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Some properties of Reinhardt domains

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Abstract. We first establish the equivalence between hyperconvexity of a fat bounded Reinhardt domain and the existence of a Stein neighbourhood basis of its closure. Next, we give a necessary and sufficient condition on a bounded Reinhardt domain D so that every holomorphic mapping from the punctured disk Δ_* into D can be extended holomorphically to a map from Δ into D.

1. Introduction. Let D be a domain in $\mathbb{C}^n$. We say that D is a Reinhardt domain if D is invariant under the action of the n-torus (for a precise definition see Section 2). Reinhardt domains are important objects in complex analysis; their pseudoconvexity, hyperconvexity, kinds of hyperbolicity, etc. have been characterized in [CCW], [Zw1], [Zw2], etc.

The aim of this paper is to study Reinhardt domains in connection with other concepts. Namely, in Theorem 3.2 we prove that a fat Reinhardt domain is hyperconvex if and only if its closure is compact Stein, i.e. has a neighbourhood basis of Stein domains. It should be remarked that there exists a fat, pseudoconvex domain in $\mathbb{C}^2$ whose closure is polynomially convex but the domain itself is *not* hyperconvex (see Proposition 3.1). On the other hand, the "worm" domains constructed by Diederich and Fornæss provide examples of hyperconvex domains whose closure is *not* compact Stein.

In Section 4, we deal with the question of extending holomorphic mapping into Reinhardt domains. Roughly speaking, we say that a domain D in $\mathbb{C}^n$ has the k- or Δ_*-extension property if every holomorphic mapping into D can be holomorphically extended through a "small" set (see Section 4 for precise definitions). The Δ_*-extension property was first studied by D. D. Thai [T] and recently by P. Thomas and D. D. Thai [TT]. In general, the k-extension property is strictly stronger than the Δ_*-extension property.

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However, we prove in Proposition 4.1 that in the class of Reinhardt domains, they are in fact equivalent. We also give a characterization of pseudoconvex Reinhardt domains with the $\varDelta_*$-extension property in Proposition 4.2.

Finally we prove in Section 5 that every pseudoconvex Reinhardt domain is the domain of existence of a bounded holomorphic function having polynomial growth. This result is closely related to earlier investigations in [JP].

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2. Basic notions and auxiliary facts. Let D be a domain in $\mathbb{C}^n$. It is said to be a *Reinhardt domain* if for every $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ we have

$$(z_1, \ldots, z_n) \in D \;\Rightarrow\; (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in D.$$

For each Reinhardt domain D in $\mathbb{C}^n$ we denote by $\log D_*$ its *logarithmic image*, more precisely

$$\log D_* = \{(\log|z_1|, \ldots, \log|z_n|) : (z_1, \ldots, z_n) \in D_*\},$$

where $D_* = \{(z_1, \ldots, z_n) \in D : z_1 \ldots z_n \neq 0\}$. We also write $\widehat{D}$ for the envelope of holomorphy of D. Next for $1 \le j \le n$ we let

$$V_j = \{z \in \mathbb{C}^n : z_j = 0\}, \qquad V = \bigcup_{1 \le j \le n} V_j.$$

The following useful criterion for pseudoconvexity of a Reinhardt domain can be found in [Zw1].

LEMMA 2.1. *Let D be a Reinhardt domain in $\mathbb{C}^n$. Then the following assertions are equivalent.*

(i) *D is pseudoconvex.*

(ii) *$\log D_*$ is convex and if $D \cap V_j \neq \emptyset$ for some $1 \le j \le n$ then*

$$(z_1, \ldots, z_{j-1}, z_j, \ldots, z_n) \in D \;\Rightarrow\; (z_1, \ldots, z_{j-1}, \lambda z_j, \ldots, z_n) \in D \;\forall |\lambda| < 1.$$

We next recall the concept of hyperconvexity. A domain D (not necessarily bounded) in $\mathbb{C}^n$ is said to be *hyperconvex* if there is a negative exhaustive continuous plurisubharmonic function for D. It is a remarkable fact that for bounded hyperconvex domains, it is enough to have a weak plurisubharmonic barrier at every boundary point. This fact is perhaps most clearly explained in [Bł]. More precisely, we have

THEOREM 2.2. *Let D be a bounded domain in $\mathbb{C}^n$. Then D is hyperconvex if and only if every boundary point ξ has a weak plurisubharmonic barrier*, *i.e. there exists a nonconstant negative plurisubharmonic function ψ on D such that*

$$\lim_{z\to\xi} \psi(z) = 0.$$

If the domain in question is pseudoconvex Reinhardt then we have a simpler criterion.

LEMMA 2.3. *Let D be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then*:

(i) *There exists a weak plurisubharmonic barrier at every point $\xi \in \partial D \setminus V$, which extends to a plurisubharmonic function in a neighbourhood of ξ in $\mathbb{C}^n$.*

(ii) *D is hyperconvex if and only if there exists a weak plurisubharmonic barrier at every point $\xi \in (\partial D) \cap V$.*

Proof. (i) The proof is implicitly contained in that of Theorem 2.14 in [CCW]; we omit the details.

(ii) follows immediately from Theorem 2.2 and (i).

For pseudoconvex Reinhardt domains we mention the following beautiful result of [Zw2]:

LEMMA 2.4. *A bounded pseudoconvex Reinhardt domain D in $\mathbb{C}^n$ is hyperconvex if and only if $D \cap V_j \neq \emptyset$ for any $j \in \{1, \ldots, n\}$ with $\overline{D} \cap V_j \neq \emptyset$.*

We need the following result about pseudoconvexity (resp. hyperconvexity) of projections of a pseudoconvex (resp. hyperconvex) Reinhardt domain. Notice that in general these properties are not preserved under projection.

