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The set of recurrent points of a continuous self-map on compact metric spaces and strong chaos

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Abstract. We discuss the existence of an uncountable strongly chaotic set of a continuous self-map on a compact metric space. It is proved that if a continuous self-map on a compact metric space has a regular shift invariant set then it has an uncountable strongly chaotic set in which each point is recurrent, but is not almost periodic.

1. Introduction. Throughout this paper, X will denote a compact metric space with metric d, and I is the closed interval $[0,1]$.

For a continuous map $f : X \to X$, we will denote the set of almost periodic points and of recurrent points of f by $A(f)$ and $R(f)$ respectively, with the usual definitions; f^n will denote the n-fold iterate of f.

For x, y in X, any real number t and positive integer n, let

$$\xi_n(f,x,y,t) = \#\{i \mid d(f^i(x), f^i(y)) < t,\ 1 \le i \le n\},$$

where we use $\#(\cdot)$ to denote the cardinality of a set. Let

$$F(f,x,y,t) = \liminf_{n\to\infty} \frac{1}{n}\,\xi_n(f,x,y,t), \qquad F^*(f,x,y,t) = \limsup_{n\to\infty} \frac{1}{n}\,\xi_n(f,x,y,t).$$

Definition 1.1. Call $x, y \in X$ a *pair of points displaying strong chaos* if

(1) $F(f,x,y,t) = 0$ for some $t > 0$,

(2) $F^*(f,x,y,t) = 1$ for any $t > 0$.

Definition 1.2. f is said to *display strong chaos* if there exists an uncountable set $D \subset X$ such that any two different points in D display strong chaos.

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For a continuous map $f : I \to I$, Schweizer and Smítal [8] have proved:

(C_1) If f has zero topological entropy, then no pair of points can form a strongly chaotic set.

(C_2) If f has positive entropy, then there exists an uncountable strongly chaotic set in which each member is an ω-limit point of f.

One may pose the following questions:

(Q_1) Is (C_1) still true for a continuous map of any compact metric space X?

(Q_2) Is there an uncountable strongly chaotic set in which each member is a recurrent point of f on compact metric spaces?

A negative answer to (Q_1) has been given in [6], where a minimal strongly chaotic sub-shift having zero topological entropy was constructed.

In this paper, a positive answer to (Q_2) is given.

In fact, we will prove

MAIN THEOREM. *Let $f : X \to X$ be continuous. If f has a regular shift invariant set, then it has an uncountable strongly chaotic set in which each point is recurrent, but is not almost periodic.*

2. Basic definitions and preparations. Let $S = \{0, 1\}$, $\Sigma = \{x = x_1x_2\ldots \mid x_i \in S,\ i = 1, 2, \ldots\}$ and define $\varrho : \Sigma \times \Sigma \to \mathbb{R}$ as follows: for any $x, y \in \Sigma$, if $x = x_1x_2\ldots$ and $y = y_1y_2\ldots$, then

$$\varrho(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1/2^k & \text{if } x \neq y \text{ and } k = \min\{n \mid x_n \neq y_n\} - 1. \end{cases}$$

It is not difficult to check that ϱ is a metric on Σ. The space (Σ, ϱ) is compact and called the *one-sided symbolic space* on two symbols.

Define $\sigma : \Sigma \to \Sigma$ by $\sigma(x_1x_2\ldots) = x_2x_3\ldots$ for any $x = x_1x_2\ldots \in \Sigma$. Then σ is continuous and called the *shift* on Σ. Call A a *tuple* (over $S = \{0, 1\}$) if it is a finite sequence of elements in S. If $A = a_1a_2\ldots a_m$ where $a_i \in S$, $1 \le i \le m$, then m is called the *length* of A, denoted by $|A| = m$.

