On “special” fibred coordinates for general and classical connections

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Abstract. Using a general connection $\Gamma$ on a fibred manifold $p : Y \to M$ and a torsion free classical linear connection $\nabla$ on $M$, we distinguish some “special” fibred coordinate systems on $Y$, and then we construct a general connection $\tilde{\mathcal{F}}(\Gamma, \nabla)$ on $Fp : FY \to FM$ for any vector bundle functor $F : Mf \to \mathcal{VB}$ of finite order.

1. Introduction. A general connection on a fibred manifold $Y \to M$ is a section $\Gamma : Y \to J^1Y$ of the first jet prolongation $J^1Y \to Y$ of $Y \to M$, which can be (equivalently) considered as the corresponding lifting map $\Gamma : Y \times_M TM \to TY$. If $p : Y \to M$ is a vector bundle and $\Gamma : Y \to J^1Y$ is a vector bundle map, then $\Gamma$ is called a linear general connection on $p : Y \to M$. A linear general connection $\Gamma : TM \to J^1TM$ on the tangent bundle $p : TM \to M$ of $M$ is called a classical linear connection on $M$, which can be (equivalently) considered as its corresponding covariant derivative $\nabla$. A classical linear connection $\nabla$ on $M$ is called torsion free if its torsion tensor is zero. More information on connections can be found in the fundamental monograph [KMS].

The present short note is devoted to studying prolongation of connections. In Section 2, given a general connection $\Gamma$ on a fibred manifold $p : Y \to M$ and a torsion free classical linear connection $\nabla$ on $M$, we distinguish some “special” so called $(\Gamma, \nabla)$-quasi-normal fibred coordinate systems on $Y$. In fact, we essentially strengthen [M3, Lemma 2]. In Section 3, applying these $(\Gamma, \nabla)$-quasi-normal fibred coordinate systems, we construct a general connection $\mathcal{F}(\Gamma, \nabla)$ on $Fp : FY \to FM$ for any vector bundle functor $F : Mf \to \mathcal{VB}$ (the concept of bundle functors can be found in [KMS]).

We recall that in [S], J. Slovák constructed a general connection $\mathcal{F}(\Gamma)$ on $Fp : FY \to FM$ from a general connection $\Gamma$ on a fibred manifold $p : Y \to M$ for any product-preserving bundle functor $F : Mf \to \mathcal{FM}$. In

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(see also [M4]), we proved that if a vector bundle functor \( F : \mathcal{M}f \to \mathcal{VB} \) (or a bundle functor \( F : \mathcal{M}f \to \mathcal{FM} \) with the point property) is not product-preserving then there is no canonical construction of a general connection on \( Fp : FY \to FM \) from a general connection \( \Gamma \) on a fibred manifold \( p : Y \to M \). Consequently, we see that an auxiliary torsion free classical linear connection \( \nabla \) on \( M \) is unavoidable to construct a general connection on \( Fp : FY \to FM \) from a general connection \( \Gamma \) on a fibred manifold \( p : Y \to M \). We also recall that in [M2], given a general connection \( \Gamma \) on a fibred manifold \( p : Y \to M \) and a \( p \)-projectable torsion free classical linear connection \( \tilde{\nabla} \) on \( Y \), we constructed a general connection \( A^F(\Gamma, \tilde{\nabla}) \) on \( Fp : FY \to FM \) for any vector bundle functor \( F : \mathcal{M}f \to \mathcal{VB} \) of finite order. Of course, torsion free classical linear connections on \( M \) are “simpler” objects than \( p \)-projectable torsion free classical linear connections on \( Y \). So, the construction \( \tilde{F}(\Gamma, \nabla) \) (presented in this note) is “more economic” than \( A^F(\Gamma, \tilde{\nabla}) \) from [M2].

All manifolds and maps we consider are assumed to be \((C^\infty)\) smooth.