LEMMA 2.5. *Let D be a pseudoconvex* (*resp. hyperconvex*) *Reinhardt domain in $\mathbb{C}^n$ and π be the projection $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_j)$, $1 \le j \le n$. Then*:

(a) *$\pi(D)$ is pseudoconvex* (*resp. hyperconvex*).

(b) *For every $a \in \pi(D)$ the set $\pi^{-1}(a) \cap D$ is a pseudoconvex* (*resp. hyperconvex*) *Reinhardt domain* (*viewed as a subset of $\mathbb{C}^{n-j}$*).

Proof. (a) It is clear that $\log(\pi(D))_* = \widetilde{\pi}(\log D_*)$, where $\widetilde{\pi}$ denotes the projection $(x_1, \ldots, x_n) \in \mathbb{R}^n \mapsto (x_1, \ldots, x_j) \in \mathbb{R}^j$. Thus $\log(\pi(D))_*$ is a convex domain in $\mathbb{R}^j$. By applying Lemmas 2.1 and 2.4 we can easily prove that $\pi(D)$ is pseudoconvex (resp. hyperconvex) if so is D.

(b) We argue similarly. It is easy to check that $\log(\pi^{-1}(a) \cap D)_* = \widetilde{\pi}^{-1}(\widetilde{a}) \cap \log D_*$, where $\widetilde{a} = (\log|a_1|, \dots, \log|a_j|)$. Therefore $\log(\pi^{-1}(a) \cap D)_*$ is convex. Now the desired conclusion follows easily from Lemmas 2.1 and 2.4.

3. Compact Stein Reinhardt sets. A compact set in $\mathbb{C}^n$ is called *compact Stein* if it has a Stein neighbourhood basis.

Let D be a domain in $\mathbb{C}^n$, and assume that $\overline{D}$ is compact Stein. By using the following criterion for pseudoconvexity: *A domain Ω in $\mathbb{C}^n$ is pseudoconvex if and only if the function $-\log d(z)$ is plurisubharmonic on Ω, where $d(z)$ is the distance from z to ∂D*, we can easily prove that D must be pseudoconvex.

Therefore it is natural to ask whether a fat domain is hyperconvex if its closure is compact Stein. Here we recall that a domain D is called *fat* if $\mathrm{Int}(\overline{D}) = D$. The answer is however "no" in general, as demonstrated in the following example which is inspired from [KR].

PROPOSITION 3.1. *There exists a bounded Hartogs domain D in $\mathbb{C}^2$ with the following properties.*

(i) *$\overline{D}$ is polynomially convex.*
(ii) $\mathrm{Int}(\overline{D}) = D$.
(iii) *D is not hyperconvex.*

Proof. Choose a locally bounded subharmonic function φ on $\mathbb{C}$ such that φ is discontinuous only at the origin in $\mathbb{C}$ (a precise construction will be given at the end of the proof). We let

$$D = \{(z, w) : |z| < 1,\ |w| < e^{-\varphi(z)}\}.$$

It is straightforward to see that $\overline{D} = D_1 \cup D_2$, where

$$D_1 = \{(z, w) : |z| \le 1,\ |w| \le e^{-\varphi(z)}\}, \qquad D_2 = \{(0, w) : |w| \le e^{-\xi}\},$$

with $\xi = \liminf_{z \to 0} f(z)$. This implies that D satisfies (ii). We must show that $D_1 \cup D_2$ is polynomially convex. For this we employ an argument close to the proof of Lemma 6.5 in [Fo].

First we prove that $\widehat{\overline{D}} \setminus \overline{D} \subset \{0\} \times \mathbb{C}$. Indeed, assume that there exists $(z_0, w_0) \in \widehat{\overline{D}} \setminus \overline{D}$ with $z_0 \neq 0$. Then since $(z_0, w_0) \notin D_1$ we deduce that $\varphi(z_0) + \log|w_0| > 0$. Now for each $\lambda > 0$ we define the function

$$u(z, w) = \log|z| + \lambda(\varphi(z) + \log|w|).$$

It is clear that u is plurisubharmonic on $\mathbb{C}^2$ and satisfies $u(z, w) \le 0$ for all $(z, w) \in \overline{D}$ and $u(z_0, w_0) > 0$ for λ large enough. In view of Theorem 4.3.4 in [Hö] (which is equivalent to the solution to the Levi problem) we get a contradiction to the fact that $(z_0, w_0) \in \widehat{\overline{D}}$. Thus $\widehat{\overline{D}} \setminus \overline{D} \subset \{0\} \times \mathbb{C}$.

Next we pick a point $(0,w^*)$ from $\widehat{\overline{D}}\cap(\{0\}\times\mathbb{C})$ which is farthest from the origin. Consider the function $f(z,w)=e^{w\overline{w^*}}$. It is easy to see that $|f|$ attains a strict maximum in the disk $\{(0,w):|w|\le|w^*|\}$ at the point $(0,w^*)$.

Now we claim that $(0,w^*)\in D_2$. Otherwise there would exist a small ball U centred at $(0,w^*)$ which is disjoint from $\overline{D}$. It follows that $(\partial U)\cap\widehat{\overline{D}}\subset\{0\}\times\mathbb{C}$. Therefore

$$|f(0,w^*)|>\sup_{(z,w)\in(\partial U)\cap\widehat{\overline{D}}}|f(z,w)|.$$

This contradicts the Rossi local maximum principle (Theorem 9.3 in [AW]). Thus $(0,w^*)\in D_2$, and therefore $\widehat{\overline{D}}$ is polynomially convex.

