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Solutions for a class of hemivariational inequalities with $p(x)$-Laplacian

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Abstract. We study a class of hemivariational inequalities with $p(x)$-Laplacian. Applying nonsmooth critical point theory for locally Lipschitz functions, we obtain the existence of solutions on interior and exterior domains.

1. Introduction and main results. Since the paper by Kováčik and Rákosník [12] where the spaces $L^{p(x)}$ and $W^{1,p(x)}$ were thoroughly studied, variable exponent Sobolev spaces have been used extensively to model various phenomena. In [17] Růžička applied them in the study of electrorheological fluids. In recent years, the differential equations and variational problems with $p(x)$-growth conditions have been extensively investigated (see for example [1, 9, 10, 14]).

Here we discuss a class of hemivariational inequalities with $p(x)$-Laplacian. Hemivariational inequalities arise in problems of mechanics and engineering, when one considers more realistic laws of nonmonotone and multivalued nature. For concrete applications, we refer to Naniewicz–Panagiotopoulos [15] and Panagiotopoulos [16]. In this paper, we study the following hemivariational inequality:

$$(1.1)\qquad \begin{cases} u \in W_0^{1,p(x)}(\Omega), \\ \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv)\,dx \\ \qquad + \int_\Omega F_x^0(x,u(x);-v(x))\,dx \ge 0, \quad \forall v \in W_0^{1,p(x)}(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a domain, p is Lipschitz continuous on $\overline{\Omega}$ and satisfies $1 < p_- \le p(x) \le p_+ < N$.

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Now we recall some basic properties of variable exponent spaces $L^{p(x)}(\Omega)$ and variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a domain. For a deeper treatment of these spaces, we refer to [5, 6, 12].

Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to [1, \infty)$ and for $p \in \mathbf{P}(\Omega)$ set

$$|u|_{p(x)} = \inf\Big\{\lambda > 0 : \int_\Omega |u/\lambda|^{p(x)}\,dx \le 1\Big\}. \tag{1.2}$$

The variable exponent space $L^{p(x)}(\Omega)$ is the class of all functions u such that $\int_\Omega |u(x)|^{p(x)}\,dx < \infty$; it is a Banach space equipped with the norm (1.2).

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is the class of all functions $u \in L^{p(x)}(\Omega)$ such that $|\nabla u| \in L^{p(x)}(\Omega)$; it can be equipped with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}. \tag{1.3}$$

We denote by $W_0^{1,p(x)}(\Omega)$ the subspace of $W^{1,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (1.3); if $\Omega \subset \mathbb{R}^N$ is a bounded domain, then $\|u\|_{1,p(x)}$ and $|\nabla u|_{p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

For all $p \in \mathbf{P}(\Omega)$, we write

$$p_+ = \sup_{x\in\Omega} p(x), \qquad p_- = \inf_{x\in\Omega} p(x),$$

and denote by $p_1 \ll p_2$ the fact that $\inf_{x\in\Omega}(p_2(x) - p_1(x)) > 0$.

Throughout this paper, we assume that $F : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function which is locally Lipschitz in the second variable and $F(x, 0) = 0$ for all $x \in \Omega$. $F_x^0(x, s; z)$ is the generalized directional derivative of $F(x, \cdot)$ at $s \in \mathbb{R}$ in direction $z \in \mathbb{R}$.

In addition, we need various conditions on F corresponding to the cases when Ω is an interior or exterior domain. Firstly, consider the case when Ω is an interior domain, i.e. Ω is bounded.

(H1) There exists $\alpha \in C(\overline{\Omega})$ with $p(x) \ll \alpha(x) \ll p^*(x)$ such that
$$|\xi| \le a_0 + a_1|t|^{\alpha(x)-1}$$
for all $(x, t) \in \Omega \times \mathbb{R}$ and $\xi \in \partial F(x, t)$, where $\partial F(x, t)$ is the generalized gradient of $F(x, \cdot)$ at $t \in \mathbb{R}$, and $a_0, a_1 > 0$.

(H2) There exists $p(x) \ll \mu$ such that $\mu F(x, t) \le -F_x^0(x, t; -t)$ for all $(x, t) \in \Omega \times \mathbb{R}$. Moreover, there exist an open set $\Omega_0 \subset \Omega$ and $a_2, a_3 > 0$ such that $F(x, t) \ge a_2|t|^\mu - a_3$ for any $(x, t) \in \Omega_0 \times \mathbb{R}$.

(H3) $\lim_{t\to 0} \max\{|\xi| : \xi \in \partial F(x, t)\}/|t|^{p(x)-1} = 0$ uniformly for almost every $x \in \Omega$.

(H4) There exists $\beta \in \mathbf{P}(\Omega)$ with $1 < \beta_- \le \beta(x) \ll p(x)$ such that
$$|\xi| \le b_0 + b_1|t|^{\beta(x)-1}$$
for all $(x, t) \in \Omega \times \mathbb{R}$ and $\xi \in \partial F(x, t)$, where $b_0, b_1 > 0$.

(H5) Let $\beta(x)$ be as in (H4). There exist $b_2 > 0$, $0 < \delta < 1$ and an open set $\Omega_0 \subset \Omega$ such that $F(x,t) \geq b_2|t|^{\beta(x)}$ for all $(x,t) \in \Omega_0 \times (0,\delta)$.

