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Noguchi-type convergence-extension theorems for (n, d)-sets

by Do Duc Thai (Hanoi), Nguyen Thi Tuyet Mai (Thainguyen) and Nguyen Thai Son (Hochiminh City)

Abstract. We introduce the notion of (n, d)-sets and show several Noguchi-type convergence-extension theorems for (n, d)-sets.

1. Introduction. The theorem of Noguchi referred to in the title of this paper can be stated as follows (see [19] or [21]):

Let M be relatively compact hyperbolically imbedded into Y. Let X be a complex manifold and A a complex hypersurface of X with only normal crossings.

If $\{f_j : X \setminus A \to M\}_{j=1}^{\infty}$ is a sequence of holomorphic mappings which converges uniformly on compact subsets of $X \setminus A$ to a holomorphic mapping $f : X \setminus A \to M$, then $\{\bar{f}_j\}_{j=1}^{\infty}$ converges uniformly on compact subsets of X to $\bar{f}$, where $\bar{f}_j : X \to Y$ and $\bar{f} : X \to Y$ are the unique holomorphic extensions of f_j and f over X.

The above theorem of Noguchi opened a new perspective in studying problems of extending holomorphic mappings, namely to study Noguchi-type convergence-extension theorems. More precisely, a "Noguchi-type convergence-extension theorem" means a theorem analogous to the theorem of Noguchi on extending holomorphic mappings, which would keep the local uniform convergence. Much attention has been given to the Noguchi theorem from the viewpoint of hyperbolic complex analysis and several Noguchi-type convergence-extension theorems for analytic hypersurfaces of complex manifolds have been obtained by various authors (see [19], [15]–[17]). It is much to be regretted, therefore, that while a substantial amount of information has been amassed concerning the Noguchi-type convergence-extension theo-

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rems for analytic hypersurfaces through the years, the present knowledge of these theorems for subsets of more general type remains extremely meagre.

For the convenient presentation, we give the following

DEFINITION 1. Let d be a real number such that $0 < d \leq 1$ and Δ be the open unit disc in $\mathbb{C}$. A subset S of Δ^n is said to be an (n, d)-*set* in Δ^n if the following are satisfied:

(i) The $(2n-2+d)$-dimensional Hausdorff measure $H_{2n-2+d}(S)$ of S is 0.

(ii) For every subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ ($i_1 < \ldots < i_k$, $1 \leq k \leq n-1$) and for every $w = (w_{i_1}, \ldots, w_{i_k}) \in \Delta^k$, either $S^w = \Delta^{n-k}$ or $H_{2n-2k-2+d}(S^w) = 0$, where

$$\begin{aligned} S^w = \{z = (&z_1, \ldots, z_{i_1-1}, z_{i_1+1}, \ldots, z_{i_k-1}, z_{i_k+1}, \ldots, z_n) \in \Delta^{n-k}; \\ &(w, z) = (z_1, \ldots, z_{i_1-1}, w_{i_1}, z_{i_1+1}, \ldots, z_{i_k-1}, w_{i_k}, z_{i_k+1}, \ldots, z_n) \in S\}. \end{aligned}$$

Example: Every complete pluripolar subset of Δ^n is an (n, d)-set. Recall that a subset S of Δ^n is said to be *complete pluripolar* if there exists a plurisubharmonic function φ on Δ^n such that $S = \{\varphi = -\infty\}$ (see [3]).

DEFINITION 2. Let X be a complex space. We say that X has the (n, d)-*convergence-extension property* (briefly X has the (n, d)-EP) if the following holds.

Let S be any closed (n, d)-set in Δ^n. Let $f_j : \Delta^n \setminus S \to X$, $j = 1, 2, \ldots$, be holomorphic mappings which converge uniformly on compact subsets of $\Delta^n \setminus S$ to a holomorphic mapping $f : \Delta^n \setminus S \to X$. Then there are unique holomorphic extensions $\bar{f}_j : \Delta^n \to X$ of f_j and $\bar{f} : \Delta^n \to X$ of f over Δ^n, and $\{\bar{f}_j\}_{j=1}^{\infty}$ converges uniformly on compact subsets of Δ^n to $\bar{f}$.

Our main aim in this paper is to present some classes of complex spaces which have the (n, d)-EP. Namely, we are going to prove the following

THEOREM 1. *Let X be a complex space and d be a real number such that $0 < d \leq 1$. Then X has the (n, d)-EP if and only if X has the $(1, d)$-EP.*

PROPOSITION 2. *Let Ω be a strictly hyperconvex domain in $\mathbb{C}^N$. Then Ω has the (n, d)-EP for any $n \geq 1$ and any $0 < d \leq 1$.*

PROPOSITION 3. *Let X be a complex space and d be a real number such that $0 < d < 1$. Let $\Omega_\varphi(X) = \{(x, \lambda) \in X \times \mathbb{C};\ |\lambda| < e^{-\varphi(x)}\}$, where $\varphi : X \to [-\infty, \infty)$ is upper semicontinuous on X. Then $\Omega_\varphi(X)$ has the (n, d)-EP iff X has the (n, d)-EP, $\varphi \in \mathrm{PSH}(X)$ and $\varphi(x) > -\infty$ for all $x \in X$.*

Finally, the proof of Theorem 1 is based on the ideas of [27], and that of Proposition 3 on the ideas of [28].