2. On quasi-normal fibred coordinate systems. Just as in [M3], let 
\[
\Phi_r : J_0^{r-1}(T^*\mathbb{R}^m \otimes \mathbb{R}^n) \to J_0^r(\mathbb{R}^m, \mathbb{R}^n)_0
\]
be the usual symmetrization 
\[
\bigoplus_{q=0}^{r-1} S^q T_0^* \mathbb{R}^m \otimes T_0^* \mathbb{R}^m \otimes \mathbb{R}^n \to \bigoplus_{q=0}^{r-1} S^{q+1} T_0^* \mathbb{R}^m \otimes \mathbb{R}^n
\]
modulo the obvious \((\text{GL}(m))\)-invariant) identifications 
\[
J_0^{r-1}(T^*\mathbb{R}^m \otimes \mathbb{R}^n) = \bigoplus_{q=0}^{r-1} S^q T_0^* \mathbb{R}^m \otimes T_0^* \mathbb{R}^m \otimes \mathbb{R}^n
\]
and 
\[
\bigoplus_{q=0}^{r-1} S^{q+1} T_0^* \mathbb{R}^m \otimes \mathbb{R}^n = J_0^r(\mathbb{R}^m, \mathbb{R}^n)_0.
\]

In other words, \( \Phi_r : J_0^{r-1}(T^*\mathbb{R}^m \otimes \mathbb{R}^n) \to J_0^r(\mathbb{R}^m, \mathbb{R}^n)_0 \) is the linear map such that 
\[
\Phi_r(j_0^{r-1}((x^{i_1} \ldots x^{i_q} dx^j)e_k)) = \frac{1}{q+1} j_0^r(x^{i_1} \ldots x^{i_q} x^j e_k)
\]
for any \( i_1, \ldots, i_q, j = 1, \ldots, m, q = 0, \ldots, r - 1 \) and \( k = 1, \ldots, n \), where \( e_k \) is the usual canonical basis in \( \mathbb{R}^n \) and \( x^1, \ldots, x^m \) are the usual coordinates on \( \mathbb{R}^m \). Then 
\[
\Phi_r(j_0^{r-1}(d\sigma)) = j_0^r(\sigma)
\]
for any \( \sigma : \mathbb{R}^m \to \mathbb{R}^n \) with \( \sigma(0) = 0 \). Clearly, \( \Phi_r \) is \((\text{GL}(m))\)-invariant and linear.
Let \( m, n, r \) be positive integers, \( \Gamma : Y \to J^1Y \) be a general connection on a fibred manifold \( p : Y \to M \) with \( \dim(m) = M \) and \( \dim(Y) = m + n \), \( \nabla \) be a torsion free classical linear connection on \( M \) and \( y_0 \in Y \) be a point with \( x_0 = p(y_0) \in M \).

**Definition 2.1.** A \((\Gamma, \nabla, y_0, r)\)-quasi-normal coordinate system on \( Y \) is a fibred chart \( \psi \) on \( Y \) with \( \psi(y_0) = (0, 0) \in \mathbb{R}^m \times \mathbb{R}^n \) covering a \( \nabla \)-normal coordinate system \( \overline{\psi} \) on \( M \) with centre \( x_0 \) such that

\[
(2.1) \quad \Phi_r\left(j_0^{r-1}\left( \sum_{|\alpha|+|\beta|\leq r-1} \sum_{j=1}^n \sum_{k=1}^m \Gamma^k_{j,\alpha\beta} x^\alpha dx^j \otimes e_k \right) \right) = 0
\]

for any \( \beta \in (\mathbb{N} \cup \{0\})^n \) with \( |\beta| \leq r - 1 \), where

\[
(2.2) \quad j_0^{r-1}\left( \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{|\alpha|+|\beta|\leq r-1} \sum_{j=1}^n \sum_{k=1}^m \Gamma^k_{j,\alpha\beta} x^\alpha y^\beta dx^j \otimes \frac{\partial}{\partial y^k} \right)
\]

is the coordinate expression of \( j_0^{r-1}(\psi_* \Gamma) \) and \( x^1, \ldots, x^m, y^1, \ldots, y^n \) are the usual coordinates on \( \mathbb{R}^m \times \mathbb{R}^n \).

We prove the following result, which is essentially stronger than the one in [M3] Lemma 2].

