A class of functions containing polyharmonic functions in $\mathbb{R}^n$

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Abstract. Some properties of the functions of the form $v(x) = \sum_{i=0}^m |x|^i h_i(x)$ in $\mathbb{R}^n$, $n \geq 2$, where each $h_i$ is a harmonic function defined outside a compact set, are obtained using the harmonic measures.

1. Introduction. Let $\Omega$ be a clamped plate with an external load density $f(x)$, $x \in \Omega$. If $\partial \Omega$ is regular for the Dirichlet problem, the solution $u(x)$ corresponding to the conditions of elasticity given by $\Delta^2 u(x) = f(x)$ on $\Omega$, $u = 0 = \partial u/\partial n$ on $\partial \Omega$, has a representation (Niculesco [16, p. 40]) $u(x) = \int_{\Omega} G_2(x, y) f(y) \, dy$. However, the calculation of $G_2(x, y)$ corresponding to a given $u$ is not simple.

This note, among other results, shows that in the particular case of $\Omega$ being in the form of a star domain with centre 0 and the equation $\Delta^2 u = f$ being reduced to the condition $\Delta u(x) = G(x) + |x|^{-1} H(x)$ where $G$ and $H$ are harmonic functions on $\Omega$, continuous on $\Omega$, we can express $u$ as $u(x) = |x|^2 h_2(x) + |x|h_1(x) + h_0(x)$ where $h_i(x)$ are harmonic functions on $\Omega$. This suggests the study of the properties of functions of the form $u(x) = \sum_{i=0}^m |x|^i h_i(x)$ in a star domain $\Omega$ with centre 0 in $\mathbb{R}^n$, $n \geq 2$, which include the polyharmonic functions of finite order on $\Omega$.

In another context, when $u(x) = \sum_{i=0}^m |x|^i h_i(x)$ is defined on the whole of $\mathbb{R}^n$, $n \geq 2$, where each $h_i$ is harmonic on $\mathbb{R}^n$, Nakai and Tada ([15, Theorem 3]) give a necessary and sufficient condition on $u$ so that each $h_i$ is a harmonic polynomial, by using the Fourier expansion method. In this note we obtain some complementary results for such functions $u$, by using harmonic measures.

2. Functions in the class $H^*(\mathbb{R}^n)$. We begin with the following definition.

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DEFINITION 2.1. Let $\Omega$ be a star domain with centre 0 in $\mathbb{R}^n$, $n \geq 2$. A continuous function $u(x)$ defined on $\Omega$ is said to be in the class $H^m(\Omega)$ if it is of the form $u(x) = \sum_{i=0}^{m} |x|^i h_i(x)$ where $h_i$, $0 \leq i \leq m$, are harmonic on $\Omega$. Write $H^*(\Omega) = \bigcup_{m \geq 0} H^m(\Omega)$.

We remark that if $u(x)$ is a polyharmonic function of order $m$ on $\Omega$ (that is, $u \in C^{2m}(\Omega)$ and $\Delta^m u = 0$), then the Almansi representation of $u$ (see Aronszajn et al. [8, Proposition 1.3]) is of the form $u(x) = \sum_{i=0}^{m-1} |x|^{2i} h_i(x)$ where $h_i(x)$ are uniquely determined harmonic functions on $\Omega$. Hence $u \in H^{2m-2}(\Omega)$, $H^0(\Omega)$ being the class of all harmonic functions on $\Omega$.

In this section, we obtain some Liouville-type theorems for functions in $H^*(\mathbb{R}^n)$. For a given continuous function $f$ on $\mathbb{R}^n$, let $D_r f$ stand for the Dirichlet solution in $|x| < r$ with boundary value $f(x)$ on $|x| = r$.

THEOREM 2.2. Let $u \in H^*(\mathbb{R}^n)$. Let $a_j$ be a sequence of real numbers increasing to $\infty$. Let $\omega$ be a nonempty open set and $e$ be a polar set in $\mathbb{R}^n$. If

$$\liminf_{j \to \infty} \frac{D_{a_j} u(z)}{a_j} = 0$$

for every $z \in \omega \setminus e$, then $u$ is harmonic on $\mathbb{R}^n$.