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2. Basic notions and auxiliary results. In this article, we shall make use of properties of complex spaces as discussed in the book of Gunning–Rossi [10].

2.1. Let X be a complex space. We say that X has the *Hartogs extension property* (briefly X has the (HEP)) if every holomorphic mapping from a Riemann domain Ω over a Stein manifold into X can be extended holomorphically to $\widehat{\Omega}$, the envelope of holomorphy of Ω.

Let $H_2(r) = \{(z_1, z_2) \in \Delta^2;\ |z_1| < r \text{ or } |z_2| > 1 - r\}$ $(0 < r < 1)$ denote the 2-dimensional Hartogs domain. It is well known ([24] or [14]) that X has the (HEP) iff every holomorphic mapping $f : H_2(r) \to X$ extends holomorphically over Δ^2.

The class of complex spaces having the (HEP) is large. It contains taut complex spaces [8], complex Lie groups [1], complete hermitian complex manifolds with non-positive holomorphic sectional curvature [24]. In particular, Ivashkovich [14] showed that a holomorphically convex Kähler manifold has the (HEP) iff it contains no rational curves. This was generalized to holomorphically convex Kähler spaces by D. D. Thai [26].

2.2. Modifying the definition of disc-convexity (see [19] or [24]), we say that a complex space X is *weakly disc-convex* if every sequence $\{f_n\} \subset H(\Delta, X)$ converges in $H(\Delta, X)$ whenever the sequence $\{f_n|_{\Delta^*}\} \subset H(\Delta^*, X)$ converges in $H(\Delta^*, X)$. Here, $H(X, Y)$ denotes the space of holomorphic mappings from a complex space X into a complex space Y equipped with the compact-open topology, and $\Delta^* = \Delta \setminus \{0\}$.

2.3. Let G be an open subset of $\mathbb{C}^n$. A function $\varphi : G \to [-\infty, \infty)$ is called *plurisubharmonic* if

(i) φ is upper semicontinuous and it is not identically $-\infty$ on any connected component of G.

(ii) For every $z_0 \in G$ and $a \in \mathbb{C}^n, a \neq 0$, and for every map $\tau : \mathbb{C} \to \mathbb{C}^n$ of the form $\tau(z) = z_0 + az$, the function $\varphi \circ \tau$ is, on every connected component of $\tau^{-1}(G)$ (which is a domain in $\mathbb{C}$), either $-\infty$ or subharmonic.

Concerning arbitrary complex spaces we make the following definition. A *plurisubharmonic function* on a complex space X is a function $\varphi : X \to [-\infty, \infty)$ having the following property. For every $x \in X$ there exists an open neighbourhood U of x and a biholomorphic map $h : U \to V$ onto a closed complex subspace V of some domain $G \subset \mathbb{C}^m$ and a plurisubharmonic function $\widetilde{\varphi} : G \to [-\infty, \infty)$ such that $\varphi|_U = \widetilde{\varphi} \circ h$ (see Peternell [22, p. 225]).

Some remarks should be made at this point. First, the definition of plurisubharmonicity does not depend on the choice of local charts. Second, Fornæss and Narasimhan proved [7] that an upper semicontinuous function $\varphi : X \to [-\infty, \infty)$, not identically $-\infty$ on any connected component of the complex space X, is plurisubharmonic iff $\varphi \circ f$ is either subharmonic or $-\infty$ for all holomorphic maps $f : \Delta \to X$.

2.4. (i) Let Z be an open set in $\mathbb{C}^n$ and $S \subset Z$ a subset. We say that S is *polar* if for any $x_0 \in S$ there are an open neighbourhood U of x_0 in Z and a subharmonic function $\varphi : U \to [-\infty, \infty)$ such that $S \cap U \subset \{\varphi = -\infty\}$.

(ii) Let Z be an open set in $\mathbb{C}^n$ and $S \subset Z$ a subset. We say that S is *pluripolar* if for any $x_0 \in S$ there are an open neighbourhood U of x_0 in Z and a plurisubharmonic function $\varphi : U \to [-\infty, \infty)$ such that $S \cap U \subset \{\varphi = -\infty\}$. If $n = 1$, then S is polar.

Note that there are polar (or pluripolar) subsets of Δ^n which are not (n, d)-sets: for instance, the subset $(\Delta \setminus \{0\}) \times \{0\}$ of Δ^2.

On the other hand, it is well known [20] that for every $d > 0$ there exists a Cantor subset $S \subset [0, 1]$ such that $H_d(S) = 0$ and $H_\delta(S) > 0$ for any $0 < \delta < d$. This implies that S is a $(1, d)$-set but S is not polar.

2.5. Let S be an (n, d)-set in Δ^n. Assume that $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ $(i_1 < \ldots < i_k, 1 \le k \le n-1)$ and $\omega = (\omega_{i_1}, \ldots, \omega_{i_k}) \in \Delta^k$ are such that $H_{2n-2k-2+d}(S^\omega) = 0$. Then S^ω is an $(n-k, d)$-set in Δ^{n-k}.

