$K_w^{(\alpha)}(z) = \frac{1}{(1-\langle z, w\rangle)^{n+\alpha+1}}, \qquad z, w \in \mathbb{B};$$

so for $h \in A^2_\alpha$ we have

$$h(w) = \langle h, K_w^{(\alpha)}\rangle_\alpha = \int_{\mathbb{B}} \frac{h(z)}{(1-\langle w, z\rangle)^{n+\alpha+1}}\, dv_\alpha(z),$$

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and for $f \in L^2(\mathbb{B}, dv_\alpha)$,

$$P_\alpha f(w) = \langle f, K_w^{(\alpha)} \rangle_\alpha = \int_{\mathbb{B}} \frac{f(z)}{(1-\langle w, z\rangle)^{n+\alpha+1}}\, dv_\alpha(z).$$

We will denote the normalized reproducing kernel for A^2_α by

$$k_w^{(\alpha)}(z) = \frac{(1-|w|^2)^{(n+\alpha+1)/2}}{(1-\langle z, w\rangle)^{n+\alpha+1}}, \qquad z, w \in \mathbb{B}.$$

In their recent papers [7] and [8] K. Stroethoff and D. Zheng studied the products of Toeplitz operators $T_f T_{\bar g}$, where $f, g \in A^2_\alpha$, densely defined on A^2_α. To state their results we need the following notation. For $w \in \mathbb{B}$ let φ_w be the automorphism of $\mathbb{B}$ of the form

$$\varphi_w(z) = \frac{w - P_w(z) - s_w Q_w(z)}{1 - \langle z, w\rangle},$$

where $s_w = (1-|w|)^{1/2}$, $P_w(z) = \frac{\langle z,w\rangle}{|w|^2} w$ if $w \neq 0$, $P_0(z) = 0$ and $Q_w = I - P_w$ (see, e.g., [4], [10] for definition and properties of the automorphism group of $\mathbb{B}$).

For $u \in L^1(\mathbb{B}, dv_\alpha)$ and $w \in \mathbb{B}$ define

$$B[u](w) = B^{(\alpha)}[u](w) = \int_{\mathbb{B}} u \circ \varphi_w(z)\, dv_\alpha(z) = \int_{\mathbb{B}} u(z) |k_w^{(\alpha)}(z)|^2\, dv_\alpha(z).$$

Stroethoff and Zheng obtained the following results for the Toeplitz products.

THEOREM 1.1 ([7], [8]). *Let $-1 < \alpha < \infty$, and let f and g be in A^2_α. If $T_f T_{\bar g}$ is bounded on A^2_α, then*

$$\sup_{w \in \mathbb{B}} B[|f|^2](w) B[|g|^2](w) < \infty.$$

THEOREM 1.2 ([7], [8]). *Let $-1 < \alpha < \infty$, and let f and g be in A^2_α. If for $\varepsilon > 0$,*

$$\sup_{w \in \mathbb{B}} B[|f|^{2+\varepsilon}](w) B[|g|^{2+\varepsilon}](w) < \infty,$$

then $T_f T_{\bar g}$ is bounded on A^2_α.

In their earlier paper Stroethoff and Zheng [6] also studied the product of Hankel operators $H_f H_g^*$, $f, g \in L^2(\mathbb{D}, dA)$, densely defined on $(A^2)^\perp$ in the setting of the unit disk. Recently, the analogous result for the unit ball has been obtained by Lu and Liu in [2]. More exactly they proved the following.

THEOREM 1.3 ([2], [6]). *Let $-1 < \alpha < \infty$, and let f and g be in $L^2(\mathbb{B}, dv_\alpha)$. If $H_f H_g^*$ is bounded, then*

$$\sup_{w \in \mathbb{B}} \|f \circ \varphi_w - P_\alpha(f \circ \varphi_w)\| \, \|g \circ \varphi_w - P_\alpha(g \circ \varphi_w)\| < \infty.$$

THEOREM 1.4 ([2], [6]). *Let $-1 < \alpha < \infty$, and let f and g be in $L^2(\mathbb{B}, dv_\alpha)$. If there is $\varepsilon > 0$ such that*

$$\sup_{w\in\mathbb{B}} \|f \circ \varphi_w - P_\alpha(f \circ \varphi_w)\|_{2+\varepsilon} \|g \circ \varphi_w - P_\alpha(g \circ \varphi_w)\|_{2+\varepsilon} < \infty,$$

then $H_f H_g^*$ is bounded.