Proof. Let $u(x) = \sum_{i=0}^{m} |x|^i h_i(x)$. Then, for fixed $z \in \omega \setminus e$ and large $j$, $D_{a_j} u(z) = \sum_{i=0}^{m} a_j^i h_i(z)$. Hence the assumption on $D_{a_j} u$ as $j \to \infty$ implies that $h_i(z) = 0$ for $1 \leq i \leq m$. Since $z$ is arbitrary in $\omega \setminus e$, $h_i$ vanishes on $\omega \setminus e$, which implies that $h_i \equiv 0$ for $1 \leq i \leq m$. Consequently, $u(x) = h_0(x)$ is harmonic on $\mathbb{R}^n$.

COROLLARY 1 (see [15, Proposition 1]). Let $u = \sum_{i=0}^{m} |x|^i h_i(x) \in H^*(\mathbb{R}^n)$. If $u(x) \to 0$ when $|x| \to \infty$, in particular if $u \equiv 0$, then each $h_i \equiv 0$.

Proof. As in the proof of Theorem 2.2 we can show that $h_i \equiv 0$ for $1 \leq i \leq m$ and $u = h_0$ on $\mathbb{R}^n$. Since $u \to 0$ at infinity, $h_0 \equiv 0$.

COROLLARY 2. Let $u \in H^*(\mathbb{R}^n)$. Let $M(r, |u|)$ denote the mean value of $|u(x)|$ on $|x| = r$. Suppose $M(r, |u|) = o(r)$ as $r \to \infty$. Then $u$ is a constant.

Proof. Write $D_r u(z) = \int u(x) d\sigma^r_z(x)$ where $\sigma^r_z$ is the harmonic measure on $|x| = r$. Recall

$$d\sigma^r_z(x) = \frac{r^{n-2} r^2 - |z|^2}{|x-z|^n} d\sigma_r(x)$$

where $d\sigma_r(x)$ is the measure on $|x| = r$, invariant with respect to the rotations of $|x| = r$ and such that $\sigma_r(|x| = r) = 1$. Suppose now $|z| \leq 1 < r = |x|$. Then

$$\frac{r^2 - |z|^2}{|x-z|^n} \leq \frac{r^2}{(r-1)^n}$$
so that
\[ |D_r u(z)| \leq \int |u(x)| \frac{r^n}{(r-1)^n} d\sigma_r(x). \]
Consequently, for large \( r \), \( |D_r u(z)| \leq 2M(r, |u|) \) so that \( \lim_{r \to \infty} D_r u(z)/r = 0 \) for \( |z| \leq 1 \). Then by the above theorem, we conclude that \( u \) is harmonic on \( \mathbb{R}^n \); this, together with the condition that \( M(r, |u|) = o(r) \), implies that \( u \) is a constant. (See [6, Corollary 3.3] for an indication of different proofs of this result.)

**Corollary 3.** Let \( u \in H^*(\mathbb{R}^n) \) be such that for a superharmonic function \( s \) on \( \mathbb{R}^n \), \( |u| \leq s \). Then \( u \) is a constant.

**Proof.** We write \( s = p + c \) where \( p \geq 0 \) is a potential on \( \mathbb{R}^n \) (\( p \equiv 0 \) is the only potential on \( \mathbb{R}^2 \)) and \( c \geq 0 \) is a constant. Since \( M(r, p) \to 0 \) as \( r \to \infty \), \( M(r, |u|) = o(r) \). Hence by Corollary 2, \( u \) is harmonic on \( \mathbb{R}^n \) and \( |u| \leq p + c \), which implies that \( |u| \leq c \) on \( \mathbb{R}^n \) so that \( u \) is a constant.