**Proposition 2.2.** Let \( m, n, r, p : Y \to M, \Gamma : Y \to J^1Y, \nabla, y_0 \in Y, x_0 = p(y_0) \in M \) be as above.

(a) There exists a \((\Gamma, \nabla, y_0, r)\)-quasi-normal fibred coordinate system \( \psi \).

(b) If \( \psi^1 \) is another \((\Gamma, \nabla, y_0, r)\)-quasi-normal coordinate system on \( Y \) then

\[
(2.3) \quad j_{y_0}^r \psi^1 = j_{y_0}^r \left( (A \times H) \circ \psi \right)
\]

for a map \( A \in GL(m) \) and a diffeomorphism \( H : \mathbb{R}^n \to \mathbb{R}^n \) preserving \( 0 \).

**Proof.** (a) Because of the existence of \( \nabla \)-normal coordinate systems and the fact that \( J^1Y \) is the orbit of \( j_0^1(0) \) under the action of the pseudo-group of local fibred diffeomorphisms, we may assume that \( Y = \mathbb{R}^m \times \mathbb{R}^n \), \( M = \mathbb{R}^m \), \( p = pr_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \) is the obvious projection, \( y_0 = (0, 0) \), \( id_{\mathbb{R}^m} \) is a \( \nabla \)-normal coordinate system with centre \( 0 \), formula (2.2) is the coordinate expression of \( j_{(0,0)}^{r-1} \Gamma \) for any \( r \), and \( \Gamma^k_{j(0)(0)} = 0 \) for \( k = 1, \ldots, n \) and \( j = 1, \ldots, m \).

We will proceed by induction on \( r \).

The case \( r = 1 \) is trivial because \( \Gamma^k_{j(0)(0)} = 0 \).

Now, we assume that there exists a \((\Gamma, \nabla, (0,0), r-1)\)-quasi-normal fibred coordinate system, \( r \geq 2 \). Replacing \( \Gamma \) by the image of \( \Gamma \) under this fibred chart (this \((\Gamma, \nabla, (0,0), r-1)\)-quasi-normal fibred coordinate system), we can assume that \( id_{\mathbb{R}^m \times \mathbb{R}^n} \) is a \((\Gamma, \nabla, (0,0), r-1)\)-quasi-normal fibred coordinate
system. Next, for any \( \beta \in (\mathbb{N} \cup \{0\})^n \) with \( |\beta| \leq r - 1 \), let \( \sigma_{\beta} = (\sigma_{\beta}^k)_{k=1}^n : \mathbb{R}^m \to \mathbb{R}^n \) be a map with \( \sigma_{\beta}(0) = 0 \) such that

\[
j_0^r(\sigma_{\beta}) = \Phi_r \left( j_0^{r-1} \left( \sum_{|\alpha|+|\beta|=r-1} \sum_{j=1}^m \sum_{k=1}^n \Gamma_{j\alpha\beta}^k x^\alpha dx^j \otimes e_k \right) \right).
\]

Define \( \psi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n \) by

\[
\psi(x, y) = \left(x, y - \sum_{|\beta| \leq r-1} \sum_{|\beta|=0} y^\beta \sigma_\beta(x) \right).
\]

It is easy to see that \( j_{(0,0)}^{r-1} \psi = \text{id} \). We prove that \( \psi \) is a \( (\Gamma, \nabla, (0,0), r) \)-quasi-normal fibred coordinate system. We see that \( \psi \) preserves

\[
j_{(0,0)}^{r-1} \left( \sum_{|\alpha|+|\beta| \leq r-1} \sum_{j=1}^m \sum_{k=1}^n \Gamma_{j\alpha\beta}^k x^\alpha y^\beta dx^j \otimes \frac{\partial}{\partial y^k} \right)
\]

because \( j_{(0,0)}^{r-1} \psi = \text{id} \), \( \psi = \text{id} \) and \( \Gamma_{j\alpha\beta}^k = 0 \). Moreover, \( \psi \) sends \( j_{(0,0)}^{r-1} (\sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i}) \) to

\[
j_{(0,0)}^{r-1} \left( \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} - \sum_{|\beta| \leq r-1} \sum_{k=1}^n \frac{\partial \sigma_\beta^k}{\partial x^i}(x)y^\beta dx^i \otimes \frac{\partial}{\partial y^k} \right).
\]

