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A class of functions containing polyharmonic functions in $\mathbb{R}^n$

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Abstract. Some properties of the functions of the form $v(x) = \sum_{i=0}^m |x|^i h_i(x)$ in $\mathbb{R}^n$, $n \geq 2$, where each h_i is a harmonic function defined outside a compact set, are obtained using the harmonic measures.

1. Introduction. Let Ω be a clamped plate with an external load density $f(x)$, $x \in \Omega$. If $\partial\Omega$ is regular for the Dirichlet problem, the solution $u(x)$ corresponding to the conditions of elasticity given by $\Delta^2 u(x) = f(x)$ on Ω, $u = 0 = \partial u/\partial n$ on $\partial\Omega$, has a representation (Nicolesco [16, p. 40]) $u(x) = \int_\Omega G_2(x,y) f(y)\,dy$. However, the calculation of $G_2(x,y)$ corresponding to a given Ω is not simple.

This note, among other results, shows that in the particular case of Ω being in the form of a star domain with centre 0 and the equation $\Delta^2 u = f$ being reduced to the condition $\Delta u(x) = G(x) + |x|^{-1} H(x)$ where G and H are harmonic functions on Ω, continuous on $\overline{\Omega}$, we can express u as $u(x) = |x|^2 h_2(x) + |x| h_1(x) + h_0(x)$ where $h_i(x)$ are harmonic functions on Ω. This suggests the study of the properties of functions of the form $u(x) = \sum_{i=0}^m |x|^i h_i(x)$ in a star domain Ω with centre 0 in $\mathbb{R}^n$, $n \geq 2$, which include the polyharmonic functions of finite order on Ω.

In another context, when $u(x) = \sum_{i=0}^m |x|^i h_i(x)$ is defined on the whole of $\mathbb{R}^n$, $n \geq 2$, where each h_i is harmonic on $\mathbb{R}^n$, Nakai and Tada ([15, Theorem 3]) give a necessary and sufficient condition on u so that each h_i is a harmonic polynomial, by using the Fourier expansion method. In this note we obtain some complementary results for such functions u, by using harmonic measures.

2. Functions in the class $H^*(\mathbb{R}^n)$. We begin with the following definition.

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DEFINITION 2.1. Let Ω be a star domain with centre 0 in $\mathbb{R}^n$, $n \geq 2$. A continuous function $u(x)$ defined on Ω is said to be in the class $H^m(\Omega)$ if it is of the form $u(x) = \sum_{i=0}^{m} |x|^i h_i(x)$ where h_i, $0 \leq i \leq m$, are harmonic on Ω. Write $H^*(\Omega) = \bigcup_{m \geq 0} H^m(\Omega)$.

We remark that if $u(x)$ is a polyharmonic function of order m on Ω (that is, $u \in C^{2m}(\Omega)$ and $\Delta^m u = 0$), then the Almansi representation of u (see Aronszajn *et al.* [8, Proposition 1.3]) is of the form $u(x) = \sum_{i=0}^{m-1} |x|^{2i} h_i(x)$ where $h_i(x)$ are uniquely determined harmonic functions on Ω. Hence $u \in H^{2m-2}(\Omega)$, $H^0(\Omega)$ being the class of all harmonic functions on Ω.

In this section, we obtain some Liouville-type theorems for functions in $H^*(\mathbb{R}^n)$. For a given continuous function f on $\mathbb{R}^n$, let $D_r f$ stand for the Dirichlet solution in $|x| < r$ with boundary value $f(x)$ on $|x| = r$.

THEOREM 2.2. *Let $u \in H^*(\mathbb{R}^n)$. Let a_j be a sequence of real numbers increasing to ∞. Let ω be a nonempty open set and e be a polar set in $\mathbb{R}^n$. If*

$$\liminf_{j \to \infty} \frac{D_{a_j} u(z)}{a_j} = 0$$

for every $z \in \omega \setminus e$, then u is harmonic on $\mathbb{R}^n$.