The above-cited results are analogous to those obtained earlier for the Hardy space H^2 (e.g., [1], [5], [9]).

Also we mention that generalizations of Theorems 1.1 and 1.2 have been obtained by J. Miao in [3].

In this paper we give a sufficient condition for boundedness of Toeplitz products which is slightly weaker than that given in Theorem 1.2. We also obtain a similar condition for Hankel products in the setting of the unit ball that is slightly weaker than Theorem 1.4.

2. Results. For $\varepsilon > 0$, $w \in \mathbb{B}$ and $u \in L^1(\mathbb{B}, dv_\alpha)$, set

$$B_\varepsilon[u](w) = B_\varepsilon^{(\alpha)}[u](w) = \int_{\mathbb{B}} u \circ \varphi_w(z) \log^{1+\varepsilon}\left(\frac{1}{1-|z|}\right) dv_\alpha(z).$$

We will prove the following.

THEOREM 2.1. *Let $-1 < \alpha < \infty$, and let $f, g \in A^2_\alpha$. If there is an $\varepsilon > 0$ such that*

$$\sup_{w\in\mathbb{B}} B_\varepsilon[|f|^2](w) B_\varepsilon[|g|^2](w) < \infty,$$

then the Toeplitz product $T_f T_{\bar{g}}$ is bounded on A^2_α.

THEOREM 2.2. *Let $-1 < \alpha < \infty$, and let $f, g \in L^2(\mathbb{B}, dv_\alpha)$. If there is an $\varepsilon > 0$ such that*

$$\sup_{w\in\mathbb{B}} \left\| [f \circ \varphi_w - P_\alpha(f \circ \varphi_w)] \log^{(1+\varepsilon)/2}\left(\frac{1}{1-|z|}\right) \right\| \times \left\| [g \circ \varphi_w - P_\alpha(g \circ \varphi_w)] \log^{(1+\varepsilon)/2}\left(\frac{1}{1-|z|}\right) \right\| < \infty,$$

then the operator $H_f H_g^*$ is bounded on $(A^2_\alpha)^\perp$.

For a multi-index $\nu = (\nu_1, \ldots, \nu_n)$ such that $|\nu| = \nu_1 + \cdots + \nu_n = m$ and f holomorphic in $\mathbb{B}$ define

$$D^\nu f = \frac{\partial f^m}{\partial z_1^{\nu_1} \ldots \partial z_n^{\nu_n}}.$$

In the proofs of the above stated theorems we will use the following lemma.

LEMMA 2.3. *Assume that $-1 < \alpha < \infty$, $n \geq 2$, $\varepsilon > 0$ and ν is a multi-index such that $|\nu| = m$. Then*

(a) *for every* $f, h \in A^2_\alpha$ *and* $w \in \mathbb{B}$,

$$|(D^\nu T_{\bar f})h(w)| \le \frac{C\{B_\varepsilon[|f|^2](w)\}^{1/2}}{(1-|w|^2)^{(n+m+\alpha+1)/2}} \times \left\{\int_{\mathbb{B}} \frac{|h(z)|^2}{|1-\langle w,z\rangle|^m} \log^{-(1+\varepsilon)}\left(\frac{1}{1-|\varphi_w(z)|}\right) dv_\alpha(z)\right\}^{1/2},$$

(b) *for* $g \in L^2(\mathbb{B}, dv_\alpha)$, $u \in (A^2_\alpha)^\perp$ *and* $w \in \mathbb{B}$,

$$|(D^\nu H^*_g u)(w)| \le \frac{C}{(1-|w|^2)^{(n+m+\alpha+1)/2}} \times \left\|(g\circ\varphi_w - P_\alpha(g\circ\varphi_w)) \log^{(1+\varepsilon)/2}\left(\frac{1}{1-|z|}\right)\right\| \times \left\{\int_{\mathbb{B}} \frac{|u(z)|^2}{|1-\langle w,z\rangle|^m} \log^{-(1+\varepsilon)}\left(\frac{1}{1-|\varphi_w(z)|}\right) dv_\alpha(z)\right\}^{1/2}.$$

