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On a Monge–Ampère type equation in the Cegrell class $\mathcal{E}_\chi$

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Abstract. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and let μ be a positive and finite measure which vanishes on all pluripolar subsets of Ω. We prove that for every continuous and strictly increasing function $\chi : (-\infty, 0) \to (-\infty, 0)$ there exists a negative plurisubharmonic function u which solves the Monge–Ampère type equation

$$-\chi(u)(dd^c u)^n = d\mu.$$

Under some additional assumption the solution u is uniquely determined.

1. Introduction. It is a classical problem in analysis to find, for a given function F, solutions u to the equation

$$(dd^c u)^n = F(z, u(z))d\mu, \tag{1.1}$$

where $(dd^c u)^n$ is the complex Monge–Ampère operator. Equations of the type (1.1) have played a significant role not only within the fields of fully nonlinear second order elliptic equations and pluripotential theory, but also in applications. We refer to [7, 8, 11, 12, 17, 20, 21] and the references therein for further information about equations of Monge–Ampère type.

Let $\mathcal{E}_0$, $\mathcal{E}_p$, $\mathcal{F}$, $\mathcal{N}$ and $\mathcal{E}$ be as in [13–15]. These are some of the so called *Cegrell classes*. The class $\mathcal{E}$ is the largest set of non-positive plurisubharmonic functions for which the complex Monge–Ampère operator is well-defined (Theorem 4.5 in [14]) and $\mathcal{N} \subset \mathcal{E}$ denotes the Cegrell class for which the smallest maximal plurisubharmonic majorant is identically equal to 0. It follows from [13–15] that $\mathcal{E}_p, \mathcal{F} \subseteq \mathcal{N}$.

These classes play a prominent role in today's pluripotential theory both in $\mathbb{C}^n$ and on compact Kähler manifolds. For further information about the Cegrell classes see e.g. [1–6, 13–15] and the references therein. In [18] (see also [10]), Guedj and Zeriahi introduced the following formalism: Let $\chi : (-\infty, 0] \to (-\infty, 0]$ be a continuous and nondecreasing function. Furthermore, let $\mathcal{E}_\chi$ contain those plurisubharmonic functions u such that there

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exists a decreasing sequence $u_j \in \mathcal{E}_0$ that converges pointwise to u on Ω, as j tends to ∞, and

$$\sup_j \int_\Omega -\chi(u_j)(dd^c u_j)^n < \infty.$$

For example, if $\chi = -(-t)^p$, then $\mathcal{E}_\chi = \mathcal{E}_p$, and if $\chi = -1$, then $\mathcal{E}_\chi = \mathcal{F}$. It should be pointed out that it is not known whether $\mathcal{E}_\chi \subseteq \mathcal{E}$ without any assumption on χ. But it was proved in [9] that if $\chi : (-\infty, 0] \to (-\infty, 0]$ is a continuous strictly increasing, convex or concave function such that $\chi(0) = 0$ and $\lim_{t\to-\infty} \chi(t) = -\infty$, then $\mathcal{E}_\chi \subset \mathcal{E}$.

The measure $(dd^c u)^n$ might have infinite total mass, i.e. $(dd^c u)^n(\Omega) = \infty$. On the other hand, if $u \in \mathcal{E}_\chi(\Omega)$, then with some additional assumptions on the function χ, the measure $(dd^c u)^n$ vanishes on all pluripolar sets in Ω, and

$$\int_\Omega -\chi(u)(dd^c u)^n < \infty.$$

Thus, $-\chi(u)(dd^c u)^n$ is a positive and finite measure defined on Ω. For this reason it is natural to consider the following Monge–Ampère type equation:

$$-\chi(u)(dd^c u)^n = d\mu,$$

where $\mu \geq 0$ is a given measure on Ω with finite total mass and that vanishes on all pluripolar subsets of Ω. In this article we prove the following theorem.

Main Theorem. *Assume that Ω is a bounded hyperconvex domain in $\mathbb{C}^n$, $n \geq 1$. Let $\chi : (-\infty, 0] \to (-\infty, 0]$ be a continuous strictly increasing function such that $\chi(0) = 0$ and $\lim_{t\to-\infty} \chi(t) = -\infty$. Furthermore, assume that $\mathcal{E}_\chi \subset \mathcal{E}$. If μ is a positive and finite measure defined on Ω such that $\mu(P) = 0$ for all pluripolar sets $P \subset \Omega$, then there exists a function $u \in \mathcal{E}_\chi$ such that*

$$-\chi(u)(dd^c u)^n = d\mu.$$

Furthermore, if $\mathcal{E}_\chi \subset \mathcal{N}$, then the solution of the above equation is uniquely determined.