Under these conditions, we get the following results.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Under hypotheses* (H1)–(H3), *problem* (1.1) *has at least one non-trivial solution.*

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Under hypotheses* (H4), (H5), *problem* (1.1) *has at least one non-trivial solution.*

We now consider the case when Ω is an exterior domain, i.e. the complement of a bounded domain.

(H6) $|\xi| \leq g(x)|t|^{\alpha(x)-1}$ for all $(x,t) \in \Omega \times \mathbb{R}$ and $\xi \in \partial F(x,t)$, where $\alpha \in \mathbf{P}(\Omega)$ with $p(x) \ll \alpha(x) \ll p^*(x)$, $g(x) \geq 0$ and $g \in L^\infty(\Omega) \cap L^{q_1(x)}(\Omega)$ with $q_1(x) = p^*(x)/(p^*(x)-\alpha(x))$.

(H7) $|\xi| \leq h(x)|t|^{\beta(x)-1}$ for all $(x,t) \in \Omega \times \mathbb{R}$ and $\xi \in \partial F(x,t)$, where $\beta \in \mathbf{P}(\Omega)$ with $1 < \beta_- \leq \beta(x) \ll p(x)$, $h(x) \geq 0$ and $h \in L^\infty(\Omega) \cap L^{q_2(x)}(\Omega)$ with $q_2(x) = p^*(x)/(p^*(x)-\beta(x))$.

With these assumptions we have the following results.

THEOREM 1.3. *Assume hypotheses* (H2), (H6) *hold and $\Omega \subset \mathbb{R}^N$ is an exterior domain. Then problem* (1.1) *has at least one non-trivial solution.*

THEOREM 1.4. *Assume hypotheses* (H5), (H7) *hold and $\Omega \subset \mathbb{R}^N$ is an exterior domain. Then problem* (1.1) *has at least one non-trivial solution.*

2. Critical point theory for locally Lipschitz functions. In this paper, our approach is mainly based on variational methods for nondifferentiable functionals, namely, locally Lipschitz functionals. For a deeper treatment of this theory, we refer to [2, 3, 4, 13]. Now we present some basic definitions and preliminary results.

Let $(X, \|\cdot\|)$ be a Banach space, X^* its topological dual, and $\varphi : X \to \mathbb{R}$ a locally Lipschitz function. The *generalized directional derivative* of φ at $u \in X$ in direction $v \in X$ is defined by

$$\varphi^0(u;v) = \limsup_{\substack{w \to u \\ t \to 0+}} \frac{\varphi(w+tv)-\varphi(w)}{t}.$$

The *generalized gradient* of φ at $u \in X$ is the set

$$\partial\varphi(u) = \{w^* \in X^* : \langle w^*, v\rangle \leq \varphi^0(u;v),\ \forall v \in X\},$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between X^* and X. A point $u \in X$ is a *critical point* of φ if $0 \in \partial\varphi(u)$. If $u \in X$ is a critical point, the value $c = \varphi(u)$ is a *critical value* of φ.

In the classical (smooth) theory, a basic analytical tool is a compactness-type condition, known as the Palais–Smale condition. In the present nonsmooth setting this condition takes the following form: A locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the *nonsmooth Palais–Smale* (P.S.) *condition* if every sequence $\{u_n\} \subset X$ such that $\varphi(u_n)$ is bounded and

$$\lambda(u_n) = \min\{\|w^*\|_{X^*} : w^* \in \partial\varphi(u_n)\} \to 0$$

as $n \to \infty$, has a convergent subsequence in X.

PROPOSITION 2.1. *If X is a reflexive Banach space, $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function which satisfies the nonsmooth* (P.S.) *condition and for some $r > 0$ and x_1, x_2 with $\|x_1 - x_2\|_X > r$, we have*

$$\max\{\varphi(x_1), \varphi(x_2)\} < \inf\{\varphi(x) : \|x - x_1\|_X = r\},$$

then there exists a critical point $y_0 \in X$ of φ such that

$$c = \varphi(y_0) \geq \inf\{\varphi(x) : \|x - x_1\|_X = r\}$$

and c is defined by the following minimax formula:

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_1,\ \gamma(1) = x_2\}$.

PROPOSITION 2.2. *If X is a reflexive Banach space, and $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function which satisfies the nonsmooth* (P.S.) *condition and is bounded from below, then $c = \inf_{u\in X} \varphi(u)$ is a critical value of φ.*

3. The case of interior domain. Throughout this section, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain and denote by c_i various positive constants. In order to discuss the problem (1.1), we need to define two functionals on $W_0^{1,p(x)}(\Omega)$:

$$\psi(u) = \int_\Omega F(x,u)\,dx,$$

$$\varphi(u) = J(u) - \psi(u) = \int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)}\,dx - \psi(u).$$

THEOREM 3.1. *Under condition* (H1) *or* (H4), *ψ is well defined and is a locally Lipschitz functional on $W_0^{1,p(x)}(\Omega)$.*

Proof. Here we only consider the case where F satisfies (H1).