Then expressing \( j_{(0,0)}^{r-1}(\psi_* \Gamma) \) by (2.2) with \( \tilde{\Gamma}_{j\alpha\beta}^k \) instead of \( \Gamma_{j\alpha\beta}^k \) we see that

\[
\Phi_r \left( j_0^{r-1} \left( \sum_{|\alpha|+|\beta| \leq r-1} \sum_{j=1}^m \sum_{k=1}^n \tilde{\Gamma}_{j\alpha\beta}^k x^\alpha dx^j \otimes e_k \right) \right)
\]

\[
= \Phi_r \left( j_0^{r-1} \left( \sum_{|\alpha|+|\beta| \leq r-1} \sum_{j=1}^m \sum_{k=1}^n \Gamma_{j\alpha\beta}^k x^\alpha dx^j \otimes e_k \right) \right) - \Phi_r(j_{(0,0)}^{r-1}(d\sigma_\beta))
\]

\[
= j_0^r(\sigma_\beta) - j_0^r(\sigma_\beta) = 0.
\]

for any \( \beta \in (\mathbb{N} \cup \{0\})^n \) with \( |\beta| \leq r - 1 \) (as \( \Phi_r(j_0^{r-1}(\sum_{|\alpha|+|\beta| \leq r-2} \Gamma_{j\alpha\beta}^k x^\alpha dx^j \otimes e_k)) = 0 \) since \( \text{id}_{\mathbb{R}^m \times \mathbb{R}^n} \) is \( (\Gamma, \nabla, (0,0), r - 1) \)-adapted).

(b) Replacing \( \Gamma \) by \( \psi_* \Gamma \) we may assume that \( \text{id}_{\mathbb{R}^m \times \mathbb{R}^n} \) is a \( (\Gamma, \nabla, (0,0), r) \)-quasi-normal fibred coordinate system. Next, we will proceed by induction on \( r \).

The case \( r = 1 \) is clear.

Now, let \( \psi^1 \) be a \( (\Gamma, \nabla, (0,0), r) \)-quasi-normal fibred coordinate system. Then (clearly) it is a \( (\tilde{\Gamma}, \nabla, (0,0), r - 1) \)-quasi-normal fibred coordinate system. Hence by the inductive assumption, \( j_{(0,0)}^{r-1} \psi^1 = j_{(0,0)}^{r-1}(A \times \tilde{H}) \) for some \( A \in \text{GL}(m) \) and some \( \tilde{H} : \mathbb{R}^n \to \mathbb{R}^n \). Clearly, \( A \times \tilde{H} \) is a \( (\Gamma, \nabla, (0,0), r) \)-quasi-normal coordinate system. Then replacing \( \Gamma \) by \( (A \times \tilde{H})_* \Gamma \) we may assume
We start with the following lemma. Let $F$ be the trivial bundle (the obvious projection). An isomorphism $\psi$ is given by $\psi(x, y) = (x, \psi^1(x, y))$, we may assume $j^{-1}_{(0,0)}\psi^1 = \text{id}$ and

$$\psi^1(x, y) = \left(x, y - \sum_{|\beta| \leq r-1} y^\beta \sigma_\beta(x)\right),$$

where $\sigma_\beta : \mathbb{R}^m \to \mathbb{R}^n$, $\sigma_\beta(0) = 0$. Then (quite similarly to the inductive step in the proof of (a)) expressing $j^{-1}_{(0,0)}(\psi^1)$ by \eqref{2.2} with $\Gamma_{j\alpha\beta}^k$ instead of $\Gamma_{j \alpha \beta}^k$ we get

$$0 = \Phi_r \left( j_{0}^{-1} \left( \sum_{|\alpha|+|\beta| \leq r-1} \sum_{j=1}^m \sum_{k=1}^n \tilde{\Gamma}^k_{j \alpha \beta} x^\alpha dx^j \otimes e_k \right) \right) = 0 - j_{0}^{-1}(\sigma_\beta)$$

for any $\beta \in (\mathbb{N} \cup \{0\})^n$ with $|\beta| \leq r - 1$. So, $j^{-1}_{(0,0)}\psi^1 = \text{id}$.  

