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Invariant scrambled sets and maximal distributional chaos

by Xinxing Wu and Peiyong Zhu (Chengdu)

Abstract. For the full shift (Σ_2, σ) on two symbols, we construct an invariant distributionally ϵ-scrambled set for all $0 < \epsilon < \operatorname{diam} \Sigma_2$ in which each point is transitive, but not weakly almost periodic.

1. Introduction. Throughout this paper, write $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. For a dynamical system (X, f), the sets of recurrent points, almost periodic points and weakly almost periodic points [Z] of f are denoted by $\mathrm{R}(f)$, $\mathrm{A}(f)$ and $\mathrm{W}(f)$, respectively.

Probably the first paper which defines 'chaos' in a mathematically rigorous way is that of Li and Yorke [LY]. Since then many other rigorous definitions of 'chaos' have been proposed. Each of these definitions tries to describe some kind of unpredictability in the evolution of the system. This was also the idea of Li and Yorke. Their fundamental observation was that in the case of the logistic equation on $[0, 1]$ it is possible to find two points with the property that during some iterations they are very close and during some other iterations the resulting values differ by almost 1 (the trajectories of these points approach two different endpoints of the interval). In [LY] it is proved that similar behavior is a common property in some class of interval maps (in particular with period 3) and since 1975 the complexity of a dynamical system has been a central topic of research.

A very important generalization of Li–Yorke chaos is that proposed by Schweizer and Smítal [SS], mainly because it is equivalent to positive topological entropy and some other concepts of chaos when restricted to the compact interval case [SS] or hyperbolic symbolic spaces [OW]. It is also remarkable that this equivalence does not transfer to higher dimensions, e.g. positive topological entropy does not imply distributional chaos in the case

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of triangular maps of the unit square [SŠ] (the same happens when the dimension is zero [P]).

In [WLY, WLCD, WCL], Wang et al. proved that there exists an uncountable distributionally scrambled set $\mathcal{T}$ satisfying $\mathcal{T} \subset \mathrm{R}(\sigma) - \mathrm{W}(\sigma) \subset \mathrm{R}(\sigma) - \mathrm{A}(\sigma)$ for the full shift σ on two symbols. In 2005, it was proved by Du [D] that an interval map f is turbulent if and only if there is an invariant scrambled set for f. Recently, Oprocha [O] constructed an invariant distributionally ϵ-scrambled set with $\epsilon = 1/8$ for the full shift σ and proved that exactly the same characterization is valid for distributional chaos. Motivated by the above results, in this paper we extend the approaches in [O, WLY, WLCD, WCL] and obtain an invariant distributionally ϵ-scrambled set with any $0 < \epsilon < \operatorname{diam} \Sigma_2$ which consists of transitive points for the full shift.

2. Preliminaries. Let (X, f) be a dynamical system with metric d. According to Li and Yorke [LY], a subset $D \subset X$ is a *scrambled set* (for f) if any different points x and y of D are proximal and not asymptotic, i.e.,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

We say f is *Li–Yorke chaotic* if there exists an uncountable scrambled set.

A dynamical system is *transitive* if for any pair of non-empty open subsets A, B of X, there exists $n \in \mathbb{N}$ such that $f^n(A) \cap B \neq \emptyset$. It is clear that the set of *transitive points* of f, i.e., points with dense orbits, is a dense G_δ subset, which will be denoted by Trans_f.

For any pair $(x, y) \in X \times X$ and for any $n \in \mathbb{N}$, the *distributional function* $F^n_{xy} : \mathbb{R} \to [0, 1]$ is defined by

$$F^n_{x,y}(t) = \frac{1}{n} |\{i : d(f^i(x), f^i(y)) < t,\ 1 \le i \le n\}|,$$

where $|A|$ denotes the cardinality of the set A. Define the *lower* and *upper distributional functions* generated by f, x and y as

$$F_{x,y}(t) = \liminf_{n\to\infty} F^n_{x,y}(t) \quad \text{and} \quad F^*_{x,y}(t) = \limsup_{n\to\infty} F^n_{x,y}(t).$$

respectively. Both $F_{x,y}$ and $F^*_{x,y}$ are non-decreasing and $F_{x,y} \le F^*_{x,y}$.