(i) *ψ is well defined.* For all $t_1, t_2 \in \mathbb{R}$, by Lebourg's mean value theorem (see [4]), there exist $\theta \in (0,1)$ and $\xi_\theta \in \partial F(x, \theta t_1 + (1-\theta)t_2)$ such that

$$F(x,t_1) - F(x,t_2) = \xi_\theta(t_1 - t_2)$$

for all $x \in \Omega$. By condition (H1), we get

$$\begin{aligned}|F(x,t_1)-F(x,t_2)| &\le (a_0+a_1|\theta t_1+(1-\theta)t_2|^{\alpha(x)-1})|t_1-t_2| \\ &\le (a_0+c_1|t_1|^{\alpha(x)-1}+c_1|t_2|^{\alpha(x)-1})|t_1-t_2|.\end{aligned}$$

We also get

$$|\psi(u)| \le \int_\Omega |F(x,u)|\,dx \le \int_\Omega (a_0+c_1|u|^{\alpha(x)-1})|u|\,dx.$$

By Theorem 1.1 in [8], we have $u \in L^1(\Omega)$ and $u \in L^{\alpha(x)}(\Omega)$ for all $u \in W_0^{1,p(x)}(\Omega)$. Hence $|\psi(u)| < \infty$.

(ii) *ψ is locally Lipschitz on $W_0^{1,p(x)}(\Omega)$.* Note that for all $u_1, u_2 \in W_0^{1,p(x)}(\Omega)$,

$$\begin{aligned}|\psi(u_1)-\psi(u_2)| &\le \int_\Omega |F(x,u_1)-F(x,u_2)|\,dx \\ &\le \int_\Omega (a_0+c_1|u_1|^{\alpha(x)-1}+c_1|u_2|^{\alpha(x)-1})|u_1-u_2|\,dx \\ &\le c_2\big|1+|u_1|^{\alpha(x)-1}+|u_2|^{\alpha(x)-1}\big|_{\alpha'(x)}\cdot|u_1-u_2|_{\alpha(x)} \\ &\le c_3\big|1+|u_1|^{\alpha(x)-1}+|u_2|^{\alpha(x)-1}\big|_{\alpha'(x)}\cdot|\nabla u_1-\nabla u_2|_{p(x)}.\end{aligned}$$

Hence it is easy to get the result. ■

THEOREM 3.2. *Under condition* (H1) *or* (H4), *for all $u, v \in W_0^{1,p(x)}(\Omega)$ we have*

$$\psi^0(u;v) \le \int_\Omega F_x^0(x,u(x);v(x))\,dx.$$

Proof. We only consider the case where F satisfies (H1).

(i) $\int_\Omega F_x^0(x,u(x);v(x))\,dx < \infty$. In fact, $F(x,\cdot)$ is continuous for all $x \in \Omega$, thus

$$\begin{aligned}&\limsup_{\substack{y\to u(x)\\ t\to 0+}} \frac{F(x,y+tv(x))-F(x,y)}{t} \\ &\quad= \limsup_{\substack{z\to 0\\ t\to 0+}} \frac{F(x,z+u(x)+tv(x))-F(x,z+u(x))}{t} \\ &\quad= \limsup_{\substack{z_n\to 0\\ t_n\to 0+}} \frac{F(x,z_n+u(x)+t_nv(x))-F(x,z_n+u(x))}{t_n},\end{aligned}$$

where z_n, t_n are rational values. As $u(x)$, $v(x)$ are measurable, we see that $F_x^0(x,u(x);v(x))$, being the "countable limsup" of measurable functionals of x, is also measurable.

We know $F_x^0(x,u(x);v(x)) = \max\{\xi\cdot v(x) : \xi \in \partial F(x,u(x))\} \triangleq \xi_x \cdot v(x)$ for all $x \in \Omega$. By condition (H1), we get

$$|F_x^0(x,u(x);v(x))| = |\xi_x \cdot v(x)| \le a_0|v(x)| + a_1|v(x)|\cdot|u(x)|^{\alpha(x)-1},$$

and so $F_x^0(x,u(x);v(x)) \in L^1(\Omega)$.

(ii) $\psi^0(u;v) \le \int_\Omega F_x^0(x,u(x);v(x))\,dx$. By the definition of $\psi^0(u;v)$, there exist $t_n \to 0+$ and $w_n \to u$ in $W_0^{1,p(x)}(\Omega)$ such that

$$\varphi^0(u;v) = \lim_{\substack{w_n\to u\\ t_n\to 0+}} \frac{\varphi(w_n+t_nv)-\varphi(w_n)}{t_n}.$$

Passing to a subsequence, still denoted by $\{w_n\}$, we may assume that $w_n(x) \to u(x)$ a.e. in Ω as $n\to\infty$. Set

$$A_n(x) = \frac{F(x,w_n(x)+t_nv(x)) - F(x,w_n(x))}{t_n},$$

$$B_n(x) = (a_0 + c_1|w_n(x)+t_nv(x)|^{\alpha(x)-1} + c_1|w_n(x)|^{\alpha(x)-1})|v(x)|,$$

$$g_n(x) = -A_n(x) + B_n(x).$$

It is easy to verify that $g_n(x) \ge 0$ for all $x\in\Omega$,

$$\limsup_{n\to\infty}\int_\Omega -g_n(x)\,dx \le \int_\Omega \limsup_{n\to\infty}(-g_n(x))\,dx.$$

Note that

$$\int_\Omega \limsup_{n\to\infty}(-g_n(x))\,dx = \int_\Omega \limsup_{n\to\infty}(A_n(x)-B_n(x))\,dx,$$

$$\int_\Omega \limsup_{n\to\infty} A_n(x)\,dx \le \int_\Omega \limsup_{\substack{y\to u(x)\\ t\to 0+}} \frac{F(x,y+tv(x))-F(x,y)}{t}\,dx = \int_\Omega F_x^0(x,u(x);v(x))\,dx,$$

$$\int_\Omega \liminf_{n\to\infty} B_n(x)\,dx = \int_\Omega (a_0 + 2c_1|u(x)|^{\alpha(x)-1})|v(x)|\,dx.$$