Indeed, this can be deduced immediately from

$$\begin{aligned}(S^\omega)^{\omega'} &= \{z \in \Delta^{n-k-p};\ (\omega', z) \in S^\omega\}\\ &= \{z \in \Delta^{n-k-p};\ (\omega, \omega', z) \in S\} = S^{(\omega,\omega')}.\end{aligned}$$

2.6. The countable union of (n, d)-sets S_j, $j = 1, 2, \ldots$, in Δ^n is also an (n, d)-set in Δ^n.

This is an immediate consequence of the equality

$$\Big(\bigcup_{j=1}^{\infty} S_j\Big)^{\omega} = \bigcup_{j=1}^{\infty} S_j^{\omega}.$$

2.7. Let X be a complex space. It is easy to see that X has the (n, d)-EP iff the restriction $R : H(\Delta^n, X) \to H(\Delta^n \setminus S, X)$ is a homeomorphism for every closed (n, d)-set S in Δ^n.

2.8. Let X be a complex space and K be a compact subset of X. The *plurisubharmonically convex hull* of K (in X) is the set

$$\widehat{K}_{\mathrm{PSH}(X)} = \{x \in X; u(x) \le \sup u(K) \text{ for all } u \in \mathrm{PSH}(X)\},$$

where $\mathrm{PSH}(X)$ is the set of all plurisubharmonic functions on X.

The complex space X is said to be *pseudoconvex* if for each compact subset K of X, $\widehat{K}_{\mathrm{PSH}(X)}$ is compact in X.

2.9. (i) Let $\Omega \subset \mathbb{C}^n$ be a domain and $u : \Omega \to \mathbb{R}$ be a $\mathcal{C}^2$-function. The *Levi form* of u at $z \in \Omega$ is the Hermitian form

$$L(u)(z)(a) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j}(z) a_i \overline{a}_j, \qquad a = (a_1, \ldots, a_n) \in \mathbb{C}^n.$$

The function u is said to be *strictly plurisubharmonic* in Ω if $L(u)(z)(a) > 0$ for all $z \in \Omega$ and $a \in \mathbb{C}^n \setminus \{0\}$.

(ii) A bounded domain $\Omega \subset \mathbb{C}^n$ is said to be *strictly pseudoconvex* if it is of the form $\Omega = \{z \in \mathbb{C}^n;\ \varrho(z) < 0\}$ and $d\varrho \neq 0$ on $\partial\Omega$, where ϱ is a $\mathcal{C}^2$-function on a neighbourhood of $\overline{\Omega}$ and ϱ is strictly plurisubharmonic in a neighbourhood of $\partial\Omega$.

2.10. (i) A domain $D \subset \mathbb{C}^n$ is called *hyperconvex* if there exists a continuous plurisubharmonic exhaustion function $\varrho : D \to (-\infty, 0)$.

(ii) A bounded domain $D \subset \mathbb{C}^n$ is said to be *strictly hyperconvex* if there exists a bounded domain Ω and a function $\varrho \in \mathcal{C}(\Omega, (-\infty, 1)) \cap \mathrm{PSH}(\Omega)$ such that $D = \{z \in \Omega;\ \varrho(z) < 0\}$, ϱ is exhaustive for Ω and for all real numbers $c \in [0, 1]$, the open set $\{z \in \Omega;\ \varrho(z) < c\}$ is connected.

It is easy to see the following implications for any bounded domain in $\mathbb{C}^n$:

$$\text{strictly pseudoconvex} \ \Rightarrow\ \text{strictly hyperconvex} \ \Rightarrow\ \text{hyperconvex} \ \Rightarrow\ \text{pseudoconvex}.$$

The converse implications are not true in general (see [4], [5], [6], [18]).

3. Proof of Theorem 1.

We need the following

LEMMA 1 (see [24]). *Let X be a complex space. If X is weakly disc-convex, then X has the (HEP).*

Proof of Theorem 1. ($\Rightarrow$) Let S be a closed $(1, d)$-set in the open unit disc Δ. Put $\widetilde{S} = S \times \Delta^{n-1}$. It is easy to see that $\widetilde{S}$ is a closed (n, d)-set in Δ^n. Assume that $f : \Delta \setminus S \to X$ is a holomorphic mapping. Consider the holomorphic mapping $\widetilde{f} : \Delta^n \setminus \widetilde{S} \to X$ given by $\widetilde{f}(z_1, z_2) = f(z_1)$. By the hypothesis, $\widetilde{f}$ extends holomorphically to $\widetilde{F}$ over Δ^n. Then $F(z_1) = \widetilde{F}(z_1, 0)$ extends f over Δ.