For an arbitrary tuple $B = b_1b_2\ldots b_n$, the set $[B] = \{x = x_1x_2\ldots \in \Sigma,\ x_i = b_i,\ 1 \le i \le n\}$ is called the *cylinder* generated by B. For any $n \ge 1$, let

$$\mathcal{B}_n = \{\, [b_1\ldots b_n] \mid b_i = 0 \text{ or } 1, 1 \le i \le n\}.$$

Then the collection $\bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a subalgebra which generates the σ-algebra of Borel subsets of Σ. Let $h : X \to \Sigma$ be a continuous map. We use $I_{[B]}$ to denote $h^{-1}[B]$ for any $[B] \in \mathcal{B}_n$.

DEFINITION 2.1. Let $f : X \to X$ be continuous. A compact set $\Lambda \subset X$ is said to be a *regular shift invariant set* for f if:

(1) $f(\Lambda) \subset \Lambda$,

(2) there exists a continuous surjection $h : \Lambda \to \Sigma$ satisfying

(a) $h \circ f|_\Lambda = \sigma \circ h$,

(b) there exists an $M > 0$ such that $\sum_{[B]\in\mathcal{B}_n} \operatorname{diam} I_{[B]} \le M$ for any $n \ge 1$.

Definition 2.2. $\mathcal{B}(X)$ is the σ-algebra of Borel subsets of X. A probability measure μ on $(X, \mathcal{B}(X))$ is an *invariant measure* for f if $\mu(f^{-1}(B)) = \mu(B)$ for any $B \in \mathcal{B}(X)$. We denote the set of all invariant measures for f by $M(X, f)$.

$\mu \in M(X, f)$ is *ergodic* (f can then also be regarded ergodic) if the only members B of $\mathcal{B}(X)$ with $f^{-1}(B) = B$ satisfy $\mu(B) = 0$ or $\mu(B) = 1$.

If μ is a unique member of $M(X, f)$, it must be ergodic [9]; we then say that f is *uniquely ergodic*.

Lemma 2.1 (see [12]). *Let $f : X \to X$ and $g : Y \to Y$ be continuous, where X and Y are compact metric spaces. If there exists a continuous surjection $h : X \to Y$ such that $g \circ h = h \circ f$, then*

(1) $h(A(f)) = A(g)$,

(2) $h(R(f)) = R(g)$.

Lemma 2.2 (see [10] or [11]). *There exists an uncountable set $\mathcal{T}$ on the one-sided symbolic space satisfying*

(1) $\mathcal{T} \subset R(\sigma) - A(\sigma)$,

(2) $\sigma|_\mathcal{T}$ *is strongly chaotic*,

(3) $\sigma|_\mathcal{T}$ *is uniquely ergodic.*

Lemma 2.3. *Let $\sigma : \Sigma \to \Sigma$ be continuous. If μ is the only invariant probability measure for $\sigma|_{R(\sigma)-A(\sigma)}$, then $\mu(\{x\}) = 0$ for any $x \in R(\sigma) - A(\sigma)$.*

Proof. Let $x \in R(\sigma) - A(\sigma)$. We first claim that $\{x\}, \sigma^{-1}(x), \sigma^{-2}(x), \ldots$ are pairwise disjoint. Assume the claim to be false; then $\sigma^{-m}(x) \cap \sigma^{-n}(x) \neq \emptyset$ for some m and n with $m > n \ge 0$. Take $y \in \sigma^{-m}(x) \cap \sigma^{-n}(x)$, so $\sigma^m(y) = \sigma^n(y) = x$. Furthermore,

$$\sigma^{m-n}(x) = \sigma^{m-n}(\sigma^n(y)) = \sigma^m(y) = x,$$

i.e. x is a periodic point, which contradicts $x \in R(\sigma) - A(\sigma)$. Since μ is an invariant probability measure for $\sigma|_{R(\sigma)-A(\sigma)}$ and the set of simple points on (Σ, ϱ) is closed, we have $\{x\} \in \mathcal{B}(\Sigma)$ and

$$\mu(\{x\}) = \mu(\sigma^{-1}(x)) = \mu(\sigma^{-2}(x)) = \ldots = \mu(\sigma^{-n}(x)).$$

By the countable additivity of μ, we get $\mu(\{x\}) = 0$. ∎

LEMMA 2.4. *Suppose* $\mathcal{T} = R(\sigma) - A(\sigma)$. *If* μ *is the only invariant probability measure for* $\sigma|_{\mathcal{T}}$, *then the sequence* $\{\mu([b_1 \dots b_n])\}$ *of real numbers converges to zero uniformly in* $b_i \in \{0,1\}$, $1 \le i \le n$, *as* $n \to \infty$.