Proof. Let $u(x) = \sum_{i=0}^{m} |x|^i h_i(x)$. Then, for fixed $z \in \omega \setminus e$ and large j, $D_{a_j} u(z) = \sum_{i=0}^{m} a_j^i h_i(z)$. Hence the assumption on $D_{a_j} u$ as $j \to \infty$ implies that $h_i(z) = 0$ for $1 \leq i \leq m$. Since z is arbitrary in $\omega \setminus e$, h_i vanishes on $\omega \setminus e$, which implies that $h_i \equiv 0$ for $1 \leq i \leq m$. Consequently, $u(x) = h_0(x)$ is harmonic on $\mathbb{R}^n$.

COROLLARY 1 (see [15, Proposition 1]). *Let $u = \sum_{i=0}^{m} |x|^i h_i(x) \in H^*(\mathbb{R}^n)$. If $u(x) \to 0$ when $|x| \to \infty$, in particular if $u \equiv 0$, then each $h_i \equiv 0$.*

Proof. As in the proof of Theorem 2.2 we can show that $h_i \equiv 0$ for $1 \leq i \leq m$ and $u = h_0$ on $\mathbb{R}^n$. Since $u \to 0$ at infinity, $h_0 \equiv 0$.

COROLLARY 2. *Let $u \in H^*(\mathbb{R}^n)$. Let $M(r, |u|)$ denote the mean value of $|u(x)|$ on $|x| = r$. Suppose $M(r, |u|) = o(r)$ as $r \to \infty$. Then u is a constant.*

Proof. Write $D_r u(z) = \int u(x)\, d\varrho_z^r(x)$ where ϱ_z^r is the harmonic measure on $|x| = r$. Recall

$$d\varrho_z^r(x) = r^{n-2} \frac{r^2 - |z|^2}{|x - z|^n}\, d\sigma_r(x)$$

where $d\sigma_r(x)$ is the measure on $|x| = r$, invariant with respect to the rotations of $|x| = r$ and such that $\sigma_r(|x| = r) = 1$. Suppose now $|z| \leq 1 < r = |x|$. Then

$$\frac{r^2 - |z|^2}{|x - z|^n} \leq \frac{r^2}{(r-1)^n}$$

so that

$$|D_r u(z)| \leq \int |u(x)| \frac{r^n}{(r-1)^n} \, d\sigma_r(x).$$

Consequently, for large r, $|D_r u(z)| \leq 2M(r, |u|)$ so that $\lim_{r\to\infty} D_r u(z)/r = 0$ for $|z| \leq 1$. Then by the above theorem, we conclude that u is harmonic on $\mathbb{R}^n$; this, together with the condition that $M(r, |u|) = o(r)$, implies that u is a constant. (See [6, Corollary 3.3] for an indication of different proofs of this result.)

Corollary 3. *Let $u \in H^*(\mathbb{R}^n)$ be such that for a superharmonic function s on $\mathbb{R}^n$, $|u| \leq s$. Then u is a constant.*

Proof. We write $s = p + c$ where $p \geq 0$ is a potential on $\mathbb{R}^n$ ($p \equiv 0$ is the only potential on $\mathbb{R}^2$) and $c \geq 0$ is a constant. Since $M(r, p) \to 0$ as $r \to \infty$, $M(r, |u|) = o(r)$. Hence by Corollary 2, u is harmonic on $\mathbb{R}^n$ and $|u| \leq p + c$, which implies that $|u| \leq c$ on $\mathbb{R}^n$ so that u is a constant.