Proof. (a) Since for any multi-index ν with $|\nu| = m$ we have

$$(D^\nu T_{\bar f} h)(w) = \frac{\Gamma(n+m+\alpha+1)}{\Gamma(n+\alpha+1)} \int_{\mathbb{B}} \frac{\overline{z^\nu f(z)} h(z)}{(1-\langle w,z\rangle)^{n+m+\alpha+1}}\, dv_\alpha(z),$$

by the Cauchy–Schwarz inequality we get

$$\begin{aligned} |(D^\nu T_{\bar f} h)(w)| &\le \frac{C}{(1-|w|^2)^{(n+\alpha+1)/2}} \\ &\times \left\{\int_{\mathbb{B}} \frac{|f(z)|^2(1-|w|^2)^{n+\alpha+1}}{|1-\langle w,z\rangle|^{2n+2\alpha+2+m}} \log^{1+\varepsilon}\left(\frac{1}{1-|\varphi_w(z)|}\right) dv_\alpha(z)\right\}^{1/2} \\ &\times \left\{\int_{\mathbb{B}} \frac{|h(z)|^2}{|1-\langle w,z\rangle|^m} \log^{-(1+\varepsilon)}\left(\frac{1}{1-|\varphi_w(z)|}\right) dv_\alpha(z)\right\}^{1/2} \\ &= \frac{C\{B_\varepsilon[|f|^2](w)\}^{1/2}}{(1-|w|^2)^{(n+m+\alpha+1)/2}} \left\{\int_{\mathbb{B}} \frac{|h(z)|^2}{|1-\langle w,z\rangle|^m} \log^{-(1+\varepsilon)}\left(\frac{1}{1-|\varphi_w(z)|}\right) dv_\alpha(z)\right\}^{1/2}, \end{aligned}$$

where the last equality follows from the change-of-variable formula.

(b) Since

$$h(z) = \frac{z^\nu P_\alpha(g\circ\varphi_w)\circ\varphi_w(z)}{(1-\langle z,w\rangle)^{n+m+\alpha+1}} \in A^2_\alpha,$$

for $u \in (A^2_\alpha)^\perp$ we have

$$\langle u, h\rangle_\alpha(w) = \int_{\mathbb{B}} \frac{u(z)\overline{z^\nu P_\alpha(g\circ\varphi_w)\circ\varphi_w(z)}}{(1-\langle w,z\rangle)^{n+m+\alpha+1}}\, dv_\alpha(z) \equiv 0.$$

Consequently,

$$
\begin{aligned}
&|(D^\nu H_g^* u)(w)| = |(D^\nu P_\alpha(\bar g u)(w) - \langle u, h\rangle_\alpha(w)| \\
&= \left| \frac{\Gamma(n+m+\alpha+1)}{\Gamma(n+\alpha+1)} \int_{\mathbb{B}} \frac{\overline{z^\nu g(z)}u(z)}{(1-\langle w,z\rangle)^{n+m+\alpha+1}}\, dv_\alpha(z) - \langle u,h\rangle_\alpha(w)\right| \\
&\le \frac{C}{(1-|w|^2)^{(n+\alpha+1)/2}} \left\{ \int_{\mathbb{B}} \frac{|u(z)|^2}{|1-\langle w,z\rangle|^m} \log^{-(1+\varepsilon)}\left(\frac{1}{1-|\varphi_w(z)|}\right) dv_\alpha(z)\right\}^{1/2} \\
&\quad\times \left\{ \int_{\mathbb{B}} \frac{|g(z)-(P_\alpha(g\circ\varphi_w)\circ\varphi_w)(z)|^2(1-|w|^2)^{n+\alpha+1}}{|1-\langle w,z\rangle|^{2n+2\alpha+2+m}} \right. \\
&\quad\left. \times \log^{1+\varepsilon}\left(\frac{1}{1-|\varphi_w(z)|}\right) dv_\alpha(z)\right\}^{1/2} \\
&= \frac{C}{(1-|w|^2)^{(n+m+\alpha+1)/2}} \left\| (g\circ\varphi_w - P_\alpha(g\circ\varphi_w)) \log^{(1+\varepsilon)/2}\left(\frac{1}{1-|z|}\right)\right\| \\
&\quad\times \left\{ \int_{\mathbb{B}} \frac{|u(z)|^2}{|1-\langle w,z\rangle|^m} \log^{-(1+\varepsilon)}\left(\frac{1}{1-|\varphi_w(z)|}\right) dv_\alpha(z)\right\}^{1/2}. \ \blacksquare
\end{aligned}
$$

In the case when $n = 1$ the unit ball is the unit disk $\mathbb{D}$ of the complex plane. In this setting one can prove the following analogous result.