Proof. For any $\varepsilon > 0$ and any $x \in \mathcal{T}$, by Lemma 2.3, there is an open neighborhood V_x of x such that $\mu(V_x) < \varepsilon$. Moreover, by the definition of $[b_1 \dots b_n]$, there exists $N > 0$ such that $\operatorname{diam}[b_1 \dots b_n] < \varepsilon$ uniformly in $b_i \in \{0,1\}$, $1 \le i \le n$, as $n \to \infty$. Thus for any $x \in [b_1 \dots b_n] \cap \mathcal{T}$, there exists $N > 0$ such that x must be contained in some V_x when $n \ge N$. So

$$\mu([b_1 \dots b_n]) = \mu([b_1 \dots b_n] \cap \mathcal{T}) < \varepsilon. \ \blacksquare$$

LEMMA 2.5 (see [7]). *Let* $f : X \to X$ *be continuous*, $x, y \in X$, $N > 0$.

(1) *If* $F(f^N, x, y, s) = 0$ *for any* $s > 0$, *then there exists a* $t > 0$ *such that* $F(f, x, y, t) = 0$.

(2) *If* $F^*(f^N, x, y, s) = 1$ *for any* $s > 0$, *then* $F^*(f, x, y, t) = 1$ *for any* $t > 0$.

3. Proof of the main theorem. By the hypothesis, f has a regular shift invariant set, denoted by Λ. Thus there is a continuous surjection $h : \Lambda \to \Sigma$ such that for any $x \in \Lambda$,

$$h \circ f(x) = \sigma \circ h(x).$$

According to Lemma 2.2, there is an uncountable set $\mathcal{T} \subset R(\sigma) - A(\sigma)$ which is strongly chaotic and $\sigma|_{\mathcal{T}}$ has the only ergodic measure μ. Set, for simplicity, $g = f|_{\Lambda}$. For any $y \in \mathcal{T}$, by Lemma 2.1 there exists $x \in R(g) - A(g)$ such that $h(x) = y$. Let

$$D = \{x \mid x \in R(g) - A(g),\ h(x) = y \text{ and } y \in \mathcal{T}\}.$$

Then $D \subset \Lambda$ and D is an uncountable set. To complete the proof, it suffices to show that D is a strongly chaotic set for f.

For any distinct $x_1, x_2 \in D$, there exist $y_1, y_2 \in \mathcal{T}$ such that $h(x_i) = y_i$ for $i = 1, 2$. Since y_1 and y_2 are in a strongly chaotic set for σ, there exists $s > 0$ and a sequence $n_k \to \infty$ such that

$$\frac{1}{n_k} \xi_{n_k}(\sigma, y_1, y_2, s) \to 0 \quad (k \to \infty). \tag{3.1}$$

Choose an $N > 0$ such that $\operatorname{diam}[B] < s$ for any $[B] \in \mathcal{B}_N$. Let

$$t = \min\{d(I_{[B]}, I_{[C]}) \mid [B], [C] \in \mathcal{B}_N \text{ and } [B] \ne [C]\},$$

where $d(I_{[B]}, I_{[C]}) = \inf\{d(p,q) \mid p \in I_{[B]},\ q \in I_{[C]}\}$. By the properties of g, $d(I_{[B]}, I_{[C]}) > 0$ for any distinct $[B], [C] \in \mathcal{B}_N$ and so $t > 0$. It is easily seen that for any $i \ge 0$,