Let us briefly state two immediate consequences of our Main Theorem. Let $\chi(t) = -(-t)^p$ ($p > 0$). Then we have: If μ is a positive and finite measure in Ω such that $\mu(P) = 0$ for all pluripolar sets $P \subset \Omega$, then there exists a unique function $u \in \mathcal{E}_p$ such that

$$(-u)^p(dd^c u)^n = d\mu.$$

Furthermore, if $\chi : (-\infty, 0] \to (-\infty, 0]$ is a continuous function such that $\chi(0) < 0$, and $\lim_{t\to-\infty} \chi(t) = -\infty$, then the existence of solution to the Monge–Ampère type equation given by

$$-\chi(u)(dd^c u)^n = d\mu$$

is a consequence of [17] under the assumption that $-\chi(t)^{-1}$ is bounded.

2. Proof of the Main Theorem

LEMMA 2.1. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. If a sequence $u_j \in \mathcal{F}$ satisfies the condition*

$$\sup_j \int_\Omega (dd^c u_j)^n < \infty,$$

and if there exists $u \in \mathrm{PSH}(\Omega)$ such that $u_j \to u$ weakly, then $u \in \mathcal{F}$.

Proof. From [14, Theorem 2.1] there exists $w_j \in \mathcal{E}_0 \cap \mathcal{C}(\bar{\Omega})$ such that $w_j \searrow u$, $j \to \infty$. Note that since $u_j \to u$ weakly, we have $u = \lim_{j\to\infty} v_j$, where

$$v_j = (\sup_{k\geq j} u_k)^*.$$

Observe that v_j is a decreasing sequence, $v_j \geq u_j$, so $v_j \in \mathcal{F}$ and from the comparison principle (see Theorem 5.15 in [14]) we have

$$\int_\Omega (dd^c v_j)^n \leq \int_\Omega (dd^c u_j)^n.$$

Define

$$\varphi_j = \max(w_j, v_j) \in \mathcal{E}_0.$$

Then φ_j is a decreasing sequence, $\varphi_j \searrow u$, and again by the same comparison principle we get

$$\sup_j \int_\Omega (dd^c \varphi_j)^n \leq \sup_j \int_\Omega (dd^c v_j)^n \leq \sup_j \int_\Omega (dd^c u_j)^n < \infty.$$

Thus, $u \in \mathcal{F}$. ∎

Next we shall prove our Main Theorem in the case of compactly supported measures.

LEMMA 2.2. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$, and let $\chi : (-\infty, 0] \to (-\infty, 0]$ be a continuous strictly increasing function such that $\lim_{t\to-\infty} \chi(t) = -\infty$ and $\chi(0) = 0$. If μ is a positive, finite, and compactly supported measure defined on Ω, such that $\mu(P) = 0$ for all pluripolar sets $P \subset \Omega$, then there exists a unique function $u \in \mathcal{F} \cap \mathcal{E}_\chi$ such that*

$$-\chi(u)(dd^c u)^n = d\mu. \tag{2.1}$$

Proof. If $\mu \equiv 0$, then it is clear that $u = 0$ is a solution of (2.1). Assume now that $\mu \not\equiv 0$. For $k \in \mathbb{N}$ consider the equation

$$(dd^c u_k)^n = \min\left(\frac{-1}{\chi(u_k)}, k\right) d\mu. \tag{2.2}$$

The function defined by

$$F_k(t) = \min\left(\frac{-1}{\chi(t)}, k\right)$$

is bounded and continuous. Therefore it follows from [17, Theorem 3.3] that there exists $u_k \in \mathcal{F}$ that satisfies (2.2). We also have

$$(dd^c u_k)^n = \min\left(\frac{-1}{\chi(u_k)}, k\right) d\mu \leq \frac{-1}{\chi(u_k)}\, d\mu.$$