3. An application to prolongation of connections. We are going to apply the result from the previous section to prolongation of connections. We start with the following lemma.

**Lemma 3.1.** Let $F : \mathcal{M} f \to \mathcal{V} \mathcal{B}$ be a vector bundle functor. Let $\text{pr}_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ be the trivial bundle (the obvious projection).

(a) The fibred manifold $F \text{pr}_1 : F(\mathbb{R}^m \times \mathbb{R}^n) \to F\mathbb{R}^m$ is isomorphic to the trivial bundle $\text{Pr}_1 : (\mathbb{R}^m \times F_0\mathbb{R}^m) \times (\mathbb{R}^n \times \ker(F_{(0,0)}\text{pr}_1)) \to \mathbb{R}^m \times F_0\mathbb{R}^m$ (the obvious projection). An isomorphism $\Phi : (\mathbb{R}^m \times F_0\mathbb{R}^m) \times (\mathbb{R}^n \times \ker(F_{(0,0)}\text{pr}_1)) \to F(\mathbb{R}^m \times \mathbb{R}^n)$ is given by

$$\Phi((x, X), (y, Y)) = F\tau_{(x,y)}(F_0i(X) + Y)$$

for any $x \in \mathbb{R}^m$, $X \in F_0\mathbb{R}^m$, $y \in \mathbb{R}^n$, $Y \in \ker(F_{(0,0)}\text{pr}_1)$, where $i : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^n$ is given by $i(x) = (x, 0)$ and $\tau_{(x,y)} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ is the translation by $(x, y)$. The inverse isomorphism is given by

$$\Phi^{-1}(v) = ((x, X), (y, Y))$$

for $v \in F_{(x,y)}(\mathbb{R}^m \times \mathbb{R}^n)$, $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, where $X = F\tau_{-x}(F\text{pr}_1(v)) \in F_0\mathbb{R}^m$ and $Y = F\tau_{-(x,y)}(v) - F_0i(X) \in \ker(F_{(0,0)}\text{pr}_1)$.

(b) Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be a local diffeomorphism. Then (under the above isomorphism)

$$F(\text{id}_{\mathbb{R}^m} \times H)((x, X), (y, Y)) = ((x, X), \tilde{H}(y, Y))$$

for any $((x, X), (y, Y)) \in F(\mathbb{R}^m \times \mathbb{R}^n)$, where $\tilde{H} : \mathbb{R}^n \times \ker(F_{(0,0)}\text{pr}_1) \to \mathbb{R}^n \times \ker(F_{(0,0)}\text{pr}_1)$ is given by

$$\tilde{H}(y, Y) = (H(y), F_{(0,0)}(\text{id}_{\mathbb{R}^m} \times (\tau_{-H(y)} \circ H \circ \tau_y))(Y)).$$

(c) Let $A \in \text{GL}(m)$. Then

$$F(A \times \text{id}_{\mathbb{R}^n})((x, X), (y, Y)) = (A_1(x, X), A_2(y, Y))$$
for any \((x, X), (y, Y)\) \in \(F(\mathbb{R}^m \times \mathbb{R}^n)\), where \(A_1 : \mathbb{R}^m \times F_0 \mathbb{R}^m \to \mathbb{R}^m \times F_0 \mathbb{R}^n\) is given by \(A_1(x, X) = (A(x), F_0 A(X))\) and \(A_2 : \mathbb{R}^n \times \ker(F_{(0,0)} \text{pr}_1) \to \mathbb{R}^n \times \ker(F_{(0,0)} \text{pr}_1)\) is given by \(A_2(y, Y) = (y, F_{(0,0)}(A \times \text{id}_{\mathbb{R}^n})(Y))\).

