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Partial integrability on Thurston manifolds

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Abstract. We determine the maximal number of independent holomorphic functions on the Thurston manifolds M^{2r+2}, $r \geq 1$, which are the first discovered compact non-Kähler almost Kähler manifolds. We follow the method which involves analyzing the torsion tensor $d\theta \bmod \theta$, where $\theta = (\theta^1, \ldots, \theta^{r+1})$ are independent $(1,0)$-forms.

1. Introduction. For an almost complex manifold (M^{2n}, J), $n \geq 1$, the integrability of J implies that there exist n independent local holomorphic functions on M^{2n}. Newlander and Nirenberg [NN] presented a potential-theoretic approach to the integrability problem on almost complex manifolds. If an almost complex structure is non-integrable, then the maximal number of independent local holomorphic functions is less than half the dimension of the manifold.

In this paper we are concerned with the partial integrability problem for generalized Thurston manifolds. In [Th], Thurston constructed examples of compact almost Kähler manifolds that are non-Kähler. His examples are now called Thurston manifolds; they have odd first Betti number. This result is connected with the classification of non-Kähler structures given in [Ab] and [W].

In [CFL], Cordero, Fernández, and de León constructed a large family of compact almost Kähler manifolds M^{2r+2}, $r \geq 1$, that are non-Kähler; then they computed the curvature, the Ricci, $*$-Ricci, and torsion tensors of the almost complex structure to detect some identities [Gr]. Their examples are called generalized Thurston manifolds. The torsion tensor in [CFL] is the Nijenhuis tensor, a skew-symmetric $(1,2)$-tensor that determines the integrability of an almost complex structure. The image bundle of the Nijenhuis tensor under a certain non-degeneracy condition also plays a crucial rôle in the partial integrability problem in the almost complex

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setting (see [Kr], [M81], and [M86]). In [MT] and [To], obstructions to local calibrations of almost complex structures are presented; Tomassini [To] gave several non-calibrable examples in $\mathbb{R}^4$ and $\mathbb{R}^6$ including the Thurston manifold M^4. These results motivated the study of compact non-Kähler almost complex structures with respect to partial integrability. It turns out that generalized Thurston manifolds are of type $(1,1)$ in the sense of Definition 2.5 (see Sect. 3). However, in general, an almost complex manifold does not have that kind of constancy in type.

Instead of the Nijenhuis tensor we use the torsion tensor $d\theta \bmod \theta$, where $\theta = (\theta^1, \ldots, \theta^n)$ are independent $(1,0)$-forms; namely, our approach provides a complex version of the method of prolongation of subbundles of a tangent bundle, initiated by Cartan [C] and Gardner [Ga], and we determine a certain type of an almost complex manifold (M^{2n}, J) (see Definitions 2.1 and 2.5). Han and Kim [HK] gave a systematic method for deciding partial integrability of almost complex manifolds. This method is an algebraic approach rather than a vector field approach by analyzing the Nijenhuis tensor; they also used a theorem of Nirenberg which relates the closedness of a subbundle of $(T^*M)^{1,0}$ and the number of independent local holomorphic functions, applying the Newlander–Nirenberg theorem and the Frobenius theorem (see [BCH] or [Tr]).

In Sect. 2, we will first recall some basics of almost complex manifolds, namely $(J, \tilde{J})$-biholomorphism, integrability, local holomorphic functions with the independence condition. Then we will consider the partial integrability problem for generalized Thurston manifolds, introduced in [CFL]; as our main result we will prove that every generalized Thurston manifold is of type $(1,1)$ (see Theorem 2.7 and Sect. 3).

2. Preliminaries

2.1. Almost complex structures and integrability. We denote by M^{2n}, $n \geq 1$, a smooth (C^∞) manifold of dimension $2n$. Let J be an almost complex structure on M^{2n}, that is, for each $p \in M^{2n}$, $J_p : T_pM^{2n} \to T_pM^{2n}$ is a smooth (in p) linear map such that $J_p \circ J_p = -\mathrm{Id}$. Then the pair (M^{2n}, J) is called an *almost complex manifold.* Let $z^j = x^j + ix^{n+j}$ be the standard coordinate functions of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. The *standard complex structure* J_{st} on $\mathbb{C}^n$ is a representative example for almost complex structures, defined by