Since μ is a positive, finite, and compactly supported measure defined on Ω and $\sup_{\operatorname{supp}\mu} u_k < c < 0$ it follows that

$$\int_\Omega \frac{-1}{\chi(u_k)}\, d\mu \leq \frac{-1}{\chi(c)}\mu(\Omega) < \infty,$$

hence

$$\sup_k \int_\Omega (-\chi(u_k))(dd^c u_k)^n \leq \mu(\Omega) < \infty.$$

We shall next prove that there exist $\alpha \in \mathcal{E}_0$ and $\beta \in \mathcal{F}$ such that

$$\beta \leq u_k \leq \alpha \qquad \text{a.e. } [d\mu],\ k \geq 2. \tag{2.3}$$

By Cegrell's decomposition theorem ([13, Theorem 6.3]) there exist functions $\phi \in \mathcal{E}_0$ and $f \in L^1((dd^c\phi)^n)$, $f \geq 0$, such that

$$\mu = f(dd^c\phi)^n.$$

Fix $a > 0$ such that $\chi(-a) \geq -1/2$. Then by Kołodziej's subsolution theorem there exists $\alpha \in \mathcal{E}_0$ such that (see [19, Theorem A])

$$(dd^c\alpha)^n = \min\left(f, \frac{a^n}{\|\phi\|^n}\right)(dd^c\phi)^n,$$

where $\|\phi\| = \sup_{z\in\Omega} |\phi(z)|$. The comparison principle (see [15, Theorem 3.7]) yields

$$\alpha \geq \frac{a}{\|\phi\|}\phi \geq -a$$

and

$$\int_{\{\alpha<u_k\}} (dd^c u_k)^n \leq \int_{\{\alpha<u_k\}} (dd^c\alpha)^n \leq \int_{\{\alpha<u_k\}} d\mu.$$

Observe that on the set $\{\alpha < u_k\}$ we have $u_k > -a$ and

$$(dd^c u_k)^n = \min\left(\frac{-1}{\chi(u_k)}, k\right) d\mu \geq \min\left(\frac{-1}{\chi(-a)}, k\right) d\mu \geq 2d\mu, \qquad k \geq 2,$$

which implies that $\mu(\{\alpha < u_k\}) = 0$, $k \geq 2$. There exists $\psi \in \mathcal{F}$ such that $(dd^c\psi)^n = d\mu$ (see [14, Lemma 5.14]). Fix $w \in \mathcal{E}_0$ and $b > 0$ such that

$$\chi(\sup_{\operatorname{supp}\mu}(\psi + bw)) < -2.$$

Let $\beta = \psi + bw$. Note that $(dd^c\beta)^n \geq d\mu$. By the comparison principle (see [15, Corollary 3.6]) we obtain

$$\int_{\{u_k<\beta\}} d\mu \leq \int_{\{u_k<\beta\}} (dd^c\beta)^n \leq \int_{\{u_k<\beta\}} (dd^c u_k)^n,$$

but on the set $\{u_k < \beta\} \cap \operatorname{supp}\mu$ we have $u_k < \beta \leq \sup_{\operatorname{supp}\mu}\beta$ and

$$(dd^c u_k)^n = \min\left(\frac{-1}{\chi(u_k)}, k\right) d\mu \leq \frac{1}{2} d\mu,$$

which means that $\mu(\{u_k < \beta\}) = 0$ for all k.

Now it follows from (2.3) that there exist a plurisubharmonic function $u \neq 0$ and a subsequence (also denoted by u_k) such that $u_k \to u$ almost everywhere $[d\mu]$. Since $u \neq 0$ it follows that

$$-\frac{1}{\chi(\sup_{\operatorname{supp}\mu} u)} < \infty.$$

By Hartogs' lemma, the functions

$$F_k(u_k) = \min(-\chi(u_k)^{-1}, k)$$

are uniformly bounded on $\operatorname{supp}\mu$ and therefore

$$\sup_k \int_\Omega (dd^c u_k)^n \leq \sup_k \int_\Omega F_k(u_k)\, d\mu < \infty.$$

Lemma 2.1 yields $u \in \mathcal{F}$.

The stability theorem proved in [17, Corollary after Theorem 2.2″] implies that the weak convergence, $u_k \to u$, is equivalent to convergence in capacity. Since $u_k \geq \beta$ and $u_k \to u$ in capacity, by [16, Theorem 1.1] we get $(dd^c u_k)^n \to (dd^c u)^n$ in the weak*-topology. Therefore the dominated convergence theorem yields

$$(dd^c u)^n = \lim_{k\to\infty} (dd^c u_k)^n = \lim_{k\to\infty} F_k(u_k) d\mu = \frac{-1}{\chi(u)}\, d\mu.$$

So we have proved that there exists a solution $u \in \mathcal{F}$ to (2.1). Then

$$\int_\Omega (-\chi(u))(dd^c u)^n < \infty$$

and it follows that $u \in \mathcal{E}_\chi$.