Remark. The above Corollary 3 expresses the classical Liouville–Picard theorem for the class $H^*(\mathbb{R}^n)$. In fact, in its standard form this theorem states that if $u \in H^0(\mathbb{R}^n)$ and $u \geq 0$, then it is a constant; this can be generalized as follows: If $u \in H^*(\mathbb{R}^n)$ is a positive superharmonic function, then u is a constant. This generalization is a consequence of the above Corollary 3. However, if we leave out the condition that u is superharmonic, we have from Armitage [7] or Futamura–Kishi–Mizuta [13] or Nakai–Tada [15] the following: If $u \in H^m(\mathbb{R}^n)$ is positive, then u is a polynomial of degree at most m. More generally, let $u(x) = \sum_{i=0}^{m} |x|^i h_i(x)$ and suppose $\liminf_{|x|\to\infty} u(x)/|x|^s \geq 0$ for some $s > m$. Then for each i, h_i is a harmonic polynomial of degree less than $s - i$. For as in [6, Lemma 2.1] we can find a locally integrable function $\varphi(x)$ on $\mathbb{R}^n$ such that $u(x) \geq \varphi(x)$ outside a compact set K and $M(r, |\varphi|) = o(r^s)$ as $r \to \infty$. Since $u \geq -|\varphi|$ on $\mathbb{R}^n \setminus K$, $M(r, u^-) = o(r^s)$ as $r \to \infty$; also $M(r, u) = O(r^m)$. Hence $M(r, |u|) = o(r^s)$. Consequently, each h_i is a harmonic polynomial of degree less than $s - i$. In particular, if $\liminf_{|x|\to\infty} u(x)/|x| \geq 0$ then u is a constant.

3. H^* functions on a star domain. Let Ω denote a star domain in $\mathbb{R}^n$, $n \geq 2$, with centre 0. Then, as in Definition 2.1, $H^m(\Omega)$ denotes the class of functions of the form $u(x) = \sum_{i=0}^{m} |x|^i h_i(x)$ where h_i are harmonic on Ω. We shall use the operator Δ^p, with integer $p \geq 1$, in the sense of distributions.

Lemma 3.1. *If $u \in C^{2p}(\Omega)$, let $f(x) = |x|^\alpha u(x)$ where $\alpha > 2p - n$. Then $\Delta^p(|x|^\alpha u(x))$ is locally integrable on Ω.*

Proof. Since $(\partial/\partial x)^\lambda |x|^\alpha$ is locally integrable on $\mathbb{R}^n$ when $\alpha > |\lambda| - n$, the lemma is evident. We need the following expression of $\Delta^p f(x)$ for later use.

For $|x| = r > 0$, let $g = \Delta^p f$ in the classical sense.

Since

$$\Delta f = \alpha(\alpha + n - 2)r^{\alpha-2}u + 2\alpha r^{\alpha-2}\sum x_i \frac{\partial u}{\partial x_i} + r^\alpha \Delta u,$$

we have $\Delta f = r^{\alpha-2}u_1$ where $u_1 \in C^{2p-2}$ (note that u_1 is harmonic if u is). Proceeding thus, we find $\Delta^j f = r^{\alpha-2j}u_j$ for $1 \le j \le p$, where $u_j \in C^{2p-2j}$. Let $g(x) = |x|^{\alpha-2p}u_p(x)$ for $|x| > 0$ and $g(0) = \limsup_{x\to 0} g(x)$. Then $g(x)$ is u.s.c. on Ω and since $\alpha - 2p + n > 0$, $g(x)$ is locally integrable on Ω and $\Delta^p f = g$ on Ω in the sense of distributions.

Remark. Let h be harmonic on Ω. Then from the above proof (replacing u by h) we see that if $\alpha > 2p - n$, then $\Delta^p(|x|^\alpha h(x)) = |x|^{\alpha-2p}H(x)$ where $H(x)$ is harmonic on Ω.

Proposition 3.2. *Let $u \in H^*(\Omega)$ and $2p \le n$. Then $\Delta^p u$ is locally integrable on Ω.*

Proof. Let $u = \sum_{i=0}^m |x|^i h_i(x)$. Then for any $i \ge 1$, $i > 2p - n$ so that $\Delta^p(|x|^i h_i(x))$ is locally integrable on Ω; for $i = 0$, $\Delta(h_0) = 0$. Hence $\Delta^p u$ is locally integrable on Ω.