Therefore

$$\int_\Omega \limsup_{n\to\infty}(-g_n(x))\,dx \le \int_\Omega F_x^0(x,u(x);v(x))\,dx - \int_\Omega (a_0+2c_1|u(x)|^{\alpha(x)-1})|v(x)|\,dx.$$

For all $(x,t)\in\Omega\times\mathbb{R}$, define $f(x,t) = |v(x)|\cdot|t|^{\alpha(x)-1}$. Then there exists $c_4>0$ such that $|f(x,t)| \le c_4(1+|v|^{p^*(x)}+|t|^{p^*(x)})$. We know that the Nemytskiĭ operator

$$N_f : L^{p^*(x)}(\Omega) \to L^1(\Omega) : u \mapsto f(x,u)$$

is continuous. By Theorem 1.1 in [8], $w_n \to u$ in $L^{p^*(x)}(\Omega)$, so

$$f(x, w_n + t_n v) \to f(x, u)$$

in $L^1(\Omega)$. Thus

$$\int_\Omega |w_n + t_n v|^{\alpha(x)-1} |v| \, dx \to \int_\Omega |u|^{\alpha(x)-1} |v| \, dx$$

and $\int_\Omega B_n(x)\, dx \to \int_\Omega (a_0 + 2c_1 |u(x)|^{\alpha(x)-1}) |v(x)| \, dx$ as $n \to \infty$. Hence

$$\limsup_{n\to\infty} \int_\Omega -g_n(x)\, dx = \psi^0(u; v) - \int_\Omega (a_0 + 2c_1 |u(x)|^{\alpha(x)-1}) |v(x)| \, dx.$$

Now the proof is complete. ■

THEOREM 3.3. *Under condition* (H1) *or* (H4), *any critical point of* φ *is a solution of* (1.1).

Proof. It is easy to verify that $J \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$. Combining this with Theorem 3.1, we find that φ is locally Lipschitz. If u is a critical point of φ, then $0 \in \partial\varphi(u)$. Thus for any $v \in W_0^{1,p(x)}(\Omega)$, $\varphi^0(u; v) \geq 0$. Noting that

$$\begin{aligned}\varphi^0(u; v) &= \langle J'(u), v\rangle + (-\psi)^0(u; v) = \langle J'(u), v\rangle + \psi^0(u; -v) \\ &\leq \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv)\, dx + \int_\Omega F_x^0(x, u(x); -v(x))\, dx,\end{aligned}$$

it is easy to get the result. ■

LEMMA 3.1. *Under conditions* (H1), (H2), φ *satisfies the* (P.S.) *condition.*

Proof. Take $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ such that $\varphi(u_n)$ is bounded and

$$\lambda(u_n) = \min\{\|w^*\|_{W^{-1,p'(x)}(\Omega)} : w^* \in \partial\varphi(u_n)\} \triangleq \|w_n^*\|_{W^{-1,p'(x)}(\Omega)} \to 0$$

as $n \to \infty$. Then

$$\varphi^0(u_n; v_n) \geq \langle w_n^*, u_n\rangle, \qquad -\varphi^0(u_n; v_n) \leq \|w_n^*\| \cdot |\nabla u_n|_{p(x)}.$$

(i) *$\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.* In fact, as $\mu \gg p(x)$, we get

$$\begin{aligned}c_5 + |\nabla u_n|_{p(x)} &\geq \varphi(u_n) - \frac{1}{\mu}\varphi^0(u_n; u_n) \\ &= \varphi(u_n) - \left\langle J'(u_n), \frac{u_n}{\mu}\right\rangle - \frac{1}{\mu}\psi^0(u_n; -u_n) \\ &\geq \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{\mu}\right) |\nabla u_n|^{p(x)}\, dx \\ &\quad - \int_\Omega \left(F(x, u_n) + \frac{1}{\mu} F_x^0(x, u_n(x); -u_n(x))\right) dx \\ &\geq \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{\mu}\right) |\nabla u_n|^{p(x)}\, dx\end{aligned}$$

when n is sufficiently large. It is easy to deduce that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

(ii) $\{u_n\}$ *has a convergent subsequence.* In fact, as $W_0^{1,p(x)}(\Omega)$ is reflexive, passing to a subsequence, still denoted by $\{u_n\}$, we may assume that there exists $u \in W_0^{1,p(x)}(\Omega)$ such that $u_n \to u$ weakly in $W_0^{1,p(x)}(\Omega)$. Then $u_n \to u$ in $L^{\alpha(x)}(\Omega)$ and in $L^{p(x)}(\Omega)$. Noting that

$$\varphi^0(u_n; u - u_n) = \langle J'(u_n), u - u_n\rangle + \psi^0(u_n; u_n - u),$$
$$\varphi^0(u; u_n - u) = \langle J'(u), u_n - u\rangle + \psi^0(u; u - u_n),$$

we get

$$\begin{aligned}\langle J'(u_n) &- J'(u), u_n - u\rangle \\ &= \psi^0(u_n; u_n - u) + \psi^0(u; u - u_n) - \varphi^0(u_n; u - u_n) - \varphi^0(u; u_n - u).\end{aligned}$$