Finally, D is not hyperconvex because φ is not continuous at the origin (see [KR]). It remains to construct a subharmonic function having the above mentioned properties. We can take

$$\varphi(z)=e^{\psi(z)},\quad\text{where}\quad\psi(z)=\sum_{n\ge1}\frac{1}{2^n}\log\left|z-\frac{1}{2^n}\right|.$$

Remark. According to Theorem 4.2.1 in [Ra], it is impossible to find a domain in $\mathbb{C}$ enjoying all requirements in Proposition 3.1. However, as the referee pointed out, there does exist a domain D in $\mathbb{C}$ satisfying (ii) and (iii) of Proposition 3.1. Indeed, consider the Zalcman type domain

$$D:=\varDelta\setminus\Big(\bigcup_{j=1}^{\infty}\overline{\varDelta}(2^{-j},2^{-j(j^2+1)})\cup\{0\}\Big),$$

where $\varDelta$ is the unit disk in $\mathbb{C}$, and $\varDelta(a,r)$ denotes the disk centred at a with radius r. Then it follows from the Wiener criterion (see [Ra, Theorem 5.4.1]) that D is not regular with respect to the Dirichlet problem (and consequently it is not hyperconvex), although D is obviously fat and $\overline{D}$ is compact Stein.

Before formulating the main result of this section we recall the following notion from [Ni] (we thank the referee for directing our attention to this reference): A bounded hyperconvex domain D in $\mathbb{C}^n$ is called *strictly hyperconvex* if there exist a bounded pseudoconvex domain $\varOmega$ in $\mathbb{C}^n$, a function $\varrho\in\mathcal{C}(\varOmega,[-\infty,1))\cap\mathrm{PSH}(\varOmega)$ such that $D=\{z\in\varOmega:\varrho(z)<0\}$, ϱ is exhaustive for $\varOmega$ and for all $c\in[0,1)$, the open set $D_c=\{z\in\varOmega:\varrho(z)<c\}$ is connected.

We have some remarks concerning this notion.

Remark. It is clear that the sets D_c for $c\in(0,1)$ small enough form a Stein neighbourhood basis for $\overline{D}$. In particular $\overline{D}$ is compact Stein. We also claim that $\mathrm{Int}(\overline{D})=D$, i.e., D is fat. Indeed, otherwise there would exist a point $z_0 \in \operatorname{Int}(\overline{D}) \setminus D$. It follows that $\varrho(z_0) = 0$. The maximum principle for plurisubharmonic functions implies that ϱ vanishes on a neighbourhood of z_0. This is clearly absurd.

We now come to the main result of this section.

THEOREM 3.2. *Let D be a bounded Reinhardt domain in $\mathbb{C}^n$. Then the following assertions are equivalent.*

(i) *D is hyperconvex.*
(ii) *D is strictly hyperconvex.*
(iii) $\operatorname{Int}(\overline{D}) = D$ *and* $\overline{D}$ *is compact Stein.*

Proof. (i)⇒(ii). By changing coordinates we may assume that the point $(1, \ldots, 1)$ lies in D. We set

$$A = \{1 \leq j \leq n : D \cap V_j = \emptyset\}.$$

Since D is hyperconvex, by Lemma 2.4 we infer that D is relatively compact in the pseudoconvex domain

$$\widetilde{D} = \mathbb{C}^n \setminus \bigcup_{j \in A} V_j.$$

It suffices to prove that there exists a continuous plurisubharmonic function u on $\widetilde{D}$ satisfying

(a) $D = \{z \in \widetilde{D} : u(z) < 0\}$.
(b) $D_t = \{z \in \widetilde{D} : u(z) < t\}$ is connected for every t.

For this we notice $\log D_*$ is a convex domain containing the origin in $\mathbb{R}^n$, thus we can define the Minkowski functional p for $\log D_*$ as follows:

$$p(x) = \inf\{\lambda > 0 : x/\lambda \in \log D_*\}, \qquad \forall x \in \mathbb{R}^n.$$

Since p is convex on $\mathbb{R}^n$ we deduce that the function

$$u(z_1, \ldots, z_n) = p(\log|z_1|, \ldots, \log|z_n|) - 1$$

is plurisubharmonic on $\mathbb{C}^n_*$ and $D_* = \{z \in \mathbb{C}^n_* : u(z) < 0\}$. Therefore if $A = \emptyset$ then the proof is finished. Otherwise, pick an arbitrary point $a \in \widetilde{D} \cap V$. If $a = 0$ then $a \in D$ by Lemma 2.1. Obviously u is bounded near 0. If $a \neq 0$ then we may assume that $a = (0, \ldots, 0, a_{k+1}, \ldots, a_n)$, where $a_j \neq 0$ for $k + 1 \leq j \leq n$. Let π_k be the projection $(x_1, \ldots, x_n) \mapsto (x_{k+1}, \ldots, x_n)$. Then $\pi_k(\log D_*)$ is a convex domain in $\mathbb{R}^{n-k}$ containing the origin. We fix $\lambda > 0$ such that

$$\left(\frac{\log|a_{k+1}|}{\lambda}, \ldots, \frac{\log|a_n|}{\lambda}\right) \in \pi_k(\log D_*).$$