Consequence. The above proposition, in particular, states that if $u \in H^*(\mathbb{R}^n)$, $n \ge 2$, then Δu is locally integrable on $\mathbb{R}^n$. This leads to an *integral representation* of u in $\mathbb{R}^n$. For that, recall that given any positive Radon measure μ on an open set ω in $\mathbb{R}^n$, $n \ge 2$, Brelot [12] shows that a subharmonic function s can be constructed on ω with associated measure μ in the local Riesz representation.

Now, for $u \in H^*(\mathbb{R}^n)$, $n \ge 2$, since Δu is locally integrable, $d\lambda(x) = \Delta u dx$ can be treated as defining the difference of two positive Radon measures on $\mathbb{R}^n$. Hence u is the difference of two subharmonic functions on $\mathbb{R}^n$. Then we can define the order of u and the order of λ as in Arsove [9] (see also [4]). If the order of λ is finite, a correspondingly modified form of the logarithmic kernel (if $n = 2$) or the Newtonian kernel (if $n \ge 3$) can be used to represent u as an integral up to an additive harmonic function which is a harmonic polynomial if the order of u is finite (see Arsove [9], and [4, Theorems 11 and 12]; see also Mizuta [14]).

Lemma 3.3. *Let H be a harmonic function on Ω. If $n + \alpha - 2 > 0$ and if $\alpha + 2i \ne 0$ for i, $0 \le i \le p - 1$, then there exists a harmonic function h on Ω such that $\Delta^p(|x|^{\alpha+2p-2}h(x)) = |x|^{\alpha-2}H(x)$.*

Proof. We prove the lemma for the case $p = 1$ by adapting the method given in Aronszajn *et al.* [8, p. 5]. The general case follows by induction. Suppose a harmonic function h exists on Ω such that $\Delta(|x|^\alpha h(x)) = |x|^{\alpha-2}H(x)$.

Then, treating h as a function of r, we should have

$$\Delta(r^\alpha h) = \alpha(n+\alpha-2)r^{\alpha-2}h + 2\alpha r^{\alpha-1}\,\frac{\partial h}{\partial r}.$$

Then

$$\alpha(n+\alpha-2)h + 2\alpha r\,\frac{\partial h}{\partial r} = H \quad \text{ on } \Omega.$$

This can be written as

$$\frac{d}{dr}[r^{(n+\alpha-2)/2}h] = \frac{H}{2\alpha}\,r^{(n+\alpha-4)/2}$$

outside 0. Since at the origin, $r^{(n+\alpha-2)/2}h = 0$, we should have

$$r^{(n+\alpha-2)/2}h(r,w) = \int_0^r \frac{1}{2\alpha}\varrho^{(n+\alpha-4)/2}H(\varrho,w)\,d\varrho,$$

where $x = (r,w)$ is represented by the spherical polar coordinates. Set $\varrho = tr$. Then

$$h(x) = \frac{1}{2\alpha}\int_0^1 t^{(n+\alpha-4)/2}H(tx)\,dt;$$

here $\Delta h = 0$ since H is harmonic. Consequently, given the harmonic function H on Ω, if we define $h(x)$ by the formula above, then $h(x)$ is harmonic on Ω, satisfying the condition $\Delta(|x|^\alpha h(x)) = |x|^{\alpha-2}H(x)$.

THEOREM 3.4. *A continuous function u on Ω is in $H^m(\Omega)$ if and only if for any integer p, $2 \le 2p \le n$, there exists a function $v \in H^{m-1}(\Omega)$ such that $\Delta^p u(x) = |x|^{1-2p}v(x)$ in the sense of distributions.*

Proof. (1) Let $u = \sum_{i=0}^m |x|^i h_i \in H^m(\Omega)$. We shall now use the Remark following Lemma 3.1 to calculate $\Delta^p u(x)$.

If $p = 1$, then $\Delta u = \sum_{i=0}^{m-2} |x|^i H_i + |x|^{-1}v_1$ where H_i $(0 \le i \le m-2)$ and v_1 are harmonic functions on Ω. Hence $\Delta u = |x|^{-1}[\sum_{i=0}^{m-2} |x|^{i+1}H_i + v_1] = |x|^{-1}s_1(x)$ where $s_1 \in H^{m-1}(\Omega)$.