$$J_{\mathrm{st}}\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^{n+j}}, \qquad J_{\mathrm{st}}\left(\frac{\partial}{\partial x^{n+j}}\right) = -\frac{\partial}{\partial x^j},$$

for each $j = 1, \ldots, n$. For almost complex manifolds (M, J) and $(\tilde{M}, \tilde{J})$, a differentiable mapping $F : M \to \tilde{M}$ is said to be $(J, \tilde{J})$-*holomorphic* if its differential dF satisfies $dF \circ J = \tilde{J} \circ dF$; if moreover F is a diffeomorphism, then it is called a $(J, \tilde{J})$-*biholomorphism.* A *complex manifold* is an almost complex manifold such that, at each point in M, there exists a neighborhood (J, J_{st})-biholomorphic to an open subset of $\mathbb{C}^n$. In this case, we call J a *complex structure.*

Let us consider the real tangent bundle TM spanned by the vector fields

$$X_1, JX_1, \dots, X_n, JX_n.$$

Then the complexified tangent bundle $\mathbb{C}TM$ has the decomposition

$$\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M,$$

where $T^{1,0}M$ ($T^{0,1}M$, respectively) is the subbundle of rank n of eigenvectors of J associated with the eigenvalue i ($-i$, respectively). A section to $T^{1,0}M$ ($T^{1,0}M$, respectively) is called a $(1,0)$-*vector field* ($(0,1)$-*vector field*, respectively). More specifically, for each $j = 1, \dots, n$,

$$Z_j = \tfrac{1}{2}(X_j - iJX_j)$$

is a $(1,0)$-vector field. The fields $Z_1, \dots, Z_n$ are linearly independent; hence, they are generators of $T^{1,0}M$. Therefore, the complex vector fields $\overline{Z}_1, \dots, \overline{Z}_n$ are generators of $T^{0,1}M$.

Definition 2.1. An almost complex structure J is said to be *integrable* if the bundle $T^{1,0}M$ is formally integrable, that is,

$$[\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subset \Gamma(T^{1,0}M),$$

which means that the Lie bracket of any two sections of $T^{1,0}M$ is again a section of $T^{1,0}M$.

The integrability of J is equivalent to the vanishing of the Nijenhuis tensor N_J:

$$N_J(X,Y) = [\tilde{X}, \tilde{Y}] + J[J\tilde{X}, \tilde{Y}] + J[\tilde{X}, \tilde{Y}] - [J\tilde{X}, J\tilde{Y}]$$

for $p \in M$ and $X, Y \in T_pM$. Here $\tilde{X}$ and $\tilde{Y}$ are vector fields that coincide with X and Y respectively at the point p.

Definition 2.2. A complex-valued function f is said to be *holomorphic* if

$$\overline{Z}_j f = 0, \quad j = 1, \dots, n.$$

Holomorphic functions $f^1, \dots, f^m$ are said to be *independent* if

$$df^1 \wedge \cdots \wedge df^m \neq 0. \tag{2.1}$$

Theorem 2.3 ([NN]). *An almost complex structure J is a complex structure if and only if J is integrable.*

The Newlander–Nirenberg theorem can be rephrased as the closedness of a certain subbundle of the complexified cotangent bundle.

Consider the exterior algebra of differential forms with complex coefficients:

$$\Omega^* = \Omega^0 \oplus \Omega^1 \oplus \cdots \oplus \Omega^{2n},$$

where Ω^0 is the ring of smooth complex-valued functions and Ω^s ($1 \leq s \leq 2n$) is the module over Ω^0 of complex-valued smooth s-forms on M.

Definition 2.4. A subalgebra $\mathcal{I}$ of Ω^* is called an *algebraic ideal* if $\mathcal{I}$ satisfies the following:

(i) $\mathcal{I} \wedge \Omega^* \subset \mathcal{I}$;
(ii) if $\phi = \sum_{s=0}^{2n} \phi_s \in \mathcal{I}$, where $\phi_s \in \Omega^s$, then each ϕ_s is in $\mathcal{I}$ (homogeneity condition).