A dynamical system (X, f) is *distributionally ϵ-chaotic* if there exists an uncountable subset $S \subset X$ such that for any distinct $x, y \in S$, we have $F^*_{x,y}(t) = 1$ for all $t > 0$ and $F_{x,y}(\epsilon) = 0$. The set S is then a *distributionally ϵ-scrambled set* and the pair (x, y) a *distributionally ϵ-chaotic pair*. If (X, f) is distributionally ϵ-chaotic for all $0 < \epsilon < \operatorname{diam} X$, we say that (X, f) exhibits *maximal distributional chaos.*

Let $\mathcal{A}$ be a finite set (an *alphabet*). An *infinite word* over $\mathcal{A}$ is a map $x : \mathbb{N} \to \mathcal{A}$. The set of all infinite words over $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathbb{N}}$. The set $\mathcal{A}$ is given the discrete topology and $\mathcal{A}^{\mathbb{N}}$ is endowed with the product topology.

This topology is metrizable and may be equivalently defined by the following metric. For any pair $x = x_1x_2\cdots$, $y = y_1y_2\cdots \in \mathcal{A}^{\mathbb{N}}$, put

$$\rho(x,y) = \sum_{n=1}^{\infty} \frac{\delta(x_n,y_n)}{2^n}, \quad \text{where} \quad \delta(x_n,y_n) = \begin{cases} 1, & x_n \neq y_n, \\ 0, & x_n = y_n. \end{cases}$$

The two-letter alphabet will be denoted as $\Sigma = \{0,1\}$. The set $\Sigma^{\mathbb{N}}$ with the above-defined metric ρ will be denoted as Σ_2.

Now let us define the *shift map* $\sigma : \Sigma_2 \to \Sigma_2$ by the formula

$$(\sigma(x))_j = x_{j+1}.$$

The pair (Σ_2, σ) is said to be the *full shift* over Σ. Every closed set $X \subset \Sigma_2$ such that $\sigma(X) \subset X$ is called a *shift*. Let $A = a_1\cdots a_n \in \Sigma^n$. Denote $\overline{A} = \overline{a_1}\cdots\overline{a_n}$ and call it the *inverse* of A, where

$$\overline{a_i} = \begin{cases} 0, & a_i = 1, \\ 1, & a_i = 0. \end{cases}$$

Lemma 2.1 ([LF, Lemma 5]). *Σ_2 has an uncountable subset E such that for any different points $x = x_1x_2\cdots$, $y = y_1y_2\cdots$ in E, $x_n = y_n$ for infinitely many n and $x_m \neq y_m$ for infinitely many m.*

3. Invariant distributionally ϵ-scrambled set in the full shift. In this section, we will construct an invariant set for the full shift on two symbols, which is distributionally ϵ-scrambled for all $0 < \epsilon < 1$. This shows that the full shift can exhibit maximal distributional chaos on an invariant subset which consists of transitive points (see Theorem 3.1).

Theorem 3.1. *There exists an uncountable invariant subset $D \subset \Sigma_2$ such that:*

(i) *$D \subset \mathrm{Trans}_\sigma - \mathrm{W}(\sigma)$.*

(ii) *D is a distributional ϵ-scrambled set for σ with any $0 < \epsilon < \operatorname{diam}\Sigma_2$.*