If $p = 2$, then $n \ge 4$ and in this case $\Delta(|x|^{-1}v_1) = |x|^{-3}v_2$ where v_2 is harmonic on Ω. This leads to the equation $\Delta^2 u = \Delta(\Delta u) = \sum_{i=0}^{m-4} |x|^i H_i' + |x|^{-1}v + |x|^{-3}v_2$ on Ω, where H_i' $(0 \le i \le m-4)$, v and v_2 are harmonic on Ω. This simplifies to the form $\Delta^2 u = |x|^{-3}s_2(x)$ where $s_2 \in H^{m-1}(\Omega)$.

This process by induction leads to the result that if $2 \le 2p \le n$, then $\Delta^p u = |x|^{-(2p-1)}s_p(x)$ where $s_p \in H^{m-1}(\Omega)$.

(2) Conversely, suppose that u is a continuous function on Ω such that for any integer p, $2 \le 2p \le n$, we have $\Delta^p u = |x|^{-(2p-1)}v$ where $v \in H^{m-1}(\Omega)$. Then, in particular for $p = 1$, by Lemma 3.3 we have

$$\Delta u = |x|^{-1}v = |x|^{-1}\sum_{i=0}^{m-1} |x|^i H_i = \sum_{i=0}^{m-1} |x|^{i-1}H_i = \sum \Delta(|x|^{i+1}h_i).$$

Hence $u = (\sum_{i=0}^{m-1} |x|^{i+1} h_i) +$ (a harmonic function on Ω); in other words, $u \in H^m(\Omega)$.

Corollary 1. *Let* $u = \sum_{i=0}^{m} |x|^i h_i(x) \in H^m(\Omega)$. *Suppose* u *is harmonic on a neighbourhood of a point in* Ω. *Then* $h_i \equiv 0$ *for* $1 \le i \le m$. *In particular*, *if* $u \equiv 0$ *on a nonempty open set in* Ω, *then* $h_i \equiv 0$ *for all* i, $0 \le i \le m$.

Proof. Let u be harmonic on a nonempty open set ω. Since $u \in H^m(\Omega)$, there exists $v_{m-1} \in H^{m-1}(\Omega)$ such that $\Delta u = |x|^{-1} v_{m-1}(x)$ on Ω. This implies $v_{m-1} = 0$ on ω. Now again by Theorem 3.4, there exists $v_{m-2} \in H^{m-2}(\Omega)$ such that $\Delta v_{m-1} = |x|^{-1} v_{m-2}(x)$, which implies that $v_{m-2} = 0$ on ω. Proceeding thus, we obtain $v_i \in H^i(\Omega)$, $0 \le i \le m-1$, such that $\Delta v_{i+1} = |x|^{-1} v_i$ on Ω and $v_i = 0$ on ω (taking $v_m = u$).

Since v_0 is harmonic on Ω and $v_0 = 0$ on ω, $v_0 \equiv 0$ on Ω. This implies v_1 is harmonic on Ω and since $v_1 = 0$ on ω, we have $v_1 \equiv 0$ on Ω. Thus proceeding, we remark that $v_i \equiv 0$ on Ω for $0 \le i \le m-1$ so that $\Delta u = 0$ on Ω; that is, u is harmonic on Ω.

Then $\sum_{i=1}^{m} |x|^i h_i(x) = u(x) - h_0(x)$ is harmonic on Ω. Choose a such that $\{x : |x| < a\} \subset \Omega$. Fix $z \in \Omega$ so that $|z| < r < a$. Let ϱ_z^r be the harmonic measure on $|x| = r$. Then

$$\sum_{i=1}^{m} \int |x|^i h_i(x)\, d\varrho_z^r(x) = \int (u - h_0)\, d\varrho_z^r,$$

which implies that

$$\sum_{i=1}^{m} r^i h_i(z) = u(z) - h_0(z).$$

Since r is arbitrary in the interval $(|z|, a)$, we have $h_i(z) = 0$ for $1 \le i \le m$ and $u(z) - h_0(z) = 0$. Since h_i and u are harmonic on Ω and z is arbitrary except for the condition $|z| < a$, we conclude $h_i \equiv 0$ on Ω for $1 \le i \le m$ and $u \equiv h_0$.