**Proof.** The proof is standard. ■

We now present the following example of prolongation of connections.

**Example 3.2.** Let \(F : \mathcal{M} f \to \mathcal{V} B\) be a vector bundle functor of order \(r\). Consider a general connection \(\Gamma\) on a fibred manifold \(p : Y \to M\) and a torsion-free classical linear connection \(\nabla\) on \(M\). Write \(\text{dim}(M) = m\) and \(\text{dim}(Y) = m + n\). We construct a general connection \(\tilde{\mathcal{F}}(\Gamma, \nabla)\) on \(F_p : F Y \to F M\) as follows. Let \(u_0 \in F_{y_0} Y\), \(y_0 \in Y\). Let \(\psi\) be a \((\Gamma, \nabla, y_0, r + 1)\)-quasi-normal fibred coordinate system (see Proposition 2.2(a)). We put

\[
\tilde{\mathcal{F}}(\Gamma, \nabla)(u_0) := J^1(\mathcal{F}(\psi^{-1})(\Theta(\mathcal{F}(\psi(u_0)))),
\]

where \(\Theta\) denotes the trivial general connection on the trivial bundle \(F\text{pr}_1 : F(\mathbb{R}^m \times \mathbb{R}^n) \to F \mathbb{R}^m\) (modulo the isomorphism from Lemma 3.1(a)). Suppose \(\psi^1\) is another \((\Gamma, \nabla, y_0, r + 1)\)-quasi-normal fibred coordinate system. Then (by Proposition 2.2(b)) \(j^{r+1}_{y_0}((\psi^1)) = j^{r+1}_{y_0}(A \times H) \circ \psi\) for some \(A \in \text{GL}(m)\) and some local diffeomorphism \(H : \mathbb{R}^n \to \mathbb{R}^n\). But \(\Theta\) is invariant with respect to \(F(A \times H)\) because of Lemma 3.1(b)–(c). Hence

\[
J^1((\psi^1)^{-1}))(\Theta(\mathcal{F}(\psi^1)(u_0))) = J^1(\mathcal{F}(\psi^{-1}))(\Theta(\mathcal{F}(\psi(u_0)));
\]

i.e. \(\tilde{\mathcal{F}}(\Gamma, \nabla)\) is correctly defined. As we can choose a family of \((\Gamma, \nabla, y_0, r + 1)\)-quasi-normal fibred coordinate systems to be smooth in \(y_0\) (it is sufficient to analyse the proof of Proposition 2.2(a)), \(\tilde{\mathcal{F}}(\Gamma, \nabla)\) is a smooth general connection on \(F_p : F Y \to F M\).

**4. A final remark.** It seems that our quasi-normal fibred coordinate systems from Proposition 2.2 may be used in the classification problems of natural operators \(A(\Gamma, \nabla)\) on pairs \((\Gamma, \nabla)\) of connections. Indeed, according to the general theorem on natural operators \([\text{KMS}]\), such natural operators are in bijection with \(G_{m,n}^r\)-invariant maps of respective type. Now, passing to quasi-normal fibred coordinate systems, we see that natural operators \(A(\Gamma, \nabla)\) are in bijection with \(\text{GL}(m) \times G_{n}^r\)-invariant maps. So, Proposition 2.2 gives a very strict reduction for natural operators on pairs \((\Gamma, \nabla)\) of connections. We hope that (for example) we may benefit from the above reduction and find all natural operators \(A\) constructing general connections \(A(\Gamma, \nabla)\) on \(J^2 Y \to M\) from general connections \(\Gamma\) on \(Y \to M\) by means of torsion-free classical linear connections \(\nabla\) on \(M\). We note that a reduction for gauge natural operators on pairs \((\Gamma, \nabla)\) of principal and classical linear connections has been described in \([\text{DM}], [\text{JV}].\) Some “special” principal coordinate systems for principal and classical connections have been described in \([\text{H}], [\text{DM}], [\text{K}].\)
References


[M4] —, *Bundle functors with the point property which admit prolongation of connections*, ibid. 97 (2010), 253–256.


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