Since $D \cap V_j \neq \emptyset$ for $1 \leq j \leq k$, Lemma 2.1 implies that $(0, \ldots, 0, b_{k+1}, \ldots, b_n) \in D$, where $\log|b_j| = (\log|a_j|)/\lambda$ for $k + 1 \leq j \leq n$. Thus we can find $\varepsilon > 0$

and a Reinhardt neighbourhood W of the point $(b_{k+1},\dots,b_n)$ such that $U\times W\subset D$, where $U=\{(z_1,\dots,z_k):|z_j|<\varepsilon \text{ for } 1\le j\le k\}$. Now we choose a neighbourhood W' of $(a_{k+1},\dots,a_n)$ so small that $(\log W')/\lambda \subset \log W_*$, where

$$\log W'=\{(\log|z_{k+1}|,\dots,\log|z_n|):(z_{k+1},\dots,z_n)\in W'\}.$$

It is obvious that $U'\times W'$ is a neighbourhood of a, where $U'=\{(z_1,\dots,z_k): |z_j|<\varepsilon^\lambda \text{ for } 1\le j\le k\}$. Furthermore, it follows from the definition of u that

$$u(z)\le \lambda-1,\qquad \forall z\in (U'\times W')\setminus V.$$

Thus u is locally bounded near every point of $\widetilde{D}\cap V$. Hence u can be extended to a plurisubharmonic function (still denoted by u) on $\widetilde{D}$. Moreover, from the above reasoning we see that for every point $a\in(\widetilde{D}\cap V)\setminus\{0\}$ the following estimate holds:

$$u(a)\le \widetilde{p}(\log|a_{i_1}|,\dots,\log|a_{i_k}|)-1, \tag{2}$$

where $a_{i_1},\dots,a_{i_k}$ $(i_1<\dots<i_k)$ are all nonzero coordinates of a and $\widetilde{p}$ is the Minkowski functional for the projection of $\log D_*$ under the map $\widetilde{\pi}_{i_1,\dots,i_k}:(z_1,\dots,z_n)\mapsto(z_{i_1},\dots,z_{i_k})$. On the other hand, let z^0 be an arbitrary point of $\widetilde{D}\cap V$ and let $z^0_{i_1},\dots,z^0_{i_k}$ be all nonzero coordinates of z^0. From the definition of u we deduce that

$$u(z)\ge \widetilde{p}(\log|z_{i_1}|,\dots,\log|z_{i_k}|),\qquad \forall z\in\mathbb{C}^n_*.$$

Since

$$u(z^0)=\limsup_{\xi\to z^0,\,\xi\in\mathbb{C}^n_*}u(\xi),$$

we deduce that

$$u(z^0)\ge \widetilde{p}(\log|z^0_{i_1}|,\dots,\log|z^0_{i_k}|)-1. \tag{3}$$

It follows from (2) and (3) that for $a\in\widetilde{D}\cap V$ we have

$$u(a)=\widetilde{p}(\log|a_{i_1}|,\dots,\log|a_{i_n}|)-1.$$

Thus u is continuous on $\widetilde{D}$ and the requirement (a) is satisfied. For (b), we notice that for any $t>0$ the set $\{z\in\mathbb{C}^n_*:u(z)<t\}$ is a pseudoconvex Reinhardt domain. It follows that the set $\{z\in\mathbb{C}^n:u(z)<t\}$ is connected for every $t>0$.

(ii)⇒(iii). See the remarks before Theorem 3.2.

(iii)⇒(i). We split the proof into two steps.

STEP 1. We will prove that $0\notin\partial D$. Assume that $0\in\partial D$. For each $k\ge1$ we define the domain

$$D_k=\{(z_1,\dots,z_n):|z_j|<1/k,\ 1\le j\le n\}\cup D.$$

It is clear that D_k is Reinhardt, and we have

$$\log(D_k)_* = \{(x_1,\dots,x_n) : x_j < -\log k,\ 1 \le j \le n\} \cup \log D_*.$$

Take $p = (p_1,\dots,p_n) \in \mathbb{C}^n_* \cap D$. It is easy to see that

$$\operatorname{conv}(\log(D_k)_*) \supset \operatorname{conv}(\{(x_1,\dots,x_n) : x_j < -\log k, 1 \le j \le n\}, \widetilde{p}),$$

where $\widetilde{p} = (\log|p_1|,\dots,\log|p_n|)$. Hence for k large enough we have

$$\operatorname{conv}(\log(D_k)_*) \supset \{(x_1,\dots,x_n) : x_j < \log|p_j|,\ 1 \le j \le n\}.$$

As $\log(\widehat{D}_k)_*$ is convex we deduce that for k large enough

$$\widehat{D}_k \supset \{(z_1,\dots,z_n) : 0 < |z_j| < |p_j|, 1 \le j \le n\}.$$

Since $\overline{D}$ is compact Stein we infer that $\overline{D}$ has a pseudoconvex Reinhardt neighbourhood basis. This implies that

$$\overline{D} \supset \{(z_1,\dots,z_n) : |z_j| < |p_j|,\ 1 \le j \le n\}.$$

Consequently, $0 \in \operatorname{Int}(\overline{D}) = D$, a contradiction.

STEP 2. We let a be an arbitrary point of $\partial D \cap V$. According to Lemma 2.3 it suffices to construct a negative plurisubharmonic function ψ on D such that $\lim_{z\to a}\psi(z) = 0$. By Step 1 we have $a \ne 0$, thus with no loss of generality we may assume that $a = (0,\dots,0,a_{j+1},\dots,a_n)$, where $a_k \ne 0$ for $j+1 \le k \le n$.