Corollary 2. *Let* $u \in H^m(\Omega)$ *and* $2 \le 2p \le n$. *Suppose* u *is* p-*harmonic* ($\Delta^p u = 0$) *on a neighbourhood of a point in* Ω. *Then* u *is* p-*harmonic on* Ω.

Proof. Suppose $\Delta^p u = 0$ on a nonempty open set ω. By Theorem 3.4, there exists a function $v \in H^{m-1}(\Omega)$ such that $\Delta^p u = |x|^{1-2p} v$ on Ω. Hence $v = 0$ on ω and consequently, by Corollary 1, $v \equiv 0$ on Ω, which means that u is p-harmonic on Ω.

Remark. We thank the referee for pointing out that in the proofs of the above two corollaries, one can use the real analyticity of u outside the origin, without having recourse to Theorem 3.4.

COROLLARY 3. *Let $u \in H^m(S)$, where $S = \{x : |x| < 1\}$ in $\mathbb{R}^n$, be such that* $\liminf_{r\to 1} M(r,|u|) = 0$. *Then $u(x) = (1-|x|)v(x)$ where $v \in H^{m-1}(S)$.*

Proof. Let $u(x) = \sum_{i=0}^{m} |x|^i h_i(x) \in H^m(S)$. Notice that by the above Corollary 1, there exist uniquely determined harmonic functions u_i on S such that

$$u(x) = \sum_{i=0}^{m} (1-|x|)^i u_i(x) \quad \text{on } S.$$

In particular,

$$h_0(x) = \sum_{i=0}^{m} u_i(x).$$

Fix z with $|z| \le 1/4$. For $|x| = r$, $1/2 \le r < 1$, let $d\varrho_z^r(x)$ denote the harmonic measure on $|x| = r$. Integrate $u(x)$ with respect to $d\varrho_z^r(x)$ to get

$$|(1-r)^m u_m(z) + \ldots + (1-r)u_1(z) + u_0(z)| = \left|\int u(z)\, d\varrho_z^r(x)\right| \le CM(r,|u|)$$

for some constant C since $|z| \le 1/4$ and $1/2 \le r < 1$ (see the proof of Corollary 2 to Theorem 2.2).

Let $r \to 1$. Since $\liminf_{r\to 1} M(r,|u|) = 0$ by hypothesis, we obtain $u_0(z) = 0$ for $|z| \le 1/4$. Hence $u_0 \equiv 0$. Consequently, $u(x) = (1-|x|)v(x)$ where

$$v(x) = \sum_{i=0}^{m-1} (1-|x|)^i u_{i+1}(x) \in H^{m-1}(S).$$

NOTE. The above corollary easily includes the result: If u is p-harmonic on S such that $\liminf_{r\to 1} M(r,|u|) = 0$, then $u(x) = (1-|x|^2)v(x)$ where v is polyharmonic on S of order $\le m-1$. This in itself is a generalization of a result of Abkar and Hedenmalm [1, pp. 321–322] proved in the complex plane using the Fourier series: If $u(z)$ is biharmonic on the unit disc in the complex plane and if $M(r,|u|) = O(1-r)$ as $r \to 1$, then $u(z) = (1-|z|^2)h(z)$ for a harmonic function on the unit disc.

4. H^* functions defined near infinity. It is not surprising that in many respects, the functions in $H^{2m-2}(\mathbb{R}^n)$ behave near infinity like the m-harmonic functions (that is, the solutions of $\Delta^m u = 0$) on $\mathbb{R}^n$. In this section, we study a class of continuous functions on $\mathbb{R}^n$ which are associated near infinity with the functions in $H^{2m-2}(\mathbb{R}^n)$ and the fundamental solution of Δ^m on $\mathbb{R}^n$. This class contains a significant collection of functions having some nice regularity properties at infinity.