Assume that a sequence $\{f_j\}_{j=1}^{\infty} \subset H(\Delta \setminus S, X)$ converges uniformly to a mapping $f_0 \in H(\Delta \setminus S, X)$ in $H(\Delta \setminus S, X)$. It is easy to see that the sequence $\{\widetilde{f}_j\}_{j=1}^{\infty} \subset H(\Delta^n \setminus \widetilde{S}, X)$ converges to a mapping $\widetilde{f}_0 \in H(\Delta^n \setminus \widetilde{S}, X)$ in $H(\Delta^n \setminus \widetilde{S}, X)$. By the hypothesis, the sequence $\{\widetilde{F}_j\}_{j=1}^{\infty} \subset H(\Delta^n, X)$ of holomorphic extensions converges to the holomorphic extension $\widetilde{F}_0 \in H(\Delta^n, X)$ in $H(\Delta^n, X)$. Therefore, the sequence $\{F_j\}_{j=1}^{\infty} \subset H(\Delta, X)$ also converges to $F_0 \in H(\Delta, X)$ in $H(\Delta, X)$.

($\Leftarrow$) The proof is by induction on the dimension n.

(i) First observe that X is weakly disc-convex. By Lemma 1, X has the (HEP).

(ii) Given $n \geq 2$. For every closed (n,d)-set S in an open set Δ^n, we put

$$S' = \{z \in \Delta^{n-1};\ \{z\} \times \Delta \subset S\}, \qquad S'' = \{w \in \Delta;\ \Delta^{n-1} \times \{w\} \subset S\}.$$

Then S' and S'' are closed in Δ^{n-1} and Δ, respectively.

We claim that S' is an $(n-1,d)$-set in Δ^{n-1} and S'' is a $(1,d)$-set in Δ.

Indeed, by the hypothesis $H_{2n-2+d}(S) = 0$, from the inclusion $(S' \times \Delta) \cup (\Delta^{n-1} \times S'') \subset S$ it follows that $H_{2(n-1)-2+d}(S') = 0$ and $H_d(S'') = 0$. Thus S'' is a $(1,d)$-set in Δ. Given now $w \in \Delta^k$ $(1 \leq k \leq n-2)$, since

$$(*) \qquad \begin{aligned} (S')^w &= \{z \in \Delta^{n-1-k};\ (w,z) \in S'\} \\ &= \{z \in \Delta^{n-1-k};\ \{(w,z)\} \times \Delta \subset S\} \end{aligned}$$

we have $(S')^w \times \Delta \subset S^w$. This shows that if $H_{2n-2k-2+d}(S^w) = 0$, then

$$H_{2(n-k-1)-2+d}((S')^w) = 0.$$

If $S^w = \Delta^{n-k}$ then from $(*)$ we have $(S')^w = \Delta^{n-k-1}$. Thus S' is an $(n-1,d)$-set in Δ^{n-1}.

On the other hand, $S^w = \{z \in \Delta^{n-1};\ (w,z) \in S\}$ is a closed $(n-1,d)$-set in Δ^{n-1} for every $w \in \Delta \setminus S''$, and $S^z = \{w \in \Delta;\ (z,w) \in S\}$ is a closed $(1,d)$-set in Δ for every $z \in \Delta^{n-1} \setminus S'$ (see 2.5).

(iii) Now assume that f is a holomorphic mapping from $(\Delta^{n-1} \times \Delta) \setminus S$ into X. For each $w \notin S''$, consider the holomorphic mapping $f^w : \Delta^{n-1} \setminus S^w \to X$ given by $f^w(z) = f(z,w)$ for all $z \in \Delta^{n-1} \setminus S^w$. By the inductive hypothesis, f^w extends to a mapping $\widetilde{f}^w \in H(\Delta^{n-1}, X)$. Similarly for each $z \notin S'$, the holomorphic mapping $f_z : \Delta \setminus S^z \to X$ given by $f_z(w) = f(z,w)$ for all $w \in \Delta \setminus S^z$ extends to a mapping $\widetilde{f}_z \in H(\Delta, X)$. Thus we can define the mappings

$$\begin{aligned} f_1 &: (\Delta^{n-1} \setminus S') \times \Delta \to X \quad \text{by} \quad f_1(z,w) = \widetilde{f}_z(w), \\ f_2 &: \Delta^{n-1} \times (\Delta \setminus S'') \to X \quad \text{by} \quad f_2(z,w) = \widetilde{f}^w(z). \end{aligned}$$

We now prove that f_1 is continuous on $(\Delta^{n-1} \setminus S') \times \Delta$. Indeed, assume that $\{(z_k, w_k)\} \subset (\Delta^{n-1} \setminus S') \times \Delta$ and $(z_k, w_k) \to (z_0, w_0) \in (\Delta^{n-1} \setminus S') \times \Delta$. Put $P = \left(\bigcup_{k=1}^{\infty} S^{z_k}\right) \cup S^{z_0}$. Then P is a closed $(1,d)$-set in Δ. Since $\{f_{z_k}\}$ converges to f_{z_0} in $H(\Delta \setminus P, X)$, by the inductive hypothesis $\{\widetilde{f}_{z_k}\}$ converges to $\widetilde{f}_{z_0}$ in $H(\Delta, X)$. Hence $\widetilde{f}_{z_k}(w_k) = f_1(z_k, w_k) \to \widetilde{f}_{z_0}(w_0) = f_1(z_0, w_0)$. Thus f_1 is continuous on $(\Delta^{n-1} \setminus S') \times \Delta$.

Similarly, f_2 is continuous on $\Delta^{n-1} \times (\Delta \setminus S'')$.