**Remark.** The above Corollary 3 expresses the classical Liouville–Picard theorem for the class \( H^*(\mathbb{R}^n) \). In fact, in its standard form this theorem states that if \( u \in H^0(\mathbb{R}^n) \) and \( u \geq 0 \), then it is a constant; this can be generalized as follows: If \( u \in H^*(\mathbb{R}^n) \) is a positive superharmonic function, then \( u \) is a constant. This generalization is a consequence of the above Corollary 3. However, if we leave out the condition that \( u \) is superharmonic, we have from Armitage [7] or Futamura–Kishi–Mizuta [13] or Nakai–Tada [15] the following: If \( u \in H^m(\mathbb{R}^n) \) is positive, then \( u \) is a polynomial of degree at most \( m \). More generally, let \( u(x) = \sum_{i=0}^m |x|^i h_i(x) \) and suppose \( \lim_{|x| \to \infty} u(x)/|x|^s \geq 0 \) for some \( s > m \). Then for each \( i \), \( h_i \) is a harmonic polynomial of degree less than \( s - i \). For as in [6, Lemma 2.1] we can find a locally integrable function \( \varphi(x) \) on \( \mathbb{R}^n \) such that \( u(x) \geq \varphi(x) \) outside a compact set \( K \) and \( M(r, |\varphi|) = o(r^s) \) as \( r \to \infty \). Since \( u \geq -|\varphi| \) on \( \mathbb{R}^n \setminus K \), \( M(r, u^-) = o(r^s) \) as \( r \to \infty \); also \( M(r, u) = O(r^m) \). Hence \( M(r, |u|) = o(r^s) \). Consequently, each \( h_i \) is a harmonic polynomial of degree less than \( s - i \). In particular, if \( \lim_{|x| \to \infty} u(x)/|x| \geq 0 \) then \( u \) is a constant.

3. **\( H^* \) functions on a star domain.** Let \( \Omega \) denote a star domain in \( \mathbb{R}^n \), \( n \geq 2 \), with centre 0. Then, as in Definition 2.1, \( H^m(\Omega) \) denotes the class of functions of the form \( u(x) = \sum_{i=0}^m |x|^i h_i(x) \) where \( h_i \) are harmonic on \( \Omega \). We shall use the operator \( \Delta^p \), with integer \( p \geq 1 \), in the sense of distributions.

**Lemma 3.1.** If \( u \in C^{2p}(\Omega) \), let \( f(x) = |x|^{\alpha} u(x) \) where \( \alpha > 2p - n \). Then \( \Delta^p(|x|^\alpha u(x)) \) is locally integrable on \( \Omega \).

**Proof.** Since \( (\partial/\partial x)^l |x|^\alpha \) is locally integrable on \( \mathbb{R}^n \) when \( \alpha > |l| - n \), the lemma is evident. We need the following expression of \( \Delta^p f(x) \) for later use.
For $|x| = r > 0$, let $g = \Delta^p f$ in the classical sense.

Since
\[ \Delta f = \alpha(\alpha + n - 2)r^{\alpha - 2}u + 2\alpha r^{\alpha - 2} \sum x_i \frac{\partial u}{\partial x_i} + r^\alpha \Delta u, \]
we have $\Delta f = r^{\alpha - 2}u_1$ where $u_1 \in C^{2p-2}$ (note that $u_1$ is harmonic if $u$ is).
Proceeding thus, we find $\Delta^j f = r^{\alpha - 2j}u_j$ for $1 \leq j \leq p$, where $u_j \in C^{2p-2j}$.
Let $g(x) = |x|^\alpha - 2pu_p(x)$ for $|x| > 0$ and $g(0) = \lim \sup_{x \to 0} g(x)$. Then $g(x)$ is u.s.c. on $\Omega$ and since $\alpha - 2p + n > 0$, $g(x)$ is locally integrable on $\Omega$ and $\Delta^p f = g$ on $\Omega$ in the sense of distributions.

**Remark.** Let $h$ be harmonic on $\Omega$. Then from the above proof (replacing $u$ by $h$) we see that if $\alpha > 2p - n$, then $\Delta^p(|x|^\alpha h(x)) = |x|^{\alpha - 2p}H(x)$ where $H(x)$ is harmonic on $\Omega$.

**Proposition 3.2.** Let $u \in H^*(\Omega)$ and $2p \leq n$. Then $\Delta^p u$ is locally integrable on $\Omega$.

**Proof.** Let $u = \sum_{i=0}^n |x|^ih_i(x)$. Then for any $i \geq 1$, $i > 2p - n$ so that $\Delta^p(|x|^ih_i(x))$ is locally integrable on $\Omega$; for $i = 0$, $\Delta(h_0) = 0$. Hence $\Delta^p u$ is locally integrable on $\Omega$.

**Consequence.** The above proposition, in particular, states that if $u \in H^*(\mathbb{R}^n)$, $n \geq 2$, then $\Delta u$ is locally integrable on $\mathbb{R}^n$. This leads to an integral representation of $u$ in $\mathbb{R}^n$. For that, recall that given any positive Radon measure $\mu$ on an open set $\omega$ in $\mathbb{R}^n$, $n \geq 2$, Brelot [12] shows that a subharmonic function $s$ can be constructed on $\omega$ with associated measure $\mu$ in the local Riesz representation.