LEMMA 2.4. *Let $-1 < \alpha < \infty$ and $\varepsilon > 0$. Then*

(a) *for every $f, h \in A^2_\alpha$ and $w \in \mathbb{D}$,*

$$
\begin{aligned}
|(T_{\bar f} h)'(w)| &\le \frac{C\{B_\varepsilon[|f|^2](w)\}^{1/2}}{(1-|w|^2)^{(\alpha+2)/2}} \\
&\quad\times \left\{ \int_{\mathbb{D}} \frac{|h(z)|^2}{|1-\bar z w|^2} \log^{-(1+\varepsilon)}\left(\frac{1}{1-|\varphi_w(z)|}\right) dv_\alpha(z)\right\}^{1/2},
\end{aligned}
$$

(b) *for $g \in L^2(\mathbb{D}, dv_\alpha)$, $u \in (A^2_\alpha)^\perp$ and $w \in \mathbb{D}$,*

$$
\begin{aligned}
|(H_g^* u)'(w)| &\le \frac{C}{(1-|w|^2)^{(\alpha+2)/2}} \left\| (g\circ\varphi_w - P_\alpha(g\circ\varphi_w)) \log^{(1+\varepsilon)/2}\left(\frac{1}{1-|z|}\right)\right\| \\
&\quad\times \left\{ \int_{\mathbb{D}} \frac{|u(z)|^2}{|1-\bar z w|^2} \log^{-(1+\varepsilon)}\left(\frac{1}{1-|\varphi_w(z)|}\right) dv_\alpha(z)\right\}^{1/2}.
\end{aligned}
$$

Proof of Theorem 2.1. Clearly, we can assume that $0 < \varepsilon < 1$. We will show that for $u, v \in A^2_\alpha$,

$$|\langle T_f T_{\bar g} u, v\rangle_\alpha| \le C\|u\|\,\|v\|.$$

We assume first that $n \ge 2$. By formula (4.11) in [8] for any positive integer m there exist complex numbers a_j, $j = 1, \ldots, 2m-1$, and b_j, $j = 1, \ldots, m$, such that the above inner product can be written as $I+II+III$, where

$$I=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2m+1)}\sum_{|\nu|=m}\int_{\mathbb{B}}(1-|w|^2)^{2m}(D^\nu T_{\bar g}u)(w)\overline{(D^\nu T_{\bar f}v)(w)}\,dv_\alpha(w),$$

$$II=\sum_{j=1}^{2m-1}a_j\sum_{|\nu|=m}\int_{\mathbb{B}}(1-|w|^2)^{2m+j}(D^\nu T_{\bar g}u)(w)\overline{(D^\nu T_{\bar f}v)(w)}\,dv_\alpha(w),$$

$$III=\sum_{j=1}^{m}b_j\int_{\mathbb{B}}(1-|w|^2)^{2m+j-1}(T_{\bar g}u)(w)\overline{(T_{\bar f}v)(w)}\,dv_\alpha(w).$$

If we assume that $m\ge n+1$, then by (a) in Lemma 2.3 and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
|II|\le C|I|&\le C\int_{\mathbb{B}}\frac{\{B_\varepsilon[|f|^2](w)\}^{1/2}}{(1-|w|^2)^{n+1-m+\alpha}}\\
&\quad\times\bigg\{\int_{\mathbb{B}}\frac{|u(z)|^2}{|1-\langle z,w\rangle|^m}\log^{-(1+\varepsilon)}\bigg(\frac{1}{1-|\varphi_w(z)|}\bigg)\,dv_\alpha(z)\bigg\}^{1/2}\\
&\quad\times\{B_\varepsilon[|g|^2](w)\}^{1/2}\\
&\quad\times\bigg\{\int_{\mathbb{B}}\frac{|v(z)|^2}{|1-\langle z,w\rangle|^m}\log^{-(1+\varepsilon)}\bigg(\frac{1}{1-|\varphi_w(z)|}\bigg)\,dv_\alpha(z)\bigg\}^{1/2}dv_\alpha(w)\\
&\le C\sup_{w\in\mathbb{B}}\{B_\varepsilon[|f|^2](w)B_\varepsilon[|g|^2](w)\}^{1/2}\\
&\quad\times\bigg\{\int_{\mathbb{B}}\int_{\mathbb{B}}\frac{|u(z)|^2}{|1-\langle z,w\rangle|^m}\log^{-(1+\varepsilon)}\bigg(\frac{1}{1-|\varphi_w(z)|}\bigg)\,dv_\alpha(z)\,dv(w)\bigg\}^{1/2}\\
&\quad\times\bigg\{\int_{\mathbb{B}}\int_{\mathbb{B}}\frac{|v(z)|^2}{|1-\langle z,w\rangle|^m}\log^{-(1+\varepsilon)}\bigg(\frac{1}{1-|\varphi_w(z)|}\bigg)\,dv_\alpha(z)\,dv(w)\bigg\}^{1/2}.
\end{aligned}$$