The homogeneity condition implies that $\mathcal{I}$ is a two-sided ideal, that is, $\Omega^* \wedge \mathcal{I} \subset \mathcal{I}$ (cf. [BC3G]). Let $\psi = (\psi^1, \ldots, \psi^l)$ be a system of smooth 1-forms on M. Let us denote by $\mathcal{I}(\psi)$, or simply (ψ), the algebraic ideal generated by ψ. Then each element of (ψ) can be written in the form

$$\sum_{k=1}^{l} \xi^k \wedge \psi^k$$

for some $\xi^k \in \Omega^*$. For two elements α and β of Ω^*, we say that

$$\alpha - \beta \equiv 0 \bmod (\psi)$$

if $\alpha - \beta \in (\psi)$. This concept can be generalized to complex differential forms. We denote by $\mathbb{C}T^*M$ the complexified cotangent bundle. Let

$$\theta^1, \ldots, \theta^n, \bar{\theta}^1, \ldots, \bar{\theta}^n$$

be the dual 1-forms of $Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n$. Then the integrability of J is equivalent to

$$d\theta^l \equiv 0 \bmod (\theta), \qquad \forall l = 1, \ldots, n,$$

where $\theta = (\theta^1, \ldots, \theta^n)$ and d is the exterior derivative. This means that (θ) is a closed ideal. If there exist n independent holomorphic functions $f^1, \ldots, f^n$ on M, then $(df^1, \ldots, df^n)$ is a closed ideal and $(\theta) = (df^1, \ldots, df^n)$.

2.2. Partial integrability and main result. Attempting to generalize the Newlander–Nirenberg theorem [NN], we are mainly concerned with the partial integrability on generalized Thurston manifolds. *Partial integrability* means that there exist m ($m \leq n$) holomorphic functions with the independence condition in (2.1). We use a complex version of Cartan–Gardner theory [C], [Ga] to determine partial integrability of generalized Thurston manifolds (cf. [BC3G] and [HK]).

For any subbundle $I \subset (T^*M)^{1,0}$ we denote by $\underline{I}$ the module over Ω^0 of smooth sections of I and by (I) the algebraic ideal of Ω^* generated by all smooth sections of I. Now we shall start with $I = I^{(0)} = (T^*M)^{1,0}$. We consider the composition of the exterior derivative $d : \underline{I} \to \Omega^2$ with the projection,

$$\underline{I} \xrightarrow{d} \Omega^2 \xrightarrow{\pi} \Omega^2/(I).$$

Let $\delta = \pi \circ d$. Then we define the submodule $\underline{I}^{(1)}$ of I as $\ker\delta$. Assuming that $I^{(1)}$ has constant rank on M, $I^{(1)}$ can be seen as a subbundle of $(T^*M)^{1,0}$. This subbundle is called the *first derived system* of $(T^*M)^{1,0}$. Now we have a short exact sequence of modules over Ω^0:

$$0 \to \underline{I}^{(1)} \to \underline{I} \xrightarrow{\delta} d\underline{I}/(I) \to 0.$$

Notice that J is integrable if $I^{(1)} = I$. If $I^{(1)} \subsetneq I$, then we consider

$$\underline{I}^{(1)} \xrightarrow{d} \Omega^2 \xrightarrow{\pi} \Omega^2/(I^{(1)});$$

we define the submodule $\underline{I}^{(2)}$ as $\ker\delta$ for $\delta = \pi \circ d$. Assuming that $\underline{I}^{(k-1)}$ has constant rank on M, we define inductively the *kth derived system* $I^{(k)}$ by

$$0 \to \underline{I}^{(k)} \to \underline{I}^{(k-1)} \xrightarrow{\delta} d\underline{I}^{(k-1)}/(I^{(k-1)}) \to 0.$$

Let ν be the smallest non-negative integer with $I^{(\nu)} = I^{(\nu+1)}$. Then we have a sequence of subbundles

$$(T^*M)^{1,0} := I = I^{(0)} \supset I^{(1)} \supset \cdots \supset I^{(\nu-1)} \supset I^{(\nu)}; \tag{2.2}$$

each $I^{(\nu)}$ is a closed subbundle of I.

Definition 2.5. An almost complex manifold (M, J) is of *type* (ν, q) if $I^{(\nu)}$ has rank q.