Now, for $u \in H^*(\mathbb{R}^n)$, $n \geq 2$, since $\Delta u$ is locally integrable, $d\lambda(x) = \Delta u dx$ can be treated as defining the difference of two positive Radon measures on $\mathbb{R}^n$. Hence $u$ is the difference of two subharmonic functions on $\mathbb{R}^n$. Then we can define the order of $u$ and the order of $\lambda$ as in Arsove [9] (see also [4]). If the order of $\lambda$ is finite, a correspondingly modified form of the logarithmic kernel (if $n = 2$) or the Newtonian kernel (if $n \geq 3$) can be used to represent $u$ as an integral up to an additive harmonic function which is a harmonic polynomial if the order of $u$ is finite (see Arsove [9], and [4, Theorems 11 and 12]; see also Mizuta [14]).

**Lemma 3.3.** Let $H$ be a harmonic function on $\Omega$. If $n + \alpha - 2 > 0$ and if $\alpha + 2i \neq 0$ for $i$, $0 \leq i \leq p - 1$, then there exists a harmonic function $h$ on $\Omega$ such that $\Delta^p(|x|^\alpha h(x)) = |x|^\alpha - 2H(x)$.

**Proof.** We prove the lemma for the case $p = 1$ by adapting the method given in Aronszajn et al. [8, p. 5]. The general case follows by induction. Suppose a harmonic function $h$ exists on $\Omega$ such that $\Delta(|x|^\alpha h(x)) = |x|^\alpha - 2H(x)$.
Then, treating $h$ as a function of $r$, we should have
\[ \Delta(r^\alpha h) = \alpha(n + \alpha - 2)r^{\alpha-2}h + 2\alpha r^{\alpha-1} \frac{\partial h}{\partial r}. \]

Then
\[ \alpha(n + \alpha - 2)h + 2\alpha r \frac{\partial h}{\partial r} = H \quad \text{on } \Omega. \]

This can be written as
\[ \frac{d}{dr}[r^{(n+\alpha-2)/2}h] = \frac{H}{2\alpha} r^{(n+\alpha-4)/2} \]
outside $0$. Since at the origin, $r^{(n+\alpha-2)/2}h = 0$, we should have
\[ r^{(n+\alpha-2)/2}h(r, w) = \int_0^r \frac{1}{2\alpha} \varrho^{(n+\alpha-4)/2} H(\varrho, w) d\varrho, \]
where $x = (r, w)$ is represented by the spherical polar coordinates. Set $\varrho = tr$. Then
\[ h(x) = \frac{1}{2\alpha} \int_0^1 t^{(n+\alpha-4)/2} H(tx) \, dt; \]
here $\Delta h = 0$ since $H$ is harmonic. Consequently, given the harmonic function $H$ on $\Omega$, if we define $h(x)$ by the formula above, then $h(x)$ is harmonic on $\Omega$, satisfying the condition $\Delta(|x|^{\alpha}h(x)) = |x|^{\alpha-2}H(x)$.

**Theorem 3.4.** A continuous function $u$ on $\Omega$ is in $H^m(\Omega)$ if and only if for any integer $p$, $2 \leq 2p \leq n$, there exists a function $v \in H^{m-1}(\Omega)$ such that $\Delta^p u(x) = |x|^{1-2p}v(x)$ in the sense of distributions.

**Proof.** (1) Let $u = \sum_{i=0}^m |x|^i h_i \in H^m(\Omega)$. We shall now use the Remark following Lemma 3.1 to calculate $\Delta^p u(x)$.

If $p = 1$, then $\Delta u = \sum_{i=0}^{m-2} |x|^i H_i + |x|^{-1} v_1$ where $H_i$ ($0 \leq i \leq m-2$) and $v_1$ are harmonic functions on $\Omega$. Hence $\Delta u = |x|^{-1} \sum_{i=0}^{m-2} |x|^{i+1} H_i + v_1 = |x|^{-1} s_1(x)$ where $s_1 \in H^{m-1}(\Omega)$.