Now we will show that if $m<n+2$, then

$$I_1=\int_{\mathbb{B}}\int_{\mathbb{B}}\frac{|u(z)|^2}{|1-\langle z,w\rangle|^m}\log^{-(1+\varepsilon)}\bigg(\frac{1}{1-|\varphi_w(z)|}\bigg)\,dv_\alpha(z)\,dv(w)\le C\|u\|_2^2.$$

Fubini's theorem and the change of variable $w'=\varphi_z(w)$ give

$$\begin{aligned}
I_1&=\int_{\mathbb{B}}\int_{\mathbb{B}}\frac{|u(z)|^2}{|1-\langle z,w\rangle|^m}\log^{-(1+\varepsilon)}\bigg(\frac{1}{1-|\varphi_w(z)|}\bigg)\,dv_\alpha(z)\,dv(w)\\
&=\int_{\mathbb{B}}|u(z)|^2(1-|z|^2)^{n+1-m}\\
&\quad\times\int_{\mathbb{B}}\frac{1}{|1-\langle z,w'\rangle|^{2n+2-m}}\log^{-(1+\varepsilon)}\bigg(\frac{1}{1-|w'|}\bigg)\,dv(w')\,dv_\alpha(z)
\end{aligned}$$

$$= \int_{\mathbb{B}} |u(z)|^2 (1-|z|^2)^{n+1-m} \bigg(2n \int_0^1 r^{2n-1} \log^{-(1+\varepsilon)} \bigg(\frac{1}{1-r} \bigg)$$

$$\times \int_S \frac{1}{|1-\langle r\zeta, z\rangle|^{2n+2-m}} \, d\sigma(\zeta)\, dr \bigg) \, dv_\alpha(z).$$

By Theorem 1.12 in [10],

$$\int_S \frac{1}{|1-\langle r\zeta, z\rangle|^{2n+2-m}} \, d\sigma(\zeta) \le \frac{C}{(1-r|z|)^{n+2-m}} \le \frac{C}{(1-|z|)^{n+1-m}} \frac{1}{1-r}.$$

Thus

$$I_1 \le C \int_{\mathbb{B}} |u(z)|^2 \, dv_\alpha(z) \int_0^1 \log^{-(1+\varepsilon)} \bigg(\frac{1}{1-r} \bigg) \frac{r}{1-r} \, dr.$$

To see that the last integral converges for $0<\varepsilon<1$, one can write

$$\int_0^1 \log^{-(1+\varepsilon)} \bigg(\frac{1}{1-r} \bigg) \frac{r}{1-r} dr = \int_0^1 t^{-1-\varepsilon}(1-e^{-t})\, dt + \int_1^{+\infty} t^{-1-\varepsilon}(1-e^{-t})\, dt.$$

To obtain the same estimate for $|III|$ it is enough to observe that

$$|(T_{\bar f} u)(w)| = \bigg| \int_{\mathbb{B}} \frac{\overline{f(z)} u(z)}{(1-\langle z, w\rangle)^{n+\alpha+1}} \, dv_\alpha(z) \bigg| \le \int_{\mathbb{B}} \frac{|f(z)|\,|u(z)|}{|1-\langle z, w\rangle|^{n+\alpha+1}} \, dv_\alpha(z)$$

$$= \int_{\mathbb{B}} \frac{|f(z)|\,|u(z)|}{|1-\langle z, w\rangle|^{(n+2\alpha+1)/2} |1-\langle z, w\rangle|^{(n+1)/2}}$$

$$\times \log^{(1+\varepsilon)/2} \bigg(\frac{1}{1-|\varphi_w(z)|} \bigg) \log^{-(1+\varepsilon)/2} \bigg(\frac{1}{1-|\varphi_w(z)|} \bigg) \, dv_\alpha(z)$$

$$\le \frac{1}{(1-|w|^2)^{(n+\alpha+1)/2}} \bigg\{ \int_{\mathbb{B}} \frac{|f(z)|^2 (1-|w|^2)^{n+\alpha+1} |1-\langle z, w\rangle|^{n+1}}{|1-\langle z, w\rangle|^{2n+2\alpha+2}}$$