For all $w^* \in \partial\varphi(u)$, $\varphi^0(u; u_n - u) \geq \langle w^*, u_n - u\rangle$, so

$$\liminf_{n\to\infty} \varphi^0(u; u_n - u) \geq 0.$$

As $\varphi^0(u_n; u - u_n) \geq \langle w_n^*, u - u_n\rangle \geq -c_5\|w_n^*\|$, we have

$$\liminf_{n\to\infty} \varphi^0(u_n; u - u_n) \geq 0.$$

By Theorem 3.2, we get

$$\begin{aligned}\psi^0&(u_n; u_n - u) + \psi^0(u; u - u_n) \\ &\leq \int_\Omega F_x^0(x, u_n(x); u_n(x) - u(x))\,dx + \int_\Omega F_x^0(x, u(x); u(x) - u_n(x))\,dx \\ &\leq \int_\Omega \max\{\xi\cdot(u_n(x) - u(x)) : \xi \in \partial F(x, u_n(x))\}\,dx \\ &\quad + \int_\Omega \max\{\xi\cdot(u(x) - u_n(x)) : \xi \in \partial F(x, u(x))\}\,dx \\ &\leq \int_\Omega c_6(1 + |u_n|^{\alpha(x)-1} + |u|^{\alpha(x)-1})|u_n - u|\,dx \\ &\leq c_7|1 + |u_n|^{\alpha(x)-1} + |u|^{\alpha(x)-1}|_{\alpha'(x)} \cdot |u_n - u|_{\alpha(x)} \leq c_8|u_n - u|_{\alpha(x)} \to 0.\end{aligned}$$

Thus $\limsup_{n\to\infty}\langle J'(u_n) - J'(u), u_n - u\rangle \leq 0$. Similar to Theorem 3.1 in [1], we conclude that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. ∎

LEMMA 3.2. *We have* $\varphi(0) = 0$. *Under conditions* (H1), (H3), *there exist* $r_1, s_1 > 0$ *such that* $\varphi(u) > 0$ *for* $0 < |\nabla u|_{p(x)} \leq r_1$ *and* $\varphi(u) > s_1$ *for* $|\nabla u|_{p(x)} = r_1$.

Proof. It is easy to show that $\varphi(0) = 0$. By condition (H3), for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\max\{|\xi| : \xi \in \partial F(x, t)\} \leq \varepsilon|t|^{p(x)-1}$$

for all $|t| < \delta$ and $x \in \Omega$. By Lebourg's mean value theorem, there exist $\theta \in (0,1)$ and $\xi_\theta \in \partial F(x,\, \theta t)$ such that $F(x,t) = \xi_\theta t$. Combining this with condition (H1), we see that for all $\varepsilon > 0$, there exists $c_9 > 0$ such that

$$|F(x,t)| \le \varepsilon |t|^{p(x)} + c_9 |t|^{\alpha(x)}$$

for all $(x,t) \in \Omega \times \mathbb{R}$. For $\varepsilon < 1/p_+$, we get

$$\begin{aligned}\varphi(u) &\ge \int_\Omega \left(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p_+} - \varepsilon |u|^{p(x)} - c_9 |u|^{\alpha(x)} \right) dx \\ &\ge \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p_+} - c_9 |u|^{\alpha(x)} \right) dx.\end{aligned}$$

As $p(x) \ll \alpha(x)$, similar to Theorem 3.1 in [1], we get the result. ∎

Lemma 3.3. *Under condition* (H2), *there exists* $e \in W_0^{1,p(x)}(\Omega)$ *such that* $\varphi(e) < 0$.

Proof. Fix $x_0 \in \Omega_0$ and $0 < R < 1/2$ such that $B_{2R}(x_0) \subset \Omega_0$. Let $\phi \in C_0^\infty(B_{2R}(x_0))$, $0 \le \phi(x) \le 1$, $|\nabla \phi(x)| \le 1/R$, and suppose $\phi(x) \equiv 1$ for $x \in B_R(x_0)$. For $t > 1$, it is easy to get

$$\begin{aligned}\varphi(t\phi) &= \int_{B_{2R}(x_0)} \left(\frac{|t\nabla \phi|^{p(x)} + |t\phi|^{p(x)}}{p(x)} - F(x, t\phi) \right) dx \\ &\le \int_{B_{2R}(x_0)} (c_{10} t^{p_+} - a_2 |t\phi|^\mu + a_3)\, dx.\end{aligned}$$

As $\mu \gg p(x)$, we get $\varphi(t\phi) < 0$, when t is sufficiently large. ∎

Proof of Theorem 1.1. By Lemmata 3.1–3.3 and Proposition 2.1, we easily get the result. ∎

Lemma 3.4. *Under condition* (H4), *the functional* φ *is bounded from below.*

Proof. By Lebourg's mean value theorem, there exist $\theta \in (0,1)$ and $\xi_\theta \in \partial F(x,\, \theta t)$ such that $F(x,t) = \xi_\theta t$. Hence

$$|F(x,t)| \le b_0 |t| + b_1 |t|^{\beta(x)}$$

for all $(x,t) \in \Omega \times \mathbb{R}$. Thus

$$\varphi(u) \ge \int_\Omega \left(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} - b_0 |u| - b_1 |u|^{\beta(x)} \right) dx$$

for all $u \in W_0^{1,p(x)}(\Omega)$. As $\beta(x) \ll p(x)$, similar to Theorem 3.2 in [1], there exists $c_{11} > 0$ such that

$$\varphi(u) \geq \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p_+}\, dx - c_{11}$$

for all $u \in W_0^{1,p(x)}(\Omega)$. Thus we get the result. ■

LEMMA 3.5. *Under condition* (H5), *there exists* $e \in W_0^{1,p(x)}(\Omega)$ *such that* $\varphi(e) < 0$.