If $p = 2$, then $n \geq 4$ and in this case $\Delta(|x|^{-1} v_1) = |x|^{-3} v_2$ where $v_2$ is harmonic on $\Omega$. This leads to the equation $\Delta^2 u = \Delta(\Delta u) = \sum_{i=0}^{m-4} |x|^i H'_i + |x|^{-1} v + |x|^{-3} v_2$ on $\Omega$, where $H'_i$ ($0 \leq i \leq m-4$), $v$ and $v_2$ are harmonic on $\Omega$. This simplifies to the form $\Delta^2 u = |x|^{-3} s_2(x)$ where $s_2 \in H^{m-1}(\Omega)$.

This process by induction leads to the result that if $2 \leq 2p \leq n$, then $\Delta^p u = |x|^{-(2p-1)} s_p(x)$ where $s_p \in H^{m-1}(\Omega)$.

(2) Conversely, suppose that $u$ is a continuous function on $\Omega$ such that for any integer $p$, $2 \leq 2p \leq n$, we have $\Delta^p u = |x|^{-(2p-1)} v$ where $v \in H^{m-1}(\Omega)$. Then, in particular for $p = 1$, by Lemma 3.3 we have
\[ \Delta u = |x|^{-1} v = |x|^{-1} \sum_{i=0}^{m-1} |x|^i H_i = \sum_{i=0}^{m-1} |x|^{i-1} H_i = \sum \Delta(|x|^{i+1} h_i). \]
Hence \( u = (\sum_{i=0}^{n-1} |x|^{i+1} h_i) + (\text{a harmonic function on } \Omega) \); in other words, \( u \in H^m(\Omega) \).

**Corollary 1.** Let \( u = \sum_{i=0}^{m} |x|^i h_i(x) \in H^m(\Omega) \). Suppose \( u \) is harmonic on a neighbourhood of a point in \( \Omega \). Then \( h_i \equiv 0 \) for \( 1 \leq i \leq m \). In particular, if \( u \equiv 0 \) on a nonempty open set in \( \Omega \), then \( h_i \equiv 0 \) for all \( i, 0 \leq i \leq m \).

**Proof.** Let \( u \) be harmonic on a nonempty open set \( \omega \). Since \( u \in H^m(\Omega) \), there exists \( v_{m-1} \in H^{m-1}(\Omega) \) such that \( \Delta u = |x|^{-1} v_{m-1}(x) \) on \( \Omega \). This implies \( v_{m-1} = 0 \) on \( \omega \). Now again by Theorem 3.4, there exists \( v_{m-2} \in H^{m-2}(\Omega) \) such that \( \Delta v_{m-1} = |x|^{-1} v_{m-2}(x) \), which implies that \( v_{m-2} = 0 \) on \( \omega \). Proceeding thus, we obtain \( v_i \in H^i(\Omega), 0 \leq i \leq m-1 \), such that \( \Delta v_{i+1} = |x|^{-1} v_i \) on \( \Omega \) and \( v_i = 0 \) on \( \omega \) (taking \( v_m = u \)).

Since \( v_0 \) is harmonic on \( \Omega \) and \( v_0 = 0 \) on \( \omega \), \( v_0 \equiv 0 \) on \( \Omega \). This implies \( v_1 \) is harmonic on \( \Omega \) and since \( v_1 = 0 \) on \( \omega \), we have \( v_1 \equiv 0 \) on \( \Omega \). Thus proceeding, we remark that \( v_i \equiv 0 \) on \( \Omega \) for \( 0 \leq i \leq m-1 \) so that \( \Delta u = 0 \) on \( \Omega \); that is, \( u \) is harmonic on \( \Omega \).

Then \( \sum_{i=1}^{m} |x|^i h_i(x) = u(x) - h_0(x) \) is harmonic on \( \Omega \). Choose \( a \) such that \( \{ x : |x| < a \} \subset \Omega \). Fix \( z \in \Omega \) so that \( |z| < r < a \). Let \( \varphi_z^r \) be the harmonic measure on \( |x| = r \). Then

\[
\sum_{i=1}^{m} |x|^i h_i(x) d\varphi_z^r(x) = \int (u - h_0) d\varphi_z^r,
\]

which implies that

\[
\sum_{i=1}^{m} r^i h_i(z) = u(z) - h_0(z).
\]

Since \( r \) is arbitrary in the interval \( (|z|, a) \), we have \( h_i(z) = 0 \) for \( 1 \leq i \leq m \) and \( u(z) - h_0(z) = 0 \). Since \( h_i \) and \( u \) are harmonic on \( \Omega \) and \( z \) is arbitrary except for the condition \( |z| < a \), we conclude \( h_i \equiv 0 \) on \( \Omega \) for \( 1 \leq i \leq m \) and \( u \equiv h_0 \).