For $m \ge 1$, $n \ge 2$, let e_m^n denote the fundamental solution of Δ^m on $\mathbb{R}^n$. We recall that

$$e_m^n = \begin{cases} |x|^{2m-n} & \text{if } 2m < n \text{ or } 2m-n \text{ is odd} > 0, \\ |x|^{2m-n}\log|x| & \text{if } 2m-n \text{ is even} \geq 0. \end{cases}$$

DEFINITION 4.1. A continuous function v defined outside a compact set in $\mathbb{R}^n$ is said to be in the class $H_\infty^{2m-2}(\mathbb{R}^n)$ if there exists $u \in H^{2m-2}(\mathbb{R}^n)$such that $u - v = O(e_m^n)$ near infinity.

PROPOSITION 4.2. *Let h_i $(0 \leq i \leq m-2)$ be arbitrary harmonic functions defined outside a compact set in $\mathbb{R}^n$. Then*

$$v = \sum_{i=0}^{2m-2} |x|^i h_i(x) \in H_\infty^{2m-2}(\mathbb{R}^n).$$

Proof. Recall that (see [2] or Axler *et al.* [10]) given a harmonic function h outside a compact set in $\mathbb{R}^n$, there exists a harmonic function H on $\mathbb{R}^n$ and a constant α such that outside a compact set,

$$h(x) = \begin{cases} H(x) + \alpha\log|x| + g(x) & \text{if } n = 2, \\ H(x) + g(x) & \text{if } n \geq 3, \end{cases}$$

where $g(x)$ is a harmonic function satisfying $g(x) = O(|x|^{2-n})$ near infinity. Hence for each i, $0 \leq i \leq 2m-2$, there exists a harmonic function H_i on $\mathbb{R}^n$ such that $|h_i - H_i| \leq A|x|^{2-n}$ if $n \geq 3$ and $\big|h_i - H_i - \alpha_i \log|x|\big| \leq A$ if $n = 2$, outside a compact set. Let $u(x) = \sum_{i=0}^{2m-2} |x|^i H_i(x)$. Then $u \in H^{2m-2}(\mathbb{R}^n)$ and near infinity $u - v = O(e_m^n)$. Hence $v \in H_\infty^{2m-2}(\mathbb{R}^n)$.

COROLLARY. *Let k be a compact set in a star domain Ω with centre* 0*. Suppose $u = \sum_{i=0}^{2m-2} |x|^i h_i(x)$ where h_i are harmonic on $\Omega \setminus k$. Then there exist $t \in H^{2m-2}(\Omega)$ and $s \in H_\infty^{m-2}(\mathbb{R}^n) \cap C(\mathbb{R}^n \setminus k)$, $s = O(e_m^n)$ near infinity, such that $u = s - t$ on $\Omega \setminus k$.*

Proof. We know that for each i, there exist $s_i \in H^0(\mathbb{R}^n \setminus k)$ and $t_i \in H^0(\Omega)$ such that $h_i = s_i - t_i$ on $\Omega \setminus k$ (Laurent decomposition for harmonic functions, see [2] or Axler *et al.* [10, pp. 171–175]).

Let now

$$s^1(x) = \sum_{i=0}^{2m-2} |x|^i s_i(x) \quad \text{and} \quad t^1(x) = \sum_{i=0}^{2m-2} |x|^i t_i(x).$$

Then $u = s^1 - t^1$ on $\Omega \setminus k$ where by the above proposition $s^1(x)$ is in the class $H_\infty^{2m-2}(\mathbb{R}^n)$, and $t^1(x) \in H^{2m-2}(\Omega)$. Since $s^1 \in H_\infty^{2m-2}(\mathbb{R}^n)$, there exists $v \in H^{2m-2}(\mathbb{R}^n)$ such that $s^1 - v = O(e_m^n)$ near infinity. Write now $s = s^1 - v$ and $t = t^1 - v$ to obtain the decomposition $u = s - t$ on $\Omega \setminus k$ as stated in the Corollary.