In [HK], Han and Kim gave a systematic approach to construct the closed subbundle of differentials of the maximal set of independent holomorphic functions (see Propositions 2.3 and 2.4 therein); they proved the following:

Theorem 2.6 ([HK]). *Let M^{2n}, $n \geq 2$, be a C^∞ manifold with C^∞ almost complex structure J. Let $(T^*M)^{1,0}$ be the bundle of $(1,0)$-forms. Then, under a generic assumption of non-degeneracy at each step of the construction, there exists a sequence of subbundles $(T^*M)^{1,0} := I^{(0)} \supset I^{(1)} \supset \cdots$ and a non-negative integer ν such that for $k = 0, 1, 2, \ldots,$*

(i) *$I^{(k+1)} \subsetneq I^{(k)}$ if $k < \nu$;*
(ii) *$I^{(k+1)} = I^{(k)}$ if $k \geq \nu$;*
(iii) *$dI^{(k+1)} \equiv 0 \bmod I^{(k)}$.*

Moreover, a function u is holomorphic if and only if $du \in I^{(\nu)}$, thus the number of independent holomorphic functions is equal to the rank of $I^{(\nu)}$.

For independent $(1,0)$-forms $\theta = (\theta^1, \ldots, \theta^n)$, we set

$$d\theta^l \equiv \sum_{j<k} T^l_{jk} \bar\theta^j \wedge \bar\theta^k \bmod (\theta), \tag{2.3}$$

where $j,k,l = 1,\dots,n$. Then (2.3) can be written as follows:

$$\begin{bmatrix} d\theta^1 \\ \vdots \\ d\theta^n \end{bmatrix} \equiv \underbrace{\begin{bmatrix} T^1_{12} & T^1_{13} & \cdots & T^1_{n-1,n} \\ \vdots & \vdots & \ddots & \vdots \\ T^n_{12} & T^n_{13} & \cdots & T^n_{n-1,n} \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} \bar\theta^1 \wedge \bar\theta^2 \\ \bar\theta^1 \wedge \bar\theta^3 \\ \vdots \\ \bar\theta^{n-1} \wedge \bar\theta^n \end{bmatrix} \bmod (\theta). \tag{2.4}$$

The $n \times \binom{n}{2}$ matrix $\mathcal{T}$ is called the *torsion* of J *with respect to the coframe* θ. By observing the torsion $\mathcal{T}$ and its prolongations, we determine the type (ν, q) (see [H] for prolongation theory).

Let G be the closed subgroup of $\mathrm{Gl}(r+3,\mathbb{C})$, $r \ge 1$, defined by

$$G = \left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 & a_1^{r+1} & a_1^{r+2} & 0 \\ 0 & 1 & \cdots & 0 & a_2^{r+1} & a_2^{r+2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_r^{r+1} & a_r^{r+2} & 0 \\ 0 & 0 & \cdots & 0 & 1 & a_{r+1}^{r+2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & e^{2\pi i\theta} \end{bmatrix} : a_i^j,\ \theta \in \mathbb{R} \right\}. \tag{2.5}$$

Then $G = H(1,r) \times S^1$, where $H(1,r)$ is the generalized Heisenberg group (cf. [CFG]) and S^1 is the unit circle. We denote by Γ the discrete subgroup of matrices with integer entries. Then we define the *generalized Thurston manifold* M^{2r+2} as the quotient of $H(1,r) \times S^1$ by the discrete subgroup Γ. In Section 3 we define an almost complex structure $\tilde J$ on M^{2r+2} for a fixed r, and we prove our main result:

THEOREM 2.7. *The generalized Thurston manifold* $(M^{2r+2}, \tilde J)$ *is of type* $(1,1)$.

REMARK 2.8. In the case where $r = 1$, the manifold M^4 is the Thurston manifold, introduced in [Th].

3. Types of generalized Thurston manifolds. We will show that each generalized Thurston manifold is of type $(1,1)$ for a certain almost complex complex structure $\tilde J$. Let $\{x^k, y, p^k, t\}$ be the local coordinate functions on G defined by

$$x^k(A) = a_k^{r+1}, \quad y(A) = a_{r+1}^{r+2}, \quad p^k(A) = a_k^{r+2}, \quad t(A) = \theta \quad (1 \le k \le r)$$

for all $A \in G$ in (2.5) (cf. [CFG]). Then we define

$$\alpha^k = dx^k, \quad \beta = dy, \quad \gamma^k = dp^k - x^k dy, \quad \eta = dt \quad (1 \le k \le r);$$

these 1-forms are linearly independent and left-invariant on G. Let

$$X_k = \frac{\partial}{\partial x^k}, \quad Y = \frac{\partial}{\partial y} + \sum_{j=1}^{r} x^j \frac{\partial}{\partial p^j}, \quad P_k = \frac{\partial}{\partial p^k}, \quad T = \frac{\partial}{\partial t}$$

be the vector fields dual to $\{\alpha^k, \beta, \gamma^k, \eta\}$.