**Corollary 2.** Let \( u \in H^m(\Omega) \) and \( 2 \leq 2p \leq n \). Suppose \( u \) is \( p \)-harmonic \( (\Delta^p u = 0) \) on a neighbourhood of a point in \( \Omega \). Then \( u \) is \( p \)-harmonic on \( \Omega \).

**Proof.** Suppose \( \Delta^p u = 0 \) on a nonempty open set \( \omega \). By Theorem 3.4, there exists a function \( v \in H^{m-1}(\Omega) \) such that \( \Delta^p u = |x|^{1-2pv} \) on \( \Omega \). Hence \( v = 0 \) on \( \omega \) and consequently, by Corollary 1, \( v \equiv 0 \) on \( \Omega \), which means that \( u \) is \( p \)-harmonic on \( \Omega \).

**Remark.** We thank the referee for pointing out that in the proofs of the above two corollaries, one can use the real analyticity of \( u \) outside the origin, without having recourse to Theorem 3.4.
Corollary 3. Let \( u \in H^m(S) \), where \( S = \{ x : |x| < 1 \} \) in \( \mathbb{R}^n \), be such that \( \lim \inf_{r \to 1} M(r, |u|) = 0 \). Then \( u(x) = (1-|x|)v(x) \) where \( v \in H^{m-1}(S) \).

Proof. Let \( u(x) = \sum_{i=0}^{m} |x|^i h_i(x) \in H^m(S) \). Notice that by the above Corollary 1, there exist uniquely determined harmonic functions \( u_i \) on \( S \) such that
\[
u(x) = \sum_{i=0}^{m} (1-|x|)^i u_i(x) \quad \text{on } S.
\]
In particular,
\[
h_0(x) = \sum_{i=0}^{m} u_i(x).
\]
Fix \( z \) with \( |z| \leq 1/4 \). For \( |x| = r \), \( 1/2 \leq r < 1 \), let \( d\mathcal{H}^r_x(z) \) denote the harmonic measure on \( |x| = r \). Integrate \( u(x) \) with respect to \( d\mathcal{H}^r_x(z) \) to get
\[
| (1-r)^m u_m(z) + \ldots + (1-r)u_1(z) + u_0(z) | = \left| \int u(z) d\mathcal{H}^r_x(x) \right| \leq CM(r, |u|)
\]
for some constant \( C \) since \( |z| \leq 1/4 \) and \( 1/2 \leq r < 1 \) (see the proof of Corollary 2 to Theorem 2.2).

Let \( r \to 1 \). Since \( \lim \inf_{r \to 1} M(r, |u|) = 0 \) by hypothesis, we obtain \( u_0(z) = 0 \) for \( |z| \leq 1/4 \). Hence \( u_0 \equiv 0 \). Consequently, \( u(x) = (1-|x|)v(x) \) where
\[
v(x) = \sum_{i=0}^{m-1} (1-|x|)^i u_{i+1}(x) \in H^{m-1}(S).
\]

Note. The above corollary easily includes the result: If \( u \) is \( p \)-harmonic on \( S \) such that \( \lim \inf_{r \to 1} M(r, |u|) = 0 \), then \( u(x) = (1-|x|^2)v(x) \) where \( v \) is polyharmonic on \( S \) of order \( \leq m-1 \). This in itself is a generalization of a result of Abkar and Hedenmalm [1, pp. 321–322] proved in the complex plane using the Fourier series: If \( u(z) \) is biharmonic on the unit disc in the complex plane and if \( M(r, |u|) = O(1-r) \) as \( r \to 1 \), then \( u(z) = (1-|z|^2)h(z) \) for a harmonic function on the unit disc.