Let π denote the projection $(z_1,\dots,z_n) \mapsto (z_{j+1},\dots,z_n)$. By Lemma 2.5(a), $\pi(D)$ is a Reinhardt pseudoconvex domain in $\mathbb{C}^{n-j}$. Now we claim that $\pi(a) \notin \pi(D)$. Indeed, otherwise we let $D' = \pi^{-1}(a') \cap D$, where $a' = (a_{j+1},\dots,a_n)$. By Lemma 2.5(b), D' is a pseudoconvex Reinhardt domain in $\mathbb{C}^j$. Since $\overline{D}$ is compact Stein, so is $\overline{D'}$. Notice that the origin in $\mathbb{C}^j$ lies on the boundary of D'. By repeating the argument used in Step 1 we see that $\overline{D'}$ contains a small neighbourhood of $0 \in \mathbb{C}^j$.

Let $(c_1,\dots,c_j,a_{j+1},\dots,a_n) \in D \cap \mathbb{C}^n_*$. We choose $\varepsilon,\delta > 0$ so small that

$$\begin{aligned} M_1 &= \{(x_1,\dots,x_n) : |x_j - b_j| < \delta,\ 1 \le j \le n\} \subset \log D_*,\\ M_2 &= \{(x_1,\dots,x_n) : x_1 < \log\varepsilon,\dots,x_j < \log\varepsilon\} \times \{(b_{j+1},\dots,b_n)\}\\ &\quad \subset \log(\overline{D})_*, \end{aligned}$$

and $\log\varepsilon < b_k - \delta$ for $1 \le k \le j$, where $b_1 = \log|c_1|,\dots,b_j = \log|c_j|$, $b_{j+1} = \log|a_{j+1}|,\dots,b_n = \log|a_n|$. This choice is possible because $\overline{D'}$ contains a neighbourhood of $0 \in \mathbb{C}^j$. We will prove that

$$\begin{aligned} \operatorname{conv}(M_1 \cup M_2) \supset M_3 := \{(x_1,\dots,x_n) : x_1 < \log\varepsilon,\dots,x_j < \log\varepsilon,\\ |x_{j+1} - b_{j+1}| < \delta/2,\dots,|x_n - b_n| < \delta/2\}. \end{aligned}$$

Indeed, assume that there exists a point $(y_1,\dots,y_n) \in M_3 \setminus \operatorname{conv}(M_1 \cup M_2)$. Since $\operatorname{conv}(M_1 \cup M_2)$ is an open convex domain in $\mathbb{R}^n$, by the Hahn–Banach theorem we can find $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ satisfying

$$(4) \quad \lambda_1 x_1 + \ldots + \lambda_n x_n < \lambda_1 y_1 + \ldots + \lambda_n y_n, \qquad (x_1, \ldots, x_n) \in M_1 \cup M_2.$$

Since (4) holds for every $(x_1, \ldots, x_n) \in M_2$ we deduce that $\lambda_k \geq 0$ for $1 \leq k \leq j$ and

$$\lambda_{j+1}(y_{j+1} - b_{j+1}) + \ldots + \lambda_n(y_n - b_n) > 0.$$

We choose a point $(x_1^0, \ldots, x_n^0) \in M_1$ with $x_k^0 = 2y_k - b_k$ for $j+1 \leq k \leq n$. Then from (4) we obtain

$$\begin{aligned} 0 &> \sum_{k=1}^{n} \lambda_k(x_k^0 - y_k) = \sum_{k=1}^{j} \lambda_k(x_k^0 - y_k) + \sum_{k=j+1}^{n} \lambda_k(x_k^0 - y_k) \\ &\geq \sum_{k=j+1}^{n} \lambda_k(x_k^0 - y_k) = \sum_{k=j+1}^{n} \lambda_k(y_k - b_k). \end{aligned}$$

We arrive at a contradiction. Hence $\operatorname{conv}(M_1 \cup M_2) \supset M_3$.

Now let Ω be an arbitrary Reinhardt domain containing $\overline{D}$. We see that

$$M_3 \subset \operatorname{conv}(M_1 \cup M_2) \subset \operatorname{conv}(\log \Omega_*) \subset \log(\widehat{\Omega})_*,$$

where $\widehat{\Omega}$ denotes the envelope of holomorphy of Ω. Since $\overline{D}$ is a compact Stein Reinhardt domain we infer that $\overline{D}$ has a pseudoconvex Reinhardt neighbourhood basis. It follows that $M_3 \subset \log(\overline{D})_*$. This implies that $0 \in \operatorname{Int}(\overline{D}) = D$, which is absurd. The proof is thereby concluded.

We mention a simple consequence of Theorem 3.2.

COROLLARY 3.3. *Let D be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$ satisfying* $\operatorname{Int}(\overline{D}) = D$. *Then $\overline{D}$ is polynomially convex if and only if $0 \in D$.*

Proof. It follows from the proof of Theorem 3.1 that if $0 \in D$ then $\overline{D}$ is holomorphically convex in $\mathbb{C}^n$; this means that $\overline{D}$ is polynomially convex. Conversely, assume that $\overline{D}$ is polynomially convex. By changing coordinates we may assume that $(1, \ldots, 1) \in D$. Since D is Reinhardt the Shilov boundary of the unit polydisk is contained in D. As $\overline{D}$ is polynomially convex it must contain the whole polydisk, in particular $0 \in D$.