Since $(\Delta^{n-1} \times \Delta) \setminus S$ is dense in $(\Delta^{n-1} \setminus S') \times (\Delta \setminus S'')$ and $f_1 = f_2$ on $(\Delta^{n-1} \times \Delta) \setminus S$, we have $f_1 = f_2$ on $(\Delta^{n-1} \setminus S') \times (\Delta \setminus S'')$.

This implies that the mapping f_1 satisfies the following: $(f_1)_z = \widetilde{f}_z \in H(\Delta, X)$ for all $z \in \Delta^{n-1} \setminus S'$ and $(f_1)^w = \widetilde{f}^w|_{\Delta^{n-1}\setminus S'} \in H(\Delta^{n-1} \setminus S', X)$ for all $w \in \Delta \setminus S''$, where $(f_1)_z$ and $(f_1)^w$ are given by $(f_1)_z(w) = (f_1)^w(z) = f_1(z, w)$. By a theorem of Shiffman [25], f_1 is holomorphic. Similarly, f_2 is also holomorphic.

We define the holomorphic mapping $\bar{f} : ((\Delta^{n-1} \setminus S') \times \Delta) \cup (\Delta^{n-1} \times (\Delta \setminus S'')) \to X$ by $\bar{f}|_{(\Delta^{n-1}\setminus S')\times\Delta} = f_1$ and $\bar{f}|_{\Delta^{n-1}\times(\Delta\setminus S'')} = f_2$.

On the other hand, since $H_{2(n-1)-2+d}(S') = H_d(S'') = 0$, by [25], we have

$$[((\Delta^{n-1} \setminus S') \times \Delta) \cup (\Delta^{n-1} \times (\Delta \setminus S''))]^\frown = \Delta^n.$$

It follows that $\bar{f}$ extends to a holomorphic mapping $\widehat{f} : \Delta^n \to X$, i.e. $\widehat{f}$ is a holomorphic extension of f over Δ^n.

(iv) Now assume that the sequence $\{f_k\} \subset H((\Delta^{n-1} \times \Delta) \setminus S, X)$ converges uniformly to $f \in H((\Delta^{n-1} \times \Delta) \setminus S, X)$. We must prove that $\widehat{f}_k \to \widehat{f}$ in $H(\Delta^{n-1} \times \Delta, X)$.

First of all, by the inductive hypothesis, we have

$$\begin{aligned} H(\Delta^n, X) &\cong H(\Delta, H(\Delta^{n-1}, X)) \cong H(\Delta, H(\Delta^{n-1} \setminus S', X)) \\ &\cong H((\Delta^{n-1} \setminus S') \times \Delta, X). \end{aligned}$$

Note that the first identification $\phi_1 : H(\Delta^n, X) \to H(\Delta, H(\Delta^{n-1}, X))$ is given by $\phi_1(h)(w) : z \mapsto h(z, w)$, the second $\phi_2 : H(\Delta, H(\Delta^{n-1}, X)) \to H(\Delta, H(\Delta^{n-1} \setminus S', X))$ by $\phi_2(h)(w) = h(w)|_{\Delta^{n-1}\setminus S'}$, and the third $\phi_3 : H(\Delta, H(\Delta^{n-1}\setminus S', X)) \to H((\Delta^{n-1}\setminus S')\times\Delta, X)$ by $\phi_3(h) : (\Delta^{n-1} \setminus S') \times \Delta \ni (z, w) \mapsto h(w)(z)$.

Thus in order to prove the above assertion, it suffices to show that $\widehat{f}_k \to \widehat{f}$ in $H((\Delta^{n-1} \setminus S') \times \Delta, X)$, i.e. we must prove that if $\{z_k\} \subset \Delta^{n-1} \setminus S'$ and $\{w_k\} \subset \Delta$ are such that $z_k \to z_0 \in \Delta^{n-1} \setminus S'$ and $w_k \to w_0 \in \Delta$ then $\widehat{f}_k(z_k, w_k) \to \widehat{f}(z_0, w_0)$.

Indeed, as before, we consider the following holomorphic mappings:

$$\begin{aligned} f_{k,z_k} &: \Delta \setminus S^{z_k} \to X, & w &\mapsto f_k(z_k, w), \\ f_{z_0} &: \Delta \setminus S^{z_0} \to X, & w &\mapsto f(z_0, w), \\ \widehat{f}_{k,z_k} &: \Delta \to X, & w &\mapsto \widehat{f}_k(z_k, w), \\ \widehat{f}_{z_0} &: \Delta \to X, & w &\mapsto \widehat{f}(z_0, w). \end{aligned}$$

Put $P = (\bigcup_{k=1}^{\infty} S^{z_k}) \cup S^{z_0}$. Then P is a closed $(1, d)$-set in Δ. Since $f_{k,z_k} \to f_{z_0}$ in $H(\Delta \setminus P, X)$, by the inductive hypothesis, we have $\widehat{f}_{k,z_k} \to \widehat{f}_{z_0}$ in $H(\Delta, X)$. Thus $\widehat{f}_{k,z_k}(w_k) = \widehat{f}_k(z_k, w_k) \to \widehat{f}_{z_0}(w_0) = \widehat{f}(z_0, w_0)$. ■

Remark 1. We now introduce some new notions.