It is proved in the proof of the Main Theorem below that if $u, v \in \mathcal{F}$ are solutions of (2.1) then $(dd^c u)^n = (dd^c v)^n$ and therefore $u = v$ (see [14, Lemma 5.14]). ∎

Proof of the Main Theorem. Assume that μ is a positive and finite measure in Ω such that $\mu(P) = 0$ for all pluripolar sets $P \subset \Omega$. Let Ω_j be a fundamental sequence of strictly pseudoconvex domains, i.e. $\Omega_j \Subset \Omega_{j+1} \Subset \Omega$ and $\bigcup_{j=1}^\infty \Omega_j = \Omega$ (see [15]). Let us define $d\mu_j = \mathbf{1}_{\Omega_j} d\mu$, where $\mathbf{1}_{\Omega_j}$ is the characteristic function for Ω_j. By Lemma 2.2 there exists a sequence $u_j \in \mathcal{F} \cap \mathcal{E}_\chi$

such that

$$-\chi(u_j)(dd^c u_j)^n = d\mu_j.$$

We shall now prove that u_j is a decreasing sequence. Let $A = \{z \in \Omega : u_j(z) < u_{j+1}(z)\}$. On the set A, we have

$$\begin{aligned}(dd^c u_j)^n &= -\chi(u_j)^{-1} d\mu_j \leq -\chi(u_{j+1})^{-1} d\mu_j \\ &\leq -\chi(u_{j+1})^{-1} d\mu_{j+1} = (dd^c u_{j+1})^n\end{aligned}$$

and by the comparison principle (see [15, Corollary 3.6]) we get

$$\int_A (dd^c u_{j+1})^n \leq \int_A (dd^c u_j)^n.$$

Hence,

$$(dd^c u_j)^n = (dd^c u_{j+1})^n \tag{2.4}$$

on A. Similarly on the set $\Omega_j \setminus A = \{z \in \Omega_j : u_j(z) \geq u_{j+1}(z)\}$ we obtain

$$\begin{aligned}(dd^c u_j)^n &= -\chi(u_j)^{-1} d\mu_j \geq -\chi(u_{j+1})^{-1} d\mu_j \\ &= -\chi(u_{j+1})^{-1} d\mu_{j+1} = (dd^c u_{j+1})^n.\end{aligned} \tag{2.5}$$

From the equalities (2.4) and (2.5) we get $(dd^c u_j)^n \geq (dd^c u_{j+1})^n$ on Ω_j. This implies that $-\chi(u_j)^{-1} d\mu_j \geq -\chi(u_{j+1})^{-1} d\mu_j$ and then $\chi(u_j) \geq \chi(u_{j+1})$ a.e. $[d\mu_j]$, so $u_j \geq u_{j+1}$ a.e. $[d\mu_j]$. Hence $\mu_j(\{u_j < u_{j+1}\}) = 0$ and $(dd^c u_j)^n = 0$ on $A \cap \Omega_j$. Since $(dd^c u_j)^n = d\mu_j = 0$ on $\Omega \setminus \Omega_j$ we finally obtain $(dd^c u_j)^n = 0$ on $A = \{u_j < u_{j+1}\}$. Now take

$$\psi \in \mathcal{E}_0 \quad \text{such that} \quad (dd^c \psi)^n = d\lambda, \tag{2.6}$$

where $d\lambda$ is the Lebesgue measure, and consider $A_k = \{z \in \Omega : u_j < u_{j+1} + k^{-1}\psi\}$. Observe that $u_{j+1} + k^{-1}\psi \in \mathcal{F}$ and $A_k \subset A$. By the comparison principle (see [15, Corollary 3.6]) we obtain

$$\int_{A_k} (dd^c(u_{j+1} + k^{-1}\psi))^n \leq \int_{A_k} (dd^c u_j)^n \leq \int_A (dd^c u_j)^n = 0,$$