Proof. Let $L_1 = \mathscr{L}_1 = 2$, $L_i = 2^{L_1+\ldots+L_{i-1}}$ and $\mathscr{L}_i = \sum_{j=1}^{i} L_j$ for $i > 1$. Let $p_1, p_2, \ldots$ be all odd prime numbers arranged in the natural order. For any $n, m \in \mathbb{N}$, set

$$\begin{aligned}
\mathscr{A}_n &= \{j \in \mathbb{N} : \mathscr{L}_{4n-1} \le j < \mathscr{L}_{4n}\}, \\
\mathscr{B}_n &= \{j \in \mathbb{N} : \mathscr{L}_{4n-3} \le j < \mathscr{L}_{4n-2}\}, \\
\mathscr{C}_{n,m} &= \{j \in \mathbb{N} : \mathscr{L}_{p_n^m-1} + (2k+1)m \le j < \mathscr{L}_{p_n^m-1} + 2(k+1)m, \\
&\qquad\qquad 1 \le 2k+1 \le [L_{p_n^m}/m] - 1\}, \\
\mathscr{D}_{n,m} &= \{j \in \mathbb{N} : \mathscr{L}_{p_n^m-1} + (2k)m \le j < \mathscr{L}_{p_n^m-1} + (2k+1)m, \\
&\qquad\qquad 0 \le 2k \le [L_{p_n^m}/m] - 1\},
\end{aligned}$$

where $[t]$ denotes the integral part of t. Take an uncountable subset $E \subset \Sigma_2$ such that for any different points $x = x_1x_2\cdots$, $y = y_1y_2\cdots \in E$, $x_n = y_n$ for infinitely many n, and $x_m \neq y_m$ for infinitely many m. By Lemma 2.1, such a subset exists. Given any fixed $z = z_1z_2\cdots \in \mathrm{Trans}_\sigma$, define $g : E \to \Sigma_2$ by $g(x) = y_1y_2\cdots$ for all $x = x_1x_2\cdots \in E$, where

$$y_j = \begin{cases} x_n, & j \in \mathscr{A}_n, n \in \mathbb{N}, \\ z_{j-\mathscr{L}_{4n-3}+1}, & j \in \mathscr{B}_n,\ n \in \mathbb{N}, \\ 1, & j \in \mathscr{C}_{n,m},\ n,m \in \mathbb{N}, \\ 0, & j \in \mathscr{D}_{n,m},\ n,m \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases}$$

Set

$$D = \bigcup_{n\in\mathbb{Z}^+} \sigma^n(g(E)).$$

Clearly, $\sigma(D) \subset D$ and D is uncountable, since E is uncountable and g is injective. Moreover, it is not difficult to check that $D \subset \mathrm{Trans}_\sigma - \mathrm{W}(\sigma)$.

Given any distinct $a, b \in D$, from the construction it is easy to see that there exist $c = c_1c_2\cdots, d = d_1d_2\cdots \in E$ and $p, q \in \mathbb{Z}^+$ such that $a = \sigma^p(g(c))$ and $b = \sigma^q(g(d))$. Let $Q = \max\{p, q\}$. To prove that (a, b) is a distributionally ϵ-chaotic pair for any $0 < \epsilon < 1$, we consider two cases:

Case 1: $c \neq d$. Observe that there exist sequences $\{m_i\}_{i=1}^\infty$, $\{n_i\}_{i=1}^\infty \subset \mathbb{N}$ such that $c_{m_i} = d_{m_i}$ and $c_{n_i} = \overline{d_{n_i}}$ for all $i \geq 1$. It follows that for any $j \in \mathscr{A}_{m_i}$, $c_{m_i} = g(c)_j = g(d)_j = d_{m_i}$, while for any $j \in \mathscr{A}_{n_i}$, $c_{n_i} = g(c)_j \neq g(d)_j = d_{n_i}$.

First, it is easy to see that the first $\mathscr{L}_{4m_i} - (j + Q + 1)$ letters of $\sigma^j(a) = \sigma^{j+p}(g(c))$ coincide with the correspondings letters of $\sigma^j(b) = \sigma^{j+q}(g(d))$ for $\mathscr{L}_{4m_i-1} \leq j \leq \mathscr{L}_{4m_i-1} + L_{4m_i}/2$, so

$$\begin{aligned} \rho(\sigma^j(a), \sigma^j(b)) &\leq \sum_{n=\mathscr{L}_{4m_i}-(j+Q)}^{\infty} \frac{1}{2^n} \leq \sum_{n=L_{4m_i}/2-Q}^{\infty} \frac{1}{2^n} \\ &= 2^{-(L_{4m_i}/2-(Q+1))} \to 0 \quad (i \to \infty). \end{aligned}$$