Proof. Fix $x_0 \in \Omega_0$ and $R > 0$ such that $B_{2R}(x_0) \subset \Omega_0$ and

$$\beta_1 = \sup_{x \in B_{2R}(x_0)} \beta(x) < p_1 = \inf_{x \in B_{2R}(x_0)} p(x).$$

Let $\phi \in C_0^\infty(B_{2R}(x_0))$, $0 \leq \phi(x) \leq 1$, $|\nabla \phi(x)| \leq 1/R$, and suppose $\phi(x) \equiv 1$ for $x \in B_R(x_0)$. For $0 < t < \min\{1, \delta\}$, by condition (H5),

$$\begin{aligned}
\varphi(t\phi) &= \int_{B_{2R}(x_0)} \left(\frac{|t\nabla \phi|^{p(x)}}{p(x)} + \frac{|t\phi|^{p(x)}}{p(x)} - F(x, t\phi) \right) dx \\
&\leq \int_{B_{2R}(x_0)} (c_{12} t^{p(x)} - b_2 (t\phi)^{\beta(x)})\, dx \\
&\leq t^{\beta_1} \int_{B_{2R}(x_0)} (c_{12} t^{p_1 - \beta_1} - b_2 \phi^{\beta(x)})\, dx.
\end{aligned}$$

As $\phi(x) \equiv 1$ for $x \in B_R(x_0)$, we have $\int_{B_{2R}(x_0)} \phi^{\beta(x)}\, dx > 0$. When t is sufficiently small, we get $\varphi(t\phi) < 0$. ■

Proof of Theorem 1.2. Similar to Lemma 3.1, it is easy to verify that the functional φ satisfies the (P.S.) condition. Combining this with Lemmata 3.4, 3.5 and Proposition 2.2, we know that

$$c = \inf_{u \in W_0^{1,p(x)}(\Omega)} \varphi(u) < 0$$

is a critical value of φ. Now the proof is complete. ■

4. The case of exterior domain. Throughout this section, we assume that $\Omega \subset \mathbb{R}^N$ is an exterior domain and denote by d_i various positive constants.

THEOREM 4.1. *Under condition* (H6) *or* (H7), ψ *is well defined and is a locally Lipschitz functional on* $W_0^{1,p(x)}(\Omega)$.

Proof. We only consider the case where the functional F satisfies (H6).

(i) ψ *is well defined.* By Lebourg's mean value theorem, for all $t_1, t_2 \in \mathbb{R}$, there exist $\theta \in (0,1)$ and $\xi_\theta \in \partial F(x, \theta t_1 + (1-\theta)t_2)$ such that

$$F(x,t_1) - F(x,t_2) = \xi_\theta(t_1 - t_2)$$

for all $x \in \Omega$. By condition (H6), we get

$$\begin{aligned}|F(x,t_1) - F(x,t_2)| &\le g(x)|\theta t_1 + (1-\theta)t_2|^{\alpha(x)-1} \cdot |t_1 - t_2| \\ &\le d_1 g(x)(|t_1|^{\alpha(x)-1} + |t_2|^{\alpha(x)-1})|t_1 - t_2|.\end{aligned}$$

By the Young inequality, we also get

$$|\psi(u)| \le \int_\Omega |F(x,u)|\,dx \le \int_\Omega d_1 g(x)|u|^{\alpha(x)}\,dx \le \int_\Omega d_2(g(x)^{q_1(x)} + |u|^{p^*(x)})\,dx.$$

Thus by Theorem 1.1 in [8], we have $|\psi(u)| < \infty$ for all $u \in W_0^{1,p(x)}(\Omega)$.

(ii) ψ *is locally Lipschitz on* $W_0^{1,p(x)}(\Omega)$. In fact, for all $u_1, u_2 \in W_0^{1,p(x)}(\Omega)$,

$$\begin{aligned}|\psi(u_1) - \psi(u_2)| &\le \int_\Omega |F(x,u_1) - F(x,u_2)|\,dx \\ &\le \int_\Omega d_1 g(x)(|u_1|^{\alpha(x)-1} + |u_2|^{\alpha(x)-1})|u_1 - u_2|\,dx \\ &\le d_3 \big|g(x)|u_1|^{\alpha(x)-1} + g(x)|u_2|^{\alpha(x)-1}\big|_{(p^*(x))'} \cdot |u_1 - u_2|_{p^*(x)} \\ &\le d_4 \big|g(x)|u_1|^{\alpha(x)-1} + g(x)|u_2|^{\alpha(x)-1}\big|_{(p^*(x))'} \cdot \|u_1 - u_2\|_{1,p(x)}.\end{aligned}$$

By the Young inequality, we get

$$\int_\Omega (g(x)|u_1|^{\alpha(x)-1})^{(p^*(x))'}\,dx \le \int_\Omega d_5(g(x)^{q_1(x)} + |u_1|^{p^*(x)})\,dx.$$

As $g \in L^{q_1(x)}(\Omega)$ and the imbedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$ is continuous, we can easily get the result. ∎

THEOREM 4.2. *Under condition* (H6) *or* (H7), *for all* $u, v \in W_0^{1,p(x)}(\Omega)$, *we have*

$$\psi^0(u;v) \le \int_\Omega F_x^0(x, u(x); v(x))\,dx.$$

Proof. (i) Similar to Theorem 3.1, we prove that $F_x^0(x,u(x);v(x))$ is measurable. Here we only consider the case where the functional F satisfies (H6).