PROPOSITION 4.3. *If v is an m-harmonic function defined outside a compact set in $\mathbb{R}^n$, then $v \in H_\infty^{2m-2}(\mathbb{R}^n)$.*

Proof. For $m = 1$, the representation for a harmonic function h outside a compact set (given in the proof of Proposition 4.2) leads to the result that $h \in H^1_\infty(\mathbb{R}^n)$.

Let us take the case $m = 2$. In this case, we start with the representation for a biharmonic function b defined near infinity in the following form (see [11, p. 19]):

$$b(x) = \begin{cases} (\alpha + \alpha_1 x_1 + \alpha_2 x_2)\log|x| + \beta|x|^2 \log|x| + B(x) + u(x) & \text{if } n = 2, \\ \beta|x| + B(x) + u(x) & \text{if } n = 3, \\ \beta \log|x| + B(x) + u(x) & \text{if } n = 4, \\ B(x) + u(x) & \text{if } n \geq 5, \end{cases}$$

where $B(x)$ is biharmonic on $\mathbb{R}^n$ and $u(x)$ is biharmonic bounded near infinity. In the case of $n \geq 5$, we can show that $|u(x)| \leq A|x|^{4-n}$ by specializing the proof (1)$\Rightarrow$(2) of [11, Theorem 10]. Consequently, since $B \in H^2(\mathbb{R}^n)$ and since $b - B = O(e^n_2)$ near infinity, we have $b \in H^2_\infty(\mathbb{R}^n)$.

Finally, for $m > 2$, we have a similar representation for an m-harmonic function defined outside a compact set (details given in a forthcoming paper [5]) which can be used to prove the proposition. The result referred to here is as follows: Let u be m-harmonic outside a compact set in $\mathbb{R}^n$. Then there exists an m-harmonic function v on $\mathbb{R}^n$ such that $u - v = O(e^n_m)$ as $|x| \to \infty$.

THEOREM 4.4. *Let $v \in H^{2m-2}_\infty(\mathbb{R}^n)$, $n > 2m \geq 2$. Suppose either one of the following conditions is satisfied*:

(1) *There exists a superharmonic function s outside a compact set such that $|v| \leq s$ near infinity.*

(2) $\lim_{|x|\to\infty} v(x)/|x| = 0$.

Then $\lim_{|x|\to\infty} v(x)$ exists and is finite.

Proof. (1) Suppose $|v| \leq s$ near infinity. Since $n \geq 3$, there exists a superharmonic function S on $\mathbb{R}^n$ such that $S - s$ is bounded near infinity (see [3]). Hence we can as well assume that s is a superharmonic function defined on the whole of $\mathbb{R}^n$ and $|v| \leq s$ near infinity.

Since $v \in H^{2m-2}_\infty(\mathbb{R}^n)$, by definition there exists $u \in H^{2m-2}(\mathbb{R}^n)$ such that $|u - v| \leq A|x|^{2m-n}$ near infinity. Hence $|u| \leq s + A|x|^{2m-n} \leq s + A$ near infinity. Then by Corollary 3 in Section 2, u is a constant α. Consequently, $\lim_{x\to\infty} v(x) = \alpha$.

(2) Suppose now $\lim_{|x|\to\infty} v(x)/|x| = 0$. Since $v \in H^{2m-2}_\infty(\mathbb{R}^n)$, there exists $u \in H^*(\mathbb{R}^n)$ such that $|u - v| \leq A|x|^{2m-n}$ near infinity. This implies that $\lim_{|x|\to\infty} u(x)/|x| = 0$. Hence, by Corollary 2 in Section 2, u is a constant α. Consequently $\lim_{|x|\to\infty} v(x) = \alpha$.

REMARK. Since every bounded continuous function v is in H^{2m-2}_∞ if $2 \leq n \leq 2m$, the above theorem is not valid if $n \leq 2m$.

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