Thus for given $t > 0$, there exists $N \in \mathbb{N}$ such that for any $i \geq N$ and $\mathscr{L}_{4m_i-1} \leq j \leq \mathscr{L}_{4m_i-1} + L_{4m_i}/2$,

$$\rho(\sigma^j(a), \sigma^j(b)) < t.$$

Consequently,

$$\begin{aligned} F^*_{a,b}(t) &= \limsup_{n\to\infty} \frac{1}{n} |\{k : \rho(\sigma^k(a), \sigma^k(b)) < t,\ 1 \leq k \leq n\}| \\ &\geq \limsup_{i\to\infty} \frac{1}{\mathscr{L}_{4m_i-1} + L_{4m_i}/2} |\mathscr{E}_i| \geq \limsup_{i\to\infty} \frac{L_{4m_i}/2}{\mathscr{L}_{4m_i-1} + L_{4m_i}/2} \\ &= \limsup_{i\to\infty} \frac{2^{\mathscr{L}_{4m_i-1}-1}}{\mathscr{L}_{4m_i-1} + 2^{\mathscr{L}_{4m_i-1}-1}} = 1, \end{aligned}$$

where

$$\mathscr{E}_i = \{k : \rho(\sigma^k(a), \sigma^k(b)) < t,\, 1 \le k \le \mathscr{L}_{4m_i-1} + L_{4m_i}/2\}.$$

Second, it is easy to see that the first $\mathscr{L}_{4n_i} - (j+Q+1)$ letters of $\sigma^j(a) = \sigma^{j+p}(g(c))$ are each different from the corresponding letters of $\sigma^j(b) = \sigma^{j+q}(g(d))$ for $\mathscr{L}_{4n_i-1} \le j \le \mathscr{L}_{4n_i-1} + L_{4n_i}/2$, so

$$\begin{aligned}\rho(\sigma^j(a), \sigma^j(b)) &\ge \sum_{n=1}^{\mathscr{L}_{4n_i}-(j+Q+1)} \frac{1}{2^n} \ge \sum_{n=1}^{L_{4n_i}/2-(Q+1)} \frac{1}{2^n} \\ &= 1 - 2^{-(L_{4n_i}/2-(Q+1))} \to 1 \quad (i \to \infty).\end{aligned}$$

For any $0 < \epsilon < 1$, there exists $M \in \mathbb{N}$ such that for any $i \ge M$ and any $\mathscr{L}_{4n_i-1} \le j \le \mathscr{L}_{4n_i-1} + L_{4n_i}/2$,

$$\rho(\sigma^j(a), \sigma^j(b)) \ge \epsilon.$$

Thus

$$\begin{aligned}F_{a,b}(\epsilon) &= \liminf_{n\to\infty} \frac{1}{n} |\{k : \rho(\sigma^k(a), \sigma^k(b)) < \epsilon,\, 1 \le k \le n\}| \\ &\le \liminf_{i\to\infty} \frac{1}{\mathscr{L}_{4n_i-1} + L_{4n_i}/2} |\mathscr{F}_i| \le \liminf_{i\to\infty} \frac{\mathscr{L}_{4n_i-1}}{\mathscr{L}_{4n_i-1} + L_{4n_i}/2} \\ &= \liminf_{i\to\infty} \frac{\mathscr{L}_{4n_i-1}}{\mathscr{L}_{4n_i-1} + 2^{\mathscr{L}_{4n_i-1}-1}} = 0,\end{aligned}$$

where

$$\mathscr{F}_i = \{k : \rho(\sigma^k(a), \sigma^k(b)) < \epsilon,\, 1 \le k \le \mathscr{L}_{4n_i-1} + L_{4n_i}/2\}.$$