$$
\begin{aligned}
&\varrho(\sigma^i(y_1), \sigma^i(y_2)) \geq s \\
&\qquad \Rightarrow\ \sigma^i(y_1) \in [B],\ \sigma^i(y_2) \in [C] \\
&\qquad\qquad\qquad\qquad \text{for some distinct } [B],[C] \in \mathcal{B}_N \text{ (since } \operatorname{diam}[B] < s) \\
&\qquad \Rightarrow\ g^i(x_1) \in I_{[B]},\ g^i(x_2) \in I_{[C]} \text{ and } d(I_{[B]}, I_{[C]}) \geq t \\
&\qquad \Rightarrow\ d(g^i(x_1), g^i(x_2)) \geq t,
\end{aligned}
$$

and therefore, for each k we have

$$\xi_{n_k}(g, x_1, x_2, t) \leq \xi_{n_k}(\sigma, y_1, y_2, s).$$

By (3.1), we get

$$\frac{1}{n_k}\, \xi_{n_k}(g, x_1, x_2, t) \to 0 \qquad (k \to \infty),$$

and hence

$$F(g, x_1, x_2, t) = 0. \tag{3.2}$$

We now prove $F^*(g, x_1, x_2, t) = 1$ for any $t > 0$. By the hypothesis, we can choose $M > 0$ such that $\sum_{[B]\in\mathcal{B}_n} \operatorname{diam} I_{[B]} \leq M$ for any fixed $n > 0$. For any given $t > 0$ and $\varepsilon > 0$, choose an integer $k > 0$ such that $tk > M$. By Lemma 2.4, we may also choose an N_1 large enough such that $\mu([B]) < \varepsilon/(2k)$ for any $[B] \in \mathcal{B}_{N_1} \cap \mathcal{T}$, i.e. for any $y \in \mathcal{T}$,

$$\lim_{n\to\infty} \frac{1}{n}\, \#\{i \mid \sigma^i(y) \in [B],\ 0 \leq i \leq n\} < \frac{\varepsilon}{2k}. \tag{3.3}$$

Put $s = 1/2^{N_1}$. Since $F^*(\sigma, y_1, y_2, s) = 1$, there exists a sequence $n_j \to \infty$ such that

$$\frac{1}{n_j}\, \xi_{n_j}(\sigma, y_1, y_2, s) \to 1 \qquad (n_j \to \infty). \tag{3.4}$$

Set, for simplicity,

$$\theta_{n_j} = \sum_{[B]\in\mathcal{B}_{N_1}\cap\mathcal{T}} \frac{1}{n_j}\, \#\{i \mid g^i(x_1), g^i(x_2) \in I_{[B]},\ 0 \leq i \leq n_j\}.$$

Noting that

$$
\begin{aligned}
&\varrho(\sigma^i(y_1), \sigma^i(y_2)) < s \\
&\qquad\qquad \Leftrightarrow\ \sigma^i(y_1), \sigma^i(y_2) \in [B] \text{ for some } [B] \in \mathcal{B}_{N_1} \cap \mathcal{T} \\
&\qquad\qquad \Leftrightarrow\ g^i(x_1), g^i(x_2) \in I_{[B]} \text{ for some } [B] \in \mathcal{B}_{N_1} \cap \mathcal{T},
\end{aligned} \tag{3.5}
$$

according to (3.4), we have

$$\theta_{n_j} \to 1 \qquad (j \to \infty). \tag{3.6}$$

Thus from (3.3), (3.5), and (3.6) we can choose N large enough such that for $n_j > N$,

$$\frac{1}{n_j}\#\{i \mid g^i(x_1), g^i(x_2) \in I_{[B]},\ 0 \le i < n_j\} < \frac{\varepsilon}{2k} \quad \text{for any } [B] \in \mathcal{B}_{N_1} \cap \mathcal{T},$$