(i) We say that a complex space X has the *strong* (n, d)*-EP* if the restriction map $R : H(\Delta^n, X) \to H(\Delta^n \setminus S, X)$ is a homeomorphism for any closed set $S \subset \Delta^n$ which is of locally finite $(2n - 2 + d)$-dimensional Hausdorff measure, where $0 < d \leq 1$. Unfortunately, we do not know whether Theorem 1 remains true for the strong (n, d)-EP with $0 < d \leq 1$.

(ii) We say that a complex space X has the *n-PEP* if the restriction map $R : H(\Delta^n, X) \to H(\Delta^n \setminus S, X)$ is homeomorphic for any closed polar set $S \subset \Delta^n$. Unfortunately, we also do not know whether Theorem 1 remains true for the n-PEP.

(iii) We say that a complex space X has the *n-PPEP* if the restriction map $R : H(\Delta^n, X) \to H(\Delta^n \setminus S, X)$ is homeomorphic for any closed pluripolar set $S \subset \Delta^n$. In [27] the first named author proved Theorem 1 for the n-PPEP.

4. Proof of Proposition 2. We need the following

Lemma 2. *Let S be a closed subset of Δ of Hausdorff measure $H_1(S) = 0$. Then for every $z_0 \in S$, there exists $r > 0$ such that*

$$\{z \in \mathbb{C};\ |z - z_0| = r\} \subset \Delta \setminus S.$$

Proof. Consider the function $\sigma : \mathbb{C} \to [0, \infty)$ given by $\sigma(z) = |z - z_0|$. Then $H_1(\sigma(S)) \leq H_1(S) = 0$. It follows that $\mathbb{R} \setminus \sigma(S)$ is dense in $\mathbb{R}$. Thus there exists $r > 0$ such that $\{z \in \mathbb{C} : |z - z_0| = r\} \subset \Delta \setminus S$. ∎

Lemma 3. *Let S be a closed subset of Δ with $H_1(S) = 0$. Then every function f holomorphic and uniformly bounded in $\Delta \setminus S$ has a holomorphic continuation into Δ.*

Proof. The proof is given in [2, A 1.4, Thm., p. 299] (see also [9, Thm. 1.4, p. 10 and Thm. 2.1, p. 64]). For the reader's convenience we repeat the details.

Since S is nowhere dense in Δ, it suffices to prove a local statement: f has a holomorphic continuation into a neighbourhood of an arbitrary point of S, which we take as the coordinate origin.

By Lemma 2, there is $r \in (0, 1)$ such that the circle $\gamma : |z| = r$ does not intersect S, and consequently $S_r = S \cap \Delta_r$ is compact, where $\Delta_r = \{|z| < r\} \subset \Delta$.

Given $z \in \Delta_r \setminus S_r$, put $\delta = \mathrm{dist}(z, S_r) > 0$. Since $H_1(S_r) = 0$, for every $\varepsilon \in (0, \delta/2)$ there exists a finite covering of S_r by discs disjoint from γ with total sum of radii $< \varepsilon$. The union of these discs is denoted by V_ε, and we put $U_\varepsilon = \Delta_r \setminus \overline{V}_\varepsilon$. Then $z \in U_\varepsilon$. Since f is holomorphic on the closure of U_ε, we have

$$f(z) = \frac{1}{2\pi i}\bigg(\int\limits_{\gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int\limits_{\partial V_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta\bigg)$$

On the other hand, we have

$$\bigg|\int\limits_{\partial V_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta\bigg| \le M \cdot 2\pi\varepsilon \cdot \frac{1}{\delta - \varepsilon} \to 0 \quad \text{as } \varepsilon \to 0, \text{ where } M = \sup_{\Delta\setminus S} |f|.$$

Thus the function f is represented on $\Delta_r \setminus S_r$ by the first integral only. But this integral is holomorphic in the whole disc Δ_r, hence it determines the required holomorphic continuation of f into a neighbourhood of 0. ■

Proof of Proposition 2. Let Ω be a strictly hyperconvex domain in $\mathbb{C}^N$.

By Theorem 1, it suffices to show that Ω has the $(1, d)$-EP. The proof will be divided into two steps: (i) we show that any holomorphic mapping $f : \Delta \setminus S \to \Omega$ extends to a holomorphic mapping $\widehat{f} : \Delta \to \Omega$; (ii) we prove the local uniform convergence of the sequence $\{\widehat{f}^k\}$.

(i) Let $f : \Delta \setminus S \to \Omega$ be any holomorphic mapping, where S is any closed $(1, d)$-set of Δ. Put $f = (f_1, \ldots, f_N)$. By Lemma 3, f_j extends to a holomorphic function $\widehat{f}_j$ on Δ. Then $\widehat{f} = (\widehat{f}_1, \ldots, \widehat{f}_N) \in H(\Delta, \overline{\Omega})$. Let ϱ be a plurisubharmonic exhaustion function of $\widetilde{\Omega}$ such that $\Omega = \{z \in \widetilde{\Omega};\ \varrho(z) < 0\}$, where $\widetilde{\Omega}$ is a bounded neighbourhood of $\overline{\Omega}$ in $\mathbb{C}^N$. Put $h = \varrho \circ \widehat{f}$. Then h is subharmonic on Δ. Since h is negative on $\Delta \setminus S$, it follows that $h \le 0$ on Δ. Suppose that there exists $z_0 \in S$ such that $\widehat{f}(z_0) \in \partial\Omega$. Then $h(z_0) = 0$. The maximum principle implies $h = 0$ on Δ. This is a contradiction, and hence $f \in H(\Delta, \Omega)$.