4. \( H^* \) functions defined near infinity. It is not surprising that in many respects, the functions in \( H^{2m-2}(\mathbb{R}^n) \) behave near infinity like the \( m \)-harmonic functions (that is, the solutions of \( \Delta^m u = 0 \)) on \( \mathbb{R}^n \). In this section, we study a class of continuous functions on \( \mathbb{R}^n \) which are associated near infinity with the functions in \( H^{2m-2}(\mathbb{R}^n) \) and the fundamental solution of \( \Delta^m \) on \( \mathbb{R}^n \). This class contains a significant collection of functions having some nice regularity properties at infinity.

For \( m \geq 1, n \geq 2 \), let \( c_m^n \) denote the fundamental solution of \( \Delta^m \) on \( \mathbb{R}^n \). We recall that
\[ e_m^n = \begin{cases} |x|^{2m-n} & \text{if } 2m < n \text{ or } 2m - n \text{ is odd } > 0, \\ |x|^{2m-n} \log |x| & \text{if } 2m - n \text{ is even } \geq 0. \end{cases} \]

**Definition 4.1.** A continuous function \( v \) defined outside a compact set in \( \mathbb{R}^n \) is said to be in the class \( H^{2m-2}(\mathbb{R}^n) \) if there exists \( u \in H^{2m-2}(\mathbb{R}^n) \) such that \( u - v = O(e_m^n) \) near infinity.

**Proposition 4.2.** Let \( h_i \) \( (0 \leq i \leq m - 2) \) be arbitrary harmonic functions defined outside a compact set in \( \mathbb{R}^n \). Then
\[ v = \sum_{i=0}^{2m-2} |x|^i h_i(x) \in H^{2m-2}(\mathbb{R}^n). \]

**Proof.** Recall that (see [2] or Axler et al. [10]) given a harmonic function \( h \) outside a compact set in \( \mathbb{R}^n \), there exists a harmonic function \( H \) on \( \mathbb{R}^n \) and a constant \( \alpha \) such that outside a compact set,
\[ h(x) = \begin{cases} H(x) + \alpha \log |x| + g(x) & \text{if } n = 2, \\ H(x) + g(x) & \text{if } n \geq 3, \end{cases} \]
where \( g(x) \) is a harmonic function satisfying \( g(x) = O(|x|^{2-n}) \) near infinity. Hence for each \( i, 0 \leq i \leq 2m - 2 \), there exists a harmonic function \( H_i \) on \( \mathbb{R}^n \) such that \( |h_i - H_i| \leq A|x|^{2-n} \) if \( n \geq 3 \) and \( |h_i - H_i - \alpha_i \log |x|| \leq A \) if \( n = 2 \), outside a compact set. Let \( u(x) = \sum_{i=0}^{2m-2} |x|^i H_i(x) \). Then \( u \in H^{2m-2}(\mathbb{R}^n) \) and near infinity \( u - v = O(e_m^n) \). Hence \( v \in H^{2m-2}(\mathbb{R}^n) \).

**Corollary.** Let \( k \) be a compact set in a star domain \( \Omega \) with centre 0. Suppose \( u = \sum_{i=0}^{2m-2} |x|^i h_i(x) \) where \( h_i \) are harmonic on \( \Omega \setminus k \). Then there exist \( t \in H^{2m-2}(\Omega) \) and \( s \in H^{2m-2}(\mathbb{R}^n) \cap C(\mathbb{R}^n \setminus k), s = O(e_m^n) \) near infinity, such that \( u = s - t \) on \( \Omega \setminus k \).

**Proof.** We know that for each \( i \), there exist \( s_i \in H^0(\mathbb{R}^n \setminus k) \) and \( t_i \in H^0(\Omega) \) such that \( h_i = s_i - t_i \) on \( \Omega \setminus k \) (Laurent decomposition for harmonic functions, see [2] or Axler et al. [10, pp. 171–175]).

Let now
\[ s^1(x) = \sum_{i=0}^{2m-2} |x|^i s_i(x) \quad \text{and} \quad t^1(x) = \sum_{i=0}^{2m-2} |x|^i t_i(x). \]
Then \( u = s^1 - t^1 \) on \( \Omega \setminus k \) where by the above proposition \( s^1(x) \) is in the class \( H^{2m-2}(\mathbb{R}^n) \), and \( t^1(x) \in H^{2m-2}(\Omega) \). Since \( s^1 \in H^{2m-2}(\mathbb{R}^n) \), there exists \( v \in H^{2m-2}(\mathbb{R}^n) \) such that \( s^1 - v = O(e_m^n) \) near infinity. Write now \( s = s^1 - v \) and \( t = t^1 - v \) to obtain the decomposition \( u = s - t \) on \( \Omega \setminus k \) as stated in the Corollary.