and then

$$0 = \int_{A_k} (dd^c(u_{j+1} + k^{-1}\psi))^n \geq \frac{1}{k^n} \int_{A_k} (dd^c \psi)^n = \frac{1}{k^n} \lambda(A_k),$$

which means that $\lambda(A_k) = 0$. Hence $\lambda(A) = 0$, since $A = \bigcup_{k=1}^{\infty} A_k$. We have proved that $u_j \geq u_{j+1}$ a.e. $[d\lambda]$, but since the functions u_j, u_{j+1} are plurisubharmonic we obtain $u_j \geq u_{j+1}$ on Ω, so u_j is a decreasing sequence. Note also that

$$\sup_j \int_\Omega -\chi(u_j)(dd^c u_j)^n \leq \int_\Omega d\mu < \infty. \tag{2.7}$$

Moreover from the standard measure theory it follows that

$$\int_{\Omega} -\chi(u_j)(dd^c u_j)^n = \int_0^\infty \chi'(-t)(dd^c u_j)^n(\{u_j < -t\})\,dt.$$

Since $u_j \in \mathcal{F}$, from [10, Corollary 2.5] we obtain

$$(dd^c u_j)^n(\{u_j < -t\}) \geq t^n C_n(\{u_j < -2t\}),$$

where C_n is the Bedford–Taylor capacity, defined in [8]. Therefore

$$\begin{aligned}(2.8)\qquad \sup_j \int_0^\infty \chi'(-t)t^n C_n(\{u_j < -2t\})\,dt &\leq \sup_j \int_{\Omega} -\chi(u_j)(dd^c u_j)^n \\ &\leq \int_{\Omega} d\mu < \infty.\end{aligned}$$

Since u_j is a decreasing sequence it follows that there exists u such that $u_j \searrow u$, and $u \in \mathrm{PSH}(\Omega)$ or $u \equiv -\infty$. Suppose that $u \equiv -\infty$. Then for any $t < 0$ we have $C_n(\{u_j < -2t\}) \to \infty$ as $j \to \infty$ and therefore

$$\sup_j \int_0^\infty \chi'(-t)t^n C_n(\{u_j < -2t\})dt = \infty,$$

which leads to a contradiction with condition (2.8). This means that $u \in \mathrm{PSH}(\Omega)$ and condition (2.7) implies that $u \in \mathcal{E}_\chi$. Since the complex Monge–Ampère operator is continuous in the class $\mathcal{E}$ with respect to decreasing sequences (see Lemma 3.2 in [15]) it follows that $(dd^c u_j)^n$ tends to $(dd^c u)^n$ in the weak*-topology. Therefore using the monotone convergence theorem we get

$$(dd^c u)^n = \lim_{j\to\infty} (dd^c u_j)^n = \lim_{j\to\infty} -\chi(u_j)^{-1}\mathbf{1}_{\Omega_j}\,d\mu = -\chi(u)^{-1}d\mu.$$

This ends the proof of the existence part of the theorem.

Now we turn to uniqueness. Suppose that there exist $u, v \in \mathcal{E}_\chi \cap \mathcal{N}$ such that $-\chi(u)(dd^c u)^n = -\chi(v)(dd^c v)^n = d\mu$. Observe that on the set $\{z \in \Omega : u(z) < v(z)\}$ we have

$$(dd^c u)^n = -\chi(u)^{-1}d\mu \leq -\chi(v)^{-1}d\mu = (dd^c v)^n.$$

Using the comparison principle (see [3, Theorem 3.1]) we obtain

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n,$$

so $(dd^c u)^n = (dd^c v)^n$ on $\{z \in \Omega : u(z) < v(z)\}$. Similarly, $(dd^c u)^n = (dd^c v)^n$ on $\{z \in \Omega : u(z) > v(z)\}$. Since μ does not put mass on pluripolar sets and $\{u = -\infty\} = \{\chi(u) = -\infty\}$ and $\{v = -\infty\} = \{\chi(v) = -\infty\}$, it follows that $(dd^c u)^n = (dd^c v)^n = 0$ on $C = \{u = -\infty\} \cup \{v = -\infty\}$. On $\{u = v\} \setminus C$ we also have

$$(dd^c u)^n = -\chi(u)^{-1} d\mu = -\chi(v)^{-1} d\mu = (dd^c v)^n.$$

Thus, $(dd^c u)^n = (dd^c v)^n$ on Ω, which implies that $u = v$ by [15, Theorem 3.1]. ∎

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