4. Extending holomorphic mappings into Reinhardt domains. In this section we deal with the extension of a holomorphic mapping into a Reinhardt domain. More precisely, we say that a Reinhardt domain D in $\mathbb{C}^n$ has the k- (resp. Δ_*-) *extension property* if every holomorphic mapping from $\Delta^k \setminus S$ (resp. $\Delta_* := \Delta \setminus \{0\}$) into D can be extended to a holomorphic map from Δ^k (resp. Δ) into D, where Δ is the unit disk in $\mathbb{C}$ and S is a closed set of the polydisk Δ^k satisfying $\mathcal{H}^{2k-1}(S) = 0$, i.e. the $(2k-1)$-Hausdorff measure of S vanishes.

It is easy to check that the k-extension property implies the Δ_*-extension property. The reverse implication is however not true in general. Indeed, we let S be a closed subset of Δ satisfying $\mathcal{H}^1(S) = 0$ but with no isolated point. We first show that $D = \Delta \setminus S$ has the Δ_*-extension property. Let $f : \Delta_* \to \Delta \setminus S$ be a holomorphic function. By the Riemann extension theorem, f extends to a holomorphic function $\widetilde{f}$ on Δ. If f is constant then we have nothing to prove, otherwise $\widetilde{f}$ is an open mapping on Δ. Since S has no isolated point we deduce that $\widetilde{f}(0) \notin S$. It follows that $\Delta \setminus S$ has the Δ_*-extension property. On the other hand, the map $f : \Delta \setminus S \to \Delta \setminus S$ defined by $f(z) = z$ is clearly not extendible through S.

The first result of this section states that these properties are in fact equivalent in the class of Reinhardt domains. More precisely we have

PROPOSITION 4.1. *Let D be a Reinhardt domain in $\mathbb{C}^n$. The following assertions are equivalent.*

(i) *D has the Δ_*-extension property.*
(ii) *D has the k-extension property.*

Proof. It suffices to prove the implication (i)$\Rightarrow$(ii). If D has the Δ_*-extension property, then by a result in [TT], D is pseudoconvex.

Next we show that D is Brody hyperbolic, i.e. there exists no nonconstant holomorphic mapping from $\mathbb{C}$ into D. Indeed, let $g : \mathbb{C} \to D$ be a holomorphic mapping. Then $\widetilde{g}(z) := g(1/z)$ is a holomorphic mapping from $\mathbb{C}_*$ into D. Since D has the Δ_*-extension property, $\widetilde{g}$ extends through the origin. Hence g is bounded on $\mathbb{C}$, and by the Liouville theorem g must be constant. Thus D is Brody hyperbolic. According to Theorem 2.5.1 in [Zw2] we may assume that D is bounded.

Next we proceed by induction on n. If $n = 1$ then D, being a bounded Reinhardt domain having the Δ_*-extension property, must be either a disk or an annulus. By the maximum principle for subharmonic functions we can prove that D has the k-extension property.

Assume that the implication (i)$\Rightarrow$(ii) holds for $n-1$; we will prove that it also holds for n. Let $f : \Delta^k \setminus S \to D$ be a holomorphic mapping, where S is a closed subset of the polydisk Δ^k with $\mathcal{H}^{2k-1}(S) = 0$. We have to prove that f extends to a holomorphic mapping $\widetilde{f} : \Delta^k \to D$. Since D is bounded, f extends to a holomorphic mapping $\widetilde{f} = (\widetilde{f}_1, \ldots, \widetilde{f}_n) : \Delta^k \to \mathbb{C}^n$ (see [Ch, Appendix]). It suffices to check that $\widetilde{f}(\alpha) \in D$ for all $\alpha \in \Delta^k$. Seeking a contradiction, assume that there exists $\alpha \in \Delta^k$ such that $\widetilde{f}(\alpha) \in \partial D$. If $\widetilde{f}(\alpha) \in (\partial D) \setminus V$, then according to Lemma 2.3(i) we can find a small ball B centred at $\widetilde{f}(\alpha)$, and a plurisubharmonic function ψ on B such that $\psi(z) < 0$ for all $z \in B \cap D$ but $\lim_{z \to \widetilde{f}(\alpha)} f(z) = 0$. Notice that the function $\widetilde{\psi} = \psi \circ \widetilde{f}$ is plurisubharmonic on a small neighbourhood of α. This leads to a contradiction, in view of the maximum principle for plurisubharmonic functions. Thus $\widetilde{f}(\alpha) \in (\partial D) \cap V$. There are two cases to be considered.

CASE 1: $\widetilde{f}_1 \dots \widetilde{f}_n \equiv 0$. We may assume that $\widetilde{f}_1 \equiv 0$. Denote by π_1 the projection $(z_1, \dots, z_n) \mapsto (z_2, \dots, z_n)$. From Lemma 2.1 we deduce that $\pi_1(D \cap \{z_1 = 0\})$ is a bounded pseudoconvex Reinhardt domain having the Δ_*-extension property. Thus by the inductive hypothesis we are done.