(ii) Let $\{f^k\}$ be a sequence in $H(\Delta \setminus S, \Omega)$ which converges locally uniformly to $f \in H(\Delta \setminus S, \Omega)$. Given $x_0 \in S$, by Lemma 2, there exists a neighbourhood V of x_0 in Δ such that $\partial V \cap S = \emptyset$. Since $\{\widehat{f}^k|_{\partial V}\}$ converges uniformly to $\widehat{f}|_{\partial V}$, the maximum principle implies the uniform convergence of $\{\widehat{f}^k|_V\}$ with limit $\widehat{f}|_V$. Thus $\widehat{f}^k \to \widehat{f}$ in $H(\Delta, \Omega)$. ■

5. Proof of Proposition 3. ($\Leftarrow$) Assume that X has the (n, d)-EP, $\varphi \in \mathrm{PSH}(X)$ and $\varphi(x) > -\infty$ for all $x \in X$. We must prove that $\Omega_\varphi(X)$ has the (n, d)-EP.

By Theorem 1, it suffices to show that $\Omega_\varphi(X)$ has the $(1, d)$-EP. The proof will be divided into two steps: (i) we show that any holomorphic mapping $f : \Delta \setminus S \to \Omega_\varphi(X)$ extends to a holomorphic mapping $\widehat{f} : \Delta \to \Omega_\varphi(X)$; (ii) we prove the local uniform convergence of the sequence $\{\widehat{f}^k\}$.

(i) Let $f = (f_1, f_2) : \Delta \setminus S \to \Omega_\varphi(X)$ be a holomorphic mapping, where S is a closed $(1,d)$-set of Δ. By the hypothesis, f_1 extends to a holomorphic mapping $\widehat{f}_1 : \Delta \to X$. Let $x_0 \in \Delta$ be an arbitrary point. Since $(\varphi \circ \widehat{f}_1)(x_0) > -\infty$, it follows from [13, Corollary 4.4.6, p. 98] that $e^{-a\varphi\circ\widehat{f}_1}$ is locally integrable at x_0 for every $a > 0$.

We now recall the following result on extension:

Theorem [11, Theorem 1, (d)]. *Suppose S is a closed subset of an open set $Z \subset \mathbb{C}^n$ and $f \in \mathrm{Hol}(Z \setminus S)$. Let $2 \le p < \infty$ and p' be the conjugate exponent $(1/p + 1/p' = 1)$. If $f \in L^p_{\mathrm{loc}}(Z)$ and $H_{2n-p'}(S)$ is locally finite, then $f \in \mathrm{Hol}(Z)$.*

Take $p > 2$ such that $(p-2)/(p-1) > d$ and take a neighbourhood U of x_0 in Δ such that

$$\int_U e^{-p\varphi\circ\widehat{f}_1(x)}\, dx < \infty.$$

From the inequality $|f_2(x)|^p < e^{-p\varphi\circ\widehat{f}_1(x)}$ for all $x \in U \setminus S$, we have $f_2 \in L^p_{\mathrm{loc}}(\Delta)$. On the other hand, since

$$H_{2-p'}(S) = H_{(p-2)/(p-1)}(S) \le C H_d(S) = 0,$$

by the above-mentioned result of Harvey–Polking, the holomorphic mapping $f_2|_{U\setminus S}$ extends holomorphically over U. Thus f_2 extends to a holomorphic function $\widehat{f}_2 : \Delta \to \mathbb{C}$. Define the holomorphic mapping

$$\widehat{f} = (\widehat{f}_1, \widehat{f}_2) : \Delta \to X \times \mathbb{C}.$$

It remains to check that $\widehat{f}(\Delta) \subset \Omega_\varphi(X)$, or equivalently,

$$\log|\widehat{f}_2(x)| + \varphi(\widehat{f}_1(x)) < 0 \quad \text{for all } x \in \Delta.$$

Given $x_0 \in S$, by Lemma 2, there exists a neighbourhood V of x_0 in Δ such that $\partial V \cap S = \emptyset$. Applying the maximum principle to the subharmonic function $\log|\widehat{f}_2| + \varphi(\widehat{f}_1)$, we get $\log|\widehat{f}_2(x_0)| + \varphi(\widehat{f}_1(x_0)) < 0$.

(ii) Let $\{f^k = (f_1^k, f_2^k)\}$ be a sequence in $H(\Delta \setminus S, \Omega_\varphi(X))$ which converges locally uniformly to $f = (f_1, f_2) \in H(\Delta \setminus S, \Omega_\varphi(X))$. By the hypothesis, $\widehat{f}_1^k \to \widehat{f}_1$ in $H(\Delta, X)$.