and

$$1 - \theta_{n_j} < \varepsilon/2. \tag{3.7}$$

On the one hand, by the definition of θ_{n_j},

$$\begin{aligned}\theta_{n_j} - \sum_{\substack{[B]\in\mathcal{B}_{N_1}\cap\mathcal{T}\\ \operatorname{diam} I_{[B]}\ge t}} \frac{\varepsilon}{2k} &\le \theta_{n_j} - \sum_{\substack{[B]\in\mathcal{B}_{N_1}\cap\mathcal{T}\\ \operatorname{diam} I_{[B]}\ge t}} \frac{1}{n_j}\#\{i \mid g^i(x_1), g^i(x_2) \in I_{[B]},\ 0 \le i < n_j\}\\ &= \sum_{\substack{[B]\in\mathcal{B}_{N_1}\cap\mathcal{T}\\ \operatorname{diam} I_{[B]}< t}} \frac{1}{n_j}\#\{i \mid g^i(x_1), g^i(x_2) \in I_{[B]},\ 0 \le i < n_j\}\\ &\le \frac{1}{n_j}\xi_{n_j}(g, x_1, x_2, t).\end{aligned} \tag{3.8}$$

On the other hand, because of the choice of k, there exist at most k different $[B]$'s with $\operatorname{diam} I_{[B]} \ge t$ in $\mathcal{B}_{N_1} \cap \mathcal{T}$. In fact, since $tk > M$, if there exists $k_1 > k$ such that k_1 different $[B]$'s satisfy $\operatorname{diam} I_{[B]} \ge t$, then $k_1 \operatorname{diam} I_{[B]} \ge tk_1 > M$. However, by the choice of M, we know that

$$M \ge \sum_{[B]\in\mathcal{B}_{N_1}\cap\mathcal{T}} \operatorname{diam} I_{[B]} \ge k_1 \operatorname{diam} I_{[B]},$$

which is contradictory. By (3.8), we have

$$\theta_{n_j} - \frac{\varepsilon}{2} = \theta_{n_j} - k\cdot\frac{\varepsilon}{2k} \le \frac{1}{n_j}\xi_{n_j}(g, x_1, x_2, t).$$

Combining this with (3.7), we see that for $n_j > N$,

$$0 \le 1 - \frac{1}{n_j}\xi_{n_j}(g, x_1, x_2, t) \le 1 - \theta_{n_j} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which gives

$$F^*(g, x_1, x_2, t) = 1. \tag{3.9}$$

By (3.2), (3.9) and the arbitrariness of x_1 and x_2, we conclude that D is an uncountable strongly chaotic set of g.

Thus, we have proved that f has an uncountable strongly chaotic set D in $R(f) - A(f)$. ■

4. Examples

EXAMPLE 4.1. Let $f \in C^0(I)$. If f has a positive topological entropy, then there exists an $N > 0$ such that f^N has a regular shift invariant set ([1]). From the theorem, we deduce that f^N has an uncountable strongly chaotic set in which each point is recurrent, but is not almost periodic. The same holds for f, since for any positive integer n, f displays strong chaos if and only if f^n does (Lemma 2.5).

EXAMPLE 4.2. Let $r_0, r_1 : S^1 \to S^1$ be irrational rotations with $r_0 \neq \pm r_1$. Define $f : \Sigma \times S^1 \to \Sigma$ by

$$f(x,t) = (\sigma(x), r_{x_1}(t))$$

for $x = x_1 x_2 \ldots \in \Sigma$, $t \in S^1$. Note that the nth iteration of f at the point $(x,t) \in \Sigma \times S^1$ is given by

$$f^n(x,t) = (\sigma^n(x), r_{x_n} \circ \ldots \circ r_{x_2} \circ r_{x_1}(t)).$$

It is easy to see that f is continuous.

Let $h : \Sigma \times S^1 \to \Sigma$ be defined by $h(x,t) = x$. We see that h satisfies (2)(a) of Definition 2.1, but not (2)(b). Indeed, we do not know if an h satisfying both (2)(a) and (2)(b) exists or not. Also we do not know if f displays strong chaos or not, since our theorem cannot be used.

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