CASE 2: $\widetilde{f}_1 \dots \widetilde{f}_n \not\equiv 0$. Let $\mathcal{Q}$ denote the complex hypersurface $(\widetilde{f}_1 \dots \widetilde{f}_n)^{-1}(0)$. First we assume that α is a regular point of $\mathcal{Q}$. Then after a local change of coordinates we may achieve that α is the origin in $\mathbb{C}^k$ and $\widetilde{f}(\Delta^* \times \Delta^{k-1}) \subset D$. As D has the Δ_*-extension property we deduce that $\widetilde{f}(\Delta^k) \subset D$. A contradiction. Thus $\alpha \in \mathcal{S}(\mathcal{Q})$, the singular locus of $\mathcal{Q}$. By using the same argument we see that $\alpha \in \mathcal{S}(\mathcal{S}(\mathcal{Q}))$. Continuing this process, we finally reach a contradiction. The desired conclusion now follows.

In this connection, we would like to mention that there exists a pseudoconvex Reinhardt domain in $\mathbb{C}^2$ having the Δ_*-extension property which is *not* hyperconvex (Example 2.11 in [CCW]). Notice that *every* hyperconvex domain has the Δ_*-extension property. Therefore the problem of finding a "good" criterion for the Δ_*-extension property of pseudoconvex Reinhardt domains is of interest. We have the following

PROPOSITION 4.2. *Let D be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then D has the Δ_*-extension property if and only if the following conditions hold*:

(a) *For any $1 \le j \le n$, $D \cap V_j$ (if not empty) has the Δ_*-extension property (as a domain in $\mathbb{C}^{n-1}$).*

(b) *Every holomorphic mapping $f : \Delta_* \to D$ of the form*

$$f(\xi) = (\lambda_1 \xi^{a_1}, \dots, \lambda_n \xi^{a_n}),$$

where $a_i \ge 0$ and $\lambda_i \ne 0$ for all i, extends through 0 to a holomorphic mapping $\widetilde{f} : \Delta \to D$.

Proof. ($\Rightarrow$) This follows immediately from the definition of the Δ_*-extension property.

($\Leftarrow$) Let $f : \Delta_* \to D$ be a holomorphic mapping. Since D is bounded, f extends holomorphically through the origin to $\widetilde{f} : \Delta \to \mathbb{C}^n$. We have to show that $\widetilde{f}(0) \in D$. Assume that $\widetilde{f}(0) \in \partial D$. If $\widetilde{f}(0) \in (\partial D) \setminus V$ then Lemma 2.3 and the maximum principle for subharmonic functions yield a contradiction. Thus $\widetilde{f}(0) \in (\partial D) \cap V$.

We consider two cases.

CASE 1: $\widetilde{f}_1 \dots \widetilde{f}_n \equiv 0$. Then the same reasoning as in Case 1 of the proof of Proposition 4.1 and (a) imply that $\widetilde{f}(0) \in D$. A contradiction.

CASE 2: $\widetilde{f}_1 \dots \widetilde{f}_n \not\equiv 0$. We choose $0 < r < 1$ so small that

$$\widetilde{f}_j(\xi) \neq 0, \qquad \forall 0 < |\xi| \leq r.$$

Thus we may write

$$\widetilde{f}_j(\xi) = \xi^{\nu_j} g_j(\xi),$$

where ν_j is the vanishing order of $\widetilde{f}_j$ at 0 and $g_j(0) \neq 0$.

Now we claim that

$$\{(\nu_1 t + \log|g_1(0)|, \dots, \nu_n t + \log|g_n(0)|) : t < \log r\} \subset \log D_*. \tag{5}$$

Indeed, if this is not true then there exists $t_0 < \log r$ such that

$$(\nu_1 t_0 + \log|g_1(0)|, \dots, \nu_n t_0 + \log|g_n(0)|) \notin \log D_*.$$

Since $\log D_*$ is an open convex domain in $\mathbb{R}^n$, by the Hahn–Banach theorem we can find $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ such that

$$\sum_{j=1}^n \lambda_j x_j < \sum_{j=1}^n \lambda_j(\nu_j t_0 + \log|g_j(0)|), \qquad \forall (x_1, \dots, x_n) \in \log D_*.$$

This implies, in particular, that for any ξ satisfying $0 < |\xi| \leq r$ we have

$$\sum_{j=1}^n \lambda_j \log|f_j(\xi)| < \sum_{j=1}^n \lambda_j(\nu_j t_0 + \log|g_j(0)|),$$

$$\sum_{j=1}^n \lambda_j(\nu_j \log|\xi| + \log|g_j(\xi)|) < \sum_{j=1}^n \lambda_j(\nu_j t_0 + \log|g_j(0)|),$$

$$\log|\xi| \cdot \Big(\sum_{j=1}^n \lambda_j \nu_j\Big) + \sum_{j=1}^n \lambda_j \log|g_j(\xi)| < t_0\Big(\sum_{j=1}^n \lambda_j \nu_j\Big) + \sum_{j=1}^n \lambda_j \log|g_j(0)|.$$

By setting

$$h(\xi) = \sum_{j=1}^n \lambda_j \log|g_j(\xi)|, \qquad 0 < |\xi| \leq r,$$

the last inequality becomes

$$h(\xi) - h(0) < (t_0 - \log|\xi|)\Big(\sum_{j=1}^n \lambda_j \nu_j\Big), \qquad \forall 0 < |\xi| \leq r.$$

By letting ξ tend to 0 we see that

$$\lambda_1 \nu_1 + \dots + \lambda_n \nu_n \geq 0.$$

As $t_0 < \log r$ we deduce that

$$h(\xi) < h(0), \qquad \forall |\xi| = r.$$

Since h is harmonic on the disk $|\xi| < r$ and continuous up to the boundary, we arrive at a contradiction to the maximum principle. Thus claim (5) is valid.