Noting that $F_x^0(x,u(x);v(x)) = \max\{\xi \cdot v(x) : \xi \in \partial F(x,u(x))\} \triangleq \xi_x \cdot v(x)$, we get

$$\begin{aligned}|F_x^0(x,u(x);v(x))| = |\xi_x \cdot v(x)| &\le g(x)|u(x)|^{\alpha(x)-1}|v(x)| \\ &\le d_6(g(x)^{q_1(x)} + |u|^{p^*(x)} + |v|^{p^*(x)}).\end{aligned}$$

Hence $F_x^0(x,u(x);v(x)) \in L^1(\Omega)$.

(ii) $\psi^0(u;v) \le \int_\Omega F_x^0(x,u(x);v(x))\,dx$. By the definition of $\psi^0(u;v)$, there exist $t_n \to 0+$ and $w_n \to u$ in $W_0^{1,p(x)}(\Omega)$ such that

$$\varphi^0(u;v) = \lim_{\substack{w_n\to u\\ t_n\to 0+}} \frac{\varphi(w_n+t_nv)-\varphi(w_n)}{t_n}.$$

Passing to a subsequence, still denoted by $\{w_n\}$, we may assume that $w_n(x)\to u(x)$ a.e. in Ω as $n\to\infty$. Set

$$\begin{aligned} A_n(x) &= \frac{F(x,w_n(x)+t_nv(x))-F(x,w_n(x))}{t_n},\\ B_n(x) &= d_1g(x)\big(|w_n(x)+t_nv(x)|^{\alpha(x)-1}+|w_n(x)|^{\alpha(x)-1}\big)|v(x)|,\\ g_n(x) &= -A_n(x)+B_n(x). \end{aligned}$$

Then similar to the proof of Theorem 3.2, we get the result. ∎

LEMMA 4.1. *Under condition* (H6), φ *satisfies the* (P.S.) *condition.*

Proof. Take $\{u_n\}\subset W_0^{1,p(x)}(\Omega)$ such that $\varphi(u_n)$ is bounded and

$$\lambda(u_n) = \min\{\|w^*\|_{W^{-1,p'(x)}(\Omega)} : w^*\in\partial\varphi(u_n)\} \triangleq \|w_n^*\|_{W^{-1,p'(x)}(\Omega)} \to 0$$

as $n\to\infty$. Then similar to Lemma 3.1, we see that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

As $W_0^{1,p(x)}(\Omega)$ is reflexive, passing to a subsequence, still denoted by $\{u_n\}$, we may assume that there exists $u\in W_0^{1,p(x)}(\Omega)$ such that $u_n\to u$ weakly in $W_0^{1,p(x)}(\Omega)$. Then $u_n\to u$ in $L^{\alpha(x)}(\Omega)$ and in $L^{p(x)}(\Omega)$. Noting that

$$\begin{aligned} \varphi^0(u_n;u-u_n) &= \langle J'(u_n),u-u_n\rangle + \psi^0(u_n;u_n-u),\\ \varphi^0(u;u_n-u) &= \langle J'(u),u_n-u\rangle + \psi^0(u;u-u_n), \end{aligned}$$

we get

$$\begin{aligned} &\langle J'(u_n)-J'(u),u_n-u\rangle\\ &\quad= \psi^0(u_n;u_n-u)+\psi^0(u;u-u_n)-\varphi^0(u_n;u-u_n)-\varphi^0(u;u_n-u). \end{aligned}$$

For all $w^*\in\partial\varphi(u)$, $\varphi^0(u;u_n-u)\ge\langle w^*,u_n-u\rangle$, so

$$\liminf_{n\to\infty}\varphi^0(u;u_n-u)\ge 0.$$

As $\varphi^0(u_n;u-u_n)\ge\langle w_n^*,u-u_n\rangle\ge -d_7\|w_n^*\|$, we have

$$\liminf_{n\to\infty}\varphi^0(u_n;u-u_n)\ge 0.$$

Moreover,

$$\begin{aligned}
&\psi^0(u_n;u_n-u)+\psi^0(u;u-u_n)\\
&\qquad\le \int_\Omega F_x^0(x,u_n(x);u_n(x)-u(x))\,dx+\int_\Omega F_x^0(x,u(x);u(x)-u_n(x))\,dx\\
&\qquad\le \int_\Omega \max\{\xi\cdot(u_n(x)-u(x)):\xi\in\partial F(x,u_n(x))\}\,dx\\
&\qquad\quad+\int_\Omega \max\{\xi\cdot(u(x)-u_n(x)):\xi\in\partial F(x,u(x))\}\,dx\\
&\qquad\le \int_\Omega g(x)(|u_n|^{\alpha(x)-1}+|u|^{\alpha(x)-1})|u_n-u|\,dx.
\end{aligned}$$

Similar to Theorem 4.3 in [11], we get

$$\int_\Omega g(x)(|u_n|^{\alpha(x)-1}+|u|^{\alpha(x)-1})|u_n-u|\,dx\to 0$$

as $n\to\infty$. Then $\limsup_{n\to\infty}\langle J'(u_n)-J'(u),u_n-u\rangle\le 0$, and similar to Theorem 3.1 in [10], it is easy to get $u_n\to u$ in $W_0^{1,p(x)}(\Omega)$. ■

Proof of Theorem 1.3. Firstly, we assert that there exist $r_2,s_2>0$ such that $\varphi(u)>0$ for $0<|\nabla u|_{p(x)}\le r_2$ and $\varphi(u)>s_2$ for $|\nabla u|_{p(x)}=r_2$.