Given $x_0 \in S$, choose a small enough neighbourhood V of $\widehat{f}_1(x_0)$ in X such that V is isomorphic to an analytic set in an open ball of $\mathbb{C}^m$. Take a relatively compact neighbourhood W of x_0 in $\widehat{f}_1^{-1}(V)$. Let k_0 be chosen such that $\widehat{f}_1^k(W) \subset V$ for every $k \ge k_0$. Then $\widehat{f}^k(W) \subset \pi^{-1}(V) \subset V \times \mathbb{C}$ for every $k \ge k_0$, where $\pi : \Omega_\varphi(X) \to X$ is the canonical projection. Without loss of generality we may assume that $V \times \mathbb{C} \subset \mathbb{C}^m \times \mathbb{C}$. By Lemma 2, we can choose a neighbourhood $\widetilde{W}$ of x_0 in W such that $\overline{\widetilde{W}} \subset W$ and $\partial\widetilde{W} \cap S = \emptyset$. Since $\{\widehat{f}^k|_{\partial\widetilde{W}}\}$ converges uniformly to $\widehat{f}|_{\partial\widetilde{W}}$, the maximum principle applied for all coordinates yields the uniform convergence of $\{\widehat{f}^k|_{\widetilde{W}}\}$ with limit $\widehat{f}|_{\widetilde{W}}$. Thus $\widehat{f}^k \to \widehat{f}$ in $H(\Delta, \Omega_\varphi(X))$.

($\Rightarrow$) Assume that $\Omega_\varphi(X)$ has the (n,d)-EP.

(i) Consider the holomorphic mapping $\theta : X \to \Omega_\varphi(X)$ given by $\theta(x) = (x, 0)$ for each $x \in X$. Then θ is a biholomorphism from X onto $\theta(X)$. Since $\theta(X)$ is a closed subspace of $\Omega_\varphi(X)$, $\theta(X)$ has the (n,d)-EP. Thus X has the (n,d)-EP.

(ii) On the other hand, since $\Omega_\varphi(X)$ has the (n,d)-EP, by Theorem 1, $\Omega_\varphi(X)$ has the $(1,d)$-EP, and hence $\Omega_\varphi(X)$ has the Δ^*-EP. Here a complex space M is said to have the *Δ^*-EP* if every holomorphic mapping $f : \Delta^* \to M$ extends holomorphically over Δ. Therefore $\Omega_\varphi(X)$ contains no complex lines [26]. If $\varphi(x_0) = -\infty$ for some $x_0 \in X$, then $\{x_0\} \times \mathbb{C} \subset \Omega_\varphi(X)$. This is impossible. Hence $\varphi \not\equiv -\infty$ on X.

(iii) We now show that φ is plurisubharmonic on X. Indeed, by a theorem of Fornæss and Narasimhan [7], it suffices to check that $\varphi \circ \sigma$ is subharmonic on Δ for every holomorphic mapping $\sigma : \Delta \to X$. Let $g = (g_1, g_2) : \Delta^* \to \Omega_\psi(\Delta)$ be an arbitrary holomorphic mapping, where $\psi := \varphi \circ \sigma$. By the Riemann extension theorem, g_1 extends to a holomorphic mapping $\widehat{g}_1 : \Delta \to \Delta$. Consider the holomorphic mapping

$$f := (\sigma \circ g_1, g_2) : \Delta^* \to \Omega_\varphi(X).$$

By the hypothesis, f extends to a holomorphic mapping

$$\widehat{f} := (\widehat{\sigma \circ g}_1, \widehat{g}_2) : \Delta \to \Omega_\varphi(X).$$

Since $\widehat{\sigma \circ g}_1 = \sigma \circ \widehat{g}_1$ on Δ^*, we have $\widehat{\sigma \circ g}_1 = \sigma \circ \widehat{g}_1$ on Δ. Then

$$|\widehat{g}_2(0)| < e^{-\varphi(\widehat{\sigma \circ g}_1(0))} = e^{-(\varphi\circ\sigma)(\widehat{g}_1(0))} = e^{-\psi(\widehat{g}_1(0))}.$$

This implies that $\widehat{g} := (\widehat{g}_1, \widehat{g}_2) : \Delta \to \Omega_\psi(\Delta)$ is a holomorphic extension of g. Thus $\Omega_\psi(\Delta)$ has the Δ^*-EP. But a domain in $\mathbb{C}^n$ with the Δ^*-EP is weakly disc-convex (simply extend the limit mapping and then apply the maximum principle for all coordinates). Thus $\Omega_\psi(\Delta)$, which is an open subset in $\mathbb{C}^2$, is weakly disc-convex. By Lemma 1, $\Omega_\psi(\Delta)$ has the (HEP). This implies the pseudoconvexity of $\Omega_\psi(\Delta)$ [23] and consequently the subharmonicity of ψ. ■

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Department of Mathematics
Hanoi University of Education
Caugiay, Hanoi, Vietnam
E-mail: ddthai@netnam.org.vn

Department of Mathematics
Thainguyen University of Education
Thainguyen, Vietnam

Department of Mathematics
University of Education of Hochiminh City
Hochiminh City, Vietnam

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