Consider the holomorphic mapping $F : \Delta_* \to \mathbb{C}^n$ defined by

$$F(\xi) = (g_1(0)\xi^{\nu_1}, \ldots, g_n(0)\xi^{\nu_n}).$$

From (5) we deduce that the mapping $G(\xi) := F(r\xi)$ satisfies (b). Thus it extends holomorphically through 0 to $\widetilde{G}$, so $\widetilde{f}(0) = \widetilde{G}(0) \in D$, contrary to assumption.

Therefore D has the Δ_*-extension property.

REMARKS. (a) Proposition 4.2 replaces a similar, though weaker result that appeared in the first version of this paper. It arised as an attempt to answer a question posed by the referee. We are grateful to him/her for bringing this question to our attention.

(b) It is possible to formulate the conclusion of Proposition 4.2 in terms of certain convex cones introduced in [Zw2, p. 28].

5. $\mathcal{O}^{(1)}$-domain of holomorphy. Let D be a domain in $\mathbb{C}^n$. We put $\delta_D = \min((1+|z|^2)^{-1/2}, d_D(z))$, where $d_D(z)$ is the distance from z to ∂D. For every $N > 0$ we define

$$\mathcal{O}^{(N)}(D, \delta_D) = \{f \in \mathcal{O}(D) : |\delta_D^N f|_\infty < \infty\},$$

where $\mathcal{O}(D)$ is the space of holomorphic functions on D. We call D an $\mathcal{O}^{(N)}(D, \delta_D)$-*domain of holomorphy* if D is the existence domain of a function in $\mathcal{O}^{(N)}(D, \delta_D)$.

It is proved in [JP] that any fat Reinhardt pseudoconvex domain in $\mathbb{C}^n$ is an $\mathcal{O}^{(N)}(D, \delta_D)$-domain of holomorphy for every $N > 0$. The following result complements this fact.

PROPOSITION 5.1. *Let D be a pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then D is an $\mathcal{O}^{(1)}(D, \delta_D)$-domain of holomorphy.*

Proof. If $\mathrm{Int}(\overline{D}) = D$ then D is an $\mathcal{O}^{(1)}(D, \delta_D)$-domain of holomorphy by the above mentioned result of [JP]. It remains to consider the case $\mathrm{Int}(\overline{D}) \neq D$.

We proceed by induction on n. For $n = 1$, D is of the form $\{z : |z| < r\}$, where $0 < r \le \infty$. Thus D is an $\mathcal{O}^{(1)}(D, \delta_D)$-domain of holomorphy.

Assume that the conclusion holds for $n - 1$. Let a be an arbitrary point from ∂D. It suffices to show that there exists a function $f_a \in \mathcal{O}^{(1)}(D, \delta_D)$ which is not extendible through a. We consider two cases.

CASE 1: $a \in \partial D \setminus V$. Then from Lemma 2.3(i) and the maximum principle for plurisubharmonic functions we deduce that $a \notin \mathrm{Int}(\overline{D})$, thus $a \in \partial(\mathrm{Int}(\overline{D}))$. Since $\mathrm{Int}(\overline{D})$ is a fat Reinhardt pseudoconvex domain we can find a function $f_a \in \mathcal{O}^{(1)}(\mathrm{Int}(\overline{D}), \delta_{\mathrm{Int}(\overline{D})})$ which is not extendible through a. Since $\mathcal{O}^{(1)}(\mathrm{Int}(\overline{D}), \delta_{\mathrm{Int}(\overline{D})}) \subset \mathcal{O}^{(1)}(D, \delta_D)$ we are done.

Case 2: $a = (a_1, \ldots, a_n) \in (\partial D) \cap V$. We may assume that $a_n = 0$. If $D \cap V_n = \emptyset$ then the function $1/z_n$ is not extendible through a. Moreover, for every $z \in D$ we have

$$\left|\delta_D(z)\,\frac{1}{z_n}\right| \le \left|\frac{d_D(z)}{z_n}\right| \le \left|\frac{d_{\mathbb{C}^n\setminus V_n}(z)}{z_n}\right| = 1.$$

This implies that $1/z_n \in \mathcal{O}^{(1)}(D, \delta_D)$. If $D \cap V_n \ne \emptyset$, then by Lemma 2.1 the map $\pi_n : (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, 0)$ sends D to $D \cap V_n$. Moreover, in view of that lemma, $D \cap V_n$ is a Reinhardt pseudoconvex domain (viewed as a subset of $\mathbb{C}^{n-1}$) and $a' = \pi_n(a) \in \partial(D \cap V_n)$. By the inductive hypothesis, there exists $g_{a'} \in \mathcal{O}^{(1)}(D \cap V_n, \delta_{D\cap V_n})$ which is not extendible through a'. It follows that $f_a = g_{a'} \circ \pi_n$ is an element of $\mathcal{O}(D)$ which is not extendible through a. It remains to prove that f_a belongs to $\mathcal{O}^{(1)}(D, \delta_D)$. To see this, notice that for $z \in D$ we get $d_D(z) \le d_{D\cap V_n}(\pi_n(z))$, and for $z \in \mathbb{C}^n$ we have $(1 + |z|^2)^{-1/2} \le (1 + |\pi_n(z)|^2)^{-1/2}$. Hence

$$\sup_{z\in D} |\delta_D(z) f_a(z)| \le \sup_{z'\in D\cap V_n} |\delta_{D\cap V_n}(z')g(z')| < \infty.$$

This implies that $f_a \in \mathcal{O}^{(1)}(D, \delta_D)$. The proof is complete.

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