In fact, by Lebourg's mean value theorem, there exist $\theta\in(0,1)$ and $\xi_\theta\in\partial F(x,\theta t)$ such that $F(x,t)=\xi_\theta t$ for all $(x,t)\in\Omega\times\mathbb{R}$. Combining this with condition (H6), we obtain

$$|F(x,t)|\le g(x)|t|^{\alpha(x)}\le d_8|t|^{\alpha(x)}$$

for all $(x,t)\in\Omega\times\mathbb{R}$. Thus

$$\begin{aligned}
\varphi(u)&\ge\int_\Omega\left(\frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p_+}-d_8|u|^{\alpha(x)}\right)dx\\
&=\int_\Omega\frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{2p_+}\,dx+\int_\Omega\left(\frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{2p_+}-d_8|u|^{\alpha(x)}\right)dx,
\end{aligned}$$

and similar to Theorem 3.1 in [10], we easily get the above assertion. Then by Lemmata 3.3, 4.1 and Proposition 2.1, we complete the proof. ■

Proof of Theorem 1.4. Firstly, we need to verify that the functional φ is coercive. By Lebourg's mean value theorem, there exist $\theta\in(0,1)$ and $\xi_\theta\in\partial F(x,\theta t)$ such that $F(x,t)=\xi_\theta t$. Hence for all $(x,t)\in\Omega\times\mathbb{R}$, we get

$$|F(x,t)|\le h(x)|t|^{\beta(x)}.$$

Thus

$$\varphi(u)\ge\int_\Omega\left(\frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}-h(x)|u|^{\beta(x)}\right)dx$$

for all $u \in W_0^{1,p(x)}(\Omega)$. As $\beta(x) \ll p(x)$, similar to Lemma 4.3 in [11], there exists $d_9 > 0$ such that

$$\varphi(u) \geq \int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{2p_+}\, dx - d_9$$

for all $u \in W_0^{1,p(x)}(\Omega)$. Then similar to the argument for Theorem 1.2, the proof is complete. ∎

5. Some examples. In this section, we give some concrete examples of functionals F satisfying the assumptions of Theorems 1.1–1.4.

EXAMPLE 5.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Define $F : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$F(x,t) = \frac{|t|^{\alpha(x)}}{\alpha(x)},$$

where $\alpha \in C(\overline{\Omega})$ with $p_+ < \alpha_-$. We can check that $-F_x^0(x,t;-t) = |t|^{\alpha(x)}$ and $\partial F(x,t) = \{|t|^{\alpha(x)-2}t\}$. If we let $\mu = \alpha_-$, it is easy to verify that F satisfies the assumptions in Theorem 1.1.

EXAMPLE 5.2. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $F : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x,t) = \frac{|t|^{\beta_-}}{\beta_-} + \frac{|t|^{\beta(x)}}{\beta(x)},$$

where $\beta \in \mathbf{P}(\Omega)$ with $1 < \beta_- \leq \beta(x) \ll p(x)$. Then we can verify that

$$\partial F(x,t) = \{|t|^{\beta_- - 2}t + |t|^{\beta(x)-2}t\}$$

and F satisfies the assumptions in Theorem 1.2.

EXAMPLE 5.3. Let $\Omega = \mathbb{R}^N \setminus B(0,1)$, where $B(0,1)$ is the closed unit ball in $\mathbb{R}^N$. Define $F : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$F(x,t) = g(x)\, \frac{|t|^{\alpha(x)}}{\alpha(x)},$$

where $\alpha \in \mathbf{P}(\Omega)$ with $p(x) \ll \alpha(x) \ll p^*(x)$, $g(x) = |x|^{-N}$. Then it is easy to see that $g \in L^\infty(\Omega) \cap L^{q_1(x)}(\Omega)$ with $q_1(x) = p^*(x)/(p^*(x) - \alpha(x))$. Thus we can apply Theorem 1.3. If we choose $\alpha \in \mathbf{P}(\Omega)$ with $1 < \alpha_- \leq \alpha(x) \ll p(x)$, then F satisfies the assumptions of Theorem 1.4.

Based on Theorems 1.1–1.4, we can solve a larger class of hemivariational inequalities which have $p(x)$-growth conditions. In particular, Theorems 1.1–1.4 are also applicable when the function $p(x)$ is a constant.

REMARK. In particular, if $F : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F(x,t) = \int_0^t f(x,s)\,ds,$$

where $\Omega \subset \mathbb{R}^N$ is a domain and $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, then the inequality (1.1) takes the form

$$\int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv)\,dx + \int_\Omega f(x,u)v\,dx = 0$$

for all $v \in W_0^{1,p(x)}(\Omega)$, i.e. $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of

$$\begin{cases} -\operatorname{div}(|\nabla w|^{p(x)-2}\nabla w) + |w|^{p(x)-2}w = f(x,w), \\ w \in W_0^{1,p(x)}(\Omega). \end{cases}$$

The existence of solutions for the above equation has been studied recently: we refer to [1, 7, 10, 11, 18].

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