**Proposition 4.3.** If \( v \) is an \( m \)-harmonic function defined outside a compact set in \( \mathbb{R}^n \), then \( v \in H^{2m-2}(\mathbb{R}^n) \).
Proof. For \( m = 1 \), the representation for a harmonic function \( h \) outside a compact set (given in the proof of Proposition 4.2) leads to the result that \( h \in H^1_\infty(\mathbb{R}^n) \).

Let us take the case \( m = 2 \). In this case, we start with the representation for a biharmonic function \( b \) defined near infinity in the following form (see [11, p. 19]):

\[
b(x) = \begin{cases} 
(\alpha + \alpha_1 x_1 + \alpha_2 x_2) \log |x| + \beta |x|^2 \log |x| + B(x) + u(x) & \text{if } n = 2, \\
\beta |x| + B(x) + u(x) & \text{if } n = 3, \\
\beta \log |x| + B(x) + u(x) & \text{if } n = 4, \\
B(x) + u(x) & \text{if } n \geq 5,
\end{cases}
\]

where \( B(x) \) is biharmonic on \( \mathbb{R}^n \) and \( u(x) \) is biharmonic bounded near infinity. In the case of \( n \geq 5 \), we can show that \( |u(x)| \leq A|x|^{4-n} \) by specializing the proof (1) \( \Rightarrow \) (2) of [11, Theorem 10]. Consequently, since \( B \in H^2(\mathbb{R}^n) \) and since \( b - B = O(e_3^0) \) near infinity, we have \( b \in H^2_\infty(\mathbb{R}^n) \).

Finally, for \( m > 2 \), we have a similar representation for an \( m \)-harmonic function defined outside a compact set (details given in a forthcoming paper [5]) which can be used to prove the proposition. The result referred to here is as follows: Let \( u \) be \( m \)-harmonic outside a compact set in \( \mathbb{R}^n \). Then there exists an \( m \)-harmonic function \( v \) on \( \mathbb{R}^n \) such that \( u - v = O(e_3^m) \) as \( |x| \to \infty \).

**Theorem 4.4.** Let \( v \in H^2m-2(\mathbb{R}^n) \), \( n > 2m \geq 2 \). Suppose either one of the following conditions is satisfied:

1. There exists a superharmonic function \( s \) outside a compact set such that \( |v| \leq s \) near infinity.
2. \( \lim_{|x| \to \infty} v(x)/|x| = 0 \).

Then \( \lim_{|x| \to \infty} v(x) \) exists and is finite.

Proof. (1) Suppose \( |v| \leq s \) near infinity. Since \( n \geq 3 \), there exists a superharmonic function \( S \) on \( \mathbb{R}^n \) such that \( S - s \) is bounded near infinity (see [3]). Hence we can as well assume that \( s \) is a superharmonic function defined on the whole of \( \mathbb{R}^n \) and \( |v| \leq s \) near infinity.

Since \( v \in H^2m-2(\mathbb{R}^n) \), by definition there exists \( u \in H^2m-2(\mathbb{R}^n) \) such that \( |u - v| \leq A|x|^{2m-n} \) near infinity. Hence \( |u| \leq s + A|x|^{2m-n} \leq s + A \) near infinity. Then by Corollary 3 in Section 2, \( u \) is a constant \( \alpha \). Consequently, \( \lim_{x \to \infty} v(x) = \alpha \).

(2) Suppose now \( \lim_{|x| \to \infty} v(x)/|x| = 0 \). Since \( v \in H^2m-2(\mathbb{R}^n) \), there exists \( u \in H^*(\mathbb{R}^n) \) such that \( |u - v| \leq A|x|^{2m-n} \) near infinity. This implies that \( \lim_{|x| \to \infty} u(x)/|x| = 0 \). Hence, by Corollary 2 in Section 2, \( u \) is a constant \( \alpha \). Consequently \( \lim_{|x| \to \infty} v(x) = \alpha \).

Remark. Since every bounded continuous function \( v \) is in \( H^2m-2 \) if \( 2 \leq n \leq 2m \), the above theorem is not valid if \( n \leq 2m \).
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