First we define an almost complex structure J on G as follows:

$$JX_k := P_k, \quad JP_k := -X_k, \quad JY := T, \quad JT := -Y \quad (1 \le k \le r).$$

In [To], Tomassini introduced this almost complex structure on M^4. Considering the canonical projection $\pi : G \to M^{2r+2}$, we define left-invariant 1-forms on M^{2r+2} by

$$\pi^*(\tilde{\alpha}^k) = \alpha^k, \quad \pi^*(\tilde{\beta}) = \beta, \quad \pi^*(\tilde{\gamma}^k) = \gamma^k, \quad \pi^*(\tilde{\eta}) = \eta \quad (1 \le k \le r).$$

Then we define an almost complex structure $\tilde{J}$ on M^{2r+2} such that

$$\pi_*(JX) = \tilde{J}(\pi_* X) \quad \text{for every } X \in \Gamma(TG).$$

Now we shall determine the torsion matrix (2.4) for $(M^{2r+2}, \tilde{J})$. By letting

$$\begin{aligned} \theta^k &= \tfrac{1}{2}\{dx^k + idp^k - ix^k dy\} \quad (1 \le k \le r), \\ \theta^{r+1} &= \tfrac{1}{2}\{dy + idt\}, \end{aligned}$$

as generators of $(1,0)$-forms on G, we descend these $(1,0)$-forms to M^{2r+2}. Then, for some local coordinate functions $\{\tilde{x}^k, \tilde{y}, \tilde{p}^k, \tilde{t}\}$ on M^{2r+2}, we define

$$\begin{aligned} \tilde{\theta}^k &:= \tfrac{1}{2}\{\tilde{\alpha}^k + i\tilde{\gamma}^k\} = \tfrac{1}{2}\{d\tilde{x}^k + id\tilde{p}^k - i\tilde{x}^k d\tilde{y}\} \quad (1 \le k \le r), \\ \tilde{\theta}^{r+1} &:= \tfrac{1}{2}\{\tilde{\beta} + i\tilde{\eta}\} = \tfrac{1}{2}\{d\tilde{y} + id\tilde{t}\}, \end{aligned} \tag{3.1}$$

as generators of $(1,0)$-forms on M^{2r+2}, that is, $I = (\tilde{\theta}^1, \ldots, \tilde{\theta}^{r+1})$ for M^{2r+2} in (2.2).

By applying the exterior derivative d to (3.1) we obtain

$$\begin{aligned} d\tilde{\theta}^k &= -\tfrac{1}{2} i d\tilde{x}^k \wedge d\tilde{y} \\ &= -\tfrac{1}{2} i\{\tilde{\theta}^k + \bar{\tilde{\theta}}^k\} \wedge \{\tilde{\theta}^{r+1} + \bar{\tilde{\theta}}^{r+1}\} \quad (1 \le k \le r), \\ d\tilde{\theta}^{r+1} &= 0. \end{aligned} \tag{3.2}$$

Now we shall evaluate $d\tilde{\theta}^k$ mod (I). For each $1 \le k \le r$, we have

$$d\tilde{\theta}^k \equiv -\tfrac{1}{2} i \bar{\tilde{\theta}}^k \wedge \bar{\tilde{\theta}}^{r+1} \bmod (\tilde{\theta}^1, \ldots, \tilde{\theta}^{r+1}). \tag{3.3}$$

From (3.3) and the second equation in (3.2), we deduce that the torsion matrix $\mathcal{T}$ has rank r on M^{2r+2}. Considering the first derived system, we let $I^{(1)} = (\tilde{\theta}^{r+1})$. It is a closed subbundle of I. By Theorem 2.6, we conclude that $(M^{2r+2}, \tilde{J})$ is of type $(1,1)$. This completes the proof of Theorem 2.7.

REMARK 3.1. The type for $(M^{2r+2}, \tilde{J})$, $r \ge 1$, is independent of the choice of dimension r.

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