$$\times \log^{(1+\varepsilon)} \bigg(\frac{1}{1-|\varphi_w(z)|} \bigg) \, dv_\alpha(z) \bigg\}^{1/2}$$

$$\times \bigg\{ \int_{\mathbb{B}} \frac{|u(z)|^2}{|1-\langle z, w\rangle|^{n+1}} \log^{-(1+\varepsilon)} \bigg(\frac{1}{1-|\varphi_w(z)|} \bigg) \, dv_\alpha(z) \bigg\}^{1/2}$$

$$\le \frac{C\{B_\varepsilon |f|^2(w)\}^{1/2}}{(1-|w|^2)^{(n+\alpha+1)/2}} \bigg\{ \int_{\mathbb{B}} \frac{|u(z)|^2}{|1-\langle z, w\rangle|^{n+1}} \log^{-(1+\varepsilon)} \bigg(\frac{1}{1-|\varphi_w(z)|} \bigg) dv_\alpha(z) \bigg\}^{1/2}.$$

In the case $n=1$, by formula (3.5) in [7] the inner product $\langle T_f T_{\bar g} u, v\rangle_\alpha$ is equal to $I + II + III$, where

$$I = \frac{\alpha+3}{\alpha+1} \int_{\mathbb{D}} (1-|w|^2)^2 (T_{\bar g} u)(w) \overline{(T_{\bar f} v)(w)} \, dv_\alpha(w),$$

$$II = \frac{1}{(\alpha+1)(\alpha+2)} \int_{\mathbb{D}} (1-|w|^2)^2 (T'_{\bar{g}}u)(w)\overline{(T'_{\bar{f}}v)(w)}\, dv_\alpha(w),$$

$$III = \frac{1}{(\alpha+1)(\alpha+3)} \int_{\mathbb{D}} (1-|w|^2)^3 (T'_{\bar{g}}u)(w)\overline{(T'_{\bar{f}}v)(w)}\, dv_\alpha(w).$$

In view of Lemma 2.4 one can proceed analogously. ■

In view of part (b) in Lemmas 2.3 and 2.4, Theorem 2.2 can be proved in much the same way.

It follows from the next lemma that Theorem 2.1 contains Theorem 1.1.

Lemma 2.5. *Let $-1 < \alpha < \infty$ and let $f, g \in A^2_\alpha$. Then for $\varepsilon > 0$ and for $w \in \mathbb{B}$,*

$$B_\varepsilon[|f|^2](w) B_\varepsilon[|g|^2](w) \le C \cdot B[|f|^{2+\varepsilon}](w) B[|g|^{2+\varepsilon}](w).$$

Proof. Let $-1 < \alpha < \infty$, $\varepsilon > 0$ and $w \in \mathbb{B}$. By Hölder's inequality we get

$$\begin{aligned} B_\varepsilon[|f|^2](w) &= \int_{\mathbb{B}} |f(z)|^2 \log^{1+\varepsilon}\left(\frac{1}{1-|\varphi_w(z)|}\right) \frac{(1-|w|^2)^{n+\alpha+1}}{|1-\langle w,z\rangle|^{2n+2\alpha+2}}\, dv_\alpha(z) \\ &\le \left\{\int_{\mathbb{B}} |f(z)|^{2+\varepsilon} \frac{(1-|w|^2)^{n+\alpha+1}}{|1-\langle w,z\rangle|^{2n+2\alpha+2}}\, dv_\alpha(z)\right\}^{2/(2+\varepsilon)} \\ &\quad \times \left\{\int_{\mathbb{B}} \log^{(1+\varepsilon)(2+\varepsilon)/\varepsilon}\left(\frac{1}{1-|\varphi_w(z)|}\right) \frac{(1-|w|^2)^{n+\alpha+1}}{|1-\langle w,z\rangle|^{2n+2\alpha+2}}\, dv_\alpha(z)\right\}^{\varepsilon/(2+\varepsilon)} \\ &= \{B[|f|^{2+\varepsilon}](w)\}^{2/(2+\varepsilon)} \left\{\int_{\mathbb{B}} \log^{(1+\varepsilon)(2+\varepsilon)/\varepsilon}\left(\frac{1}{1-|z|}\right) dv_\alpha(z)\right\}^{\varepsilon/(2+\varepsilon)}. \end{aligned}$$

Since the last integral is convergent, our claim follows. ■

Similarly one can prove that Theorem 2.2 contains Theorem 1.4.

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