Case 2: $c = d$. Then $p \ne q$. Without loss of generality, we may assume that $p < q$. It is easy to see that the first $\mathscr{L}_{4m} - (j+Q+1)$ letters of $\sigma^j(a) = \sigma^{j+p}(g(c))$ coincide with the corresponding letters of $\sigma^j(b) = \sigma^{j+q}(g(d))$ for $\mathscr{L}_{4m-1} \le j \le \mathscr{L}_{4m-1} + L_{4m}/2$, so

$$\begin{aligned}\rho(\sigma^j(a), \sigma^j(b)) &\le \sum_{n=\mathscr{L}_{4m}-(j+Q)}^{\infty} \frac{1}{2^n} \le \sum_{n=L_{4m}/2-Q}^{\infty} \frac{1}{2^n} \\ &= 2^{-(L_{4m}/2-(Q+1))} \to 0 \quad (m \to \infty).\end{aligned}$$

Thus given any $t > 0$, there exists $N' \in \mathbb{N}$ such that for any $m \ge N'$ and $\mathscr{L}_{4m-1} \le j \le \mathscr{L}_{4m-1} + L_{4m}/2$,

$$\rho(\sigma^j(a), \sigma^j(b)) < t.$$

Therefore,

$$\begin{aligned} F^*_{a,b}(t) &= \limsup_{n\to\infty} \frac{1}{n}|\{k : \rho(\sigma^k(a), \sigma^k(b)) < t,\ 1 \le k \le n\}| \\ &\ge \limsup_{m\to\infty} \frac{1}{\mathscr{L}_{4m-1} + L_{4m}/2}|\mathscr{G}_m| \ge \limsup_{m\to\infty} \frac{L_{4m}/2}{\mathscr{L}_{4m-1} + L_{4m}/2} \\ &= \limsup_{m\to\infty} \frac{2^{\mathscr{L}_{4m-1}-1}}{\mathscr{L}_{4m-1} + 2^{\mathscr{L}_{4m-1}-1}} = 1, \end{aligned}$$

where

$$\mathscr{G}_m = \{k : \rho(\sigma^k(a), \sigma^k(b)) < t,\ 1 \le k \le \mathscr{L}_{4m-1} + L_{4m}/2\}.$$

According to the construction of c, it is not difficult to check that for any $n, m \in \mathbb{N}$ and any $\mathscr{L}_{p_n^m-1} \le j < \mathscr{L}_{p_n^m-1} + L_{p_n^m}/2 - m$, the first $\mathscr{L}_{p_n^m-1} + [L_{p_n^m}/m]m - (j+m+1)$ letters of $\sigma^j(g(c))$ are each different from the corresponding letters of $\sigma^{j+m}(g(c))$. In particular, for any $\mathscr{L}_{p_n^{q-p}-1} \le j + p < \mathscr{L}_{p_n^{q-p}-1} + L_{p_n^{q-p}}/2 - (q-p)$,

$$\begin{aligned} \rho(\sigma^j(a), \sigma^j(b)) &= \rho\big(\sigma^{j+p}(g(c)), \sigma^{j+p+(q-p)}(g(c))\big) \\ &\ge \sum_{n=1}^{\mathscr{L}_{p_n^{q-p}-1} + \left[\frac{L_{p_n^{q-p}}}{q-p}\right](q-p) - (j+q-p+1)} \frac{1}{2^n} \\ &\ge \sum_{n=1}^{\left[\frac{L_{p_n^{q-p}}}{q-p}\right](q-p) - L_{p_n^{q-p}}/2 - 1 + p} \frac{1}{2^n} \ge \sum_{n=1}^{L_{p_n^{q-p}}/2 + 2p - q - 1} \frac{1}{2^n} \\ &= 1 - 2^{-(L_{p_n^{q-p}}/2 + 2p - q - 1)} \to 1 \quad (n \to \infty). \end{aligned}$$

Then for any $0 < \epsilon < 1$, there exists $M' \in \mathbb{N}$ such that for any $n \ge M'$ and $\mathscr{L}_{p_n^{q-p}-1} \le j + p < \mathscr{L}_{p_n^{q-p}-1} + L_{p_n^{q-p}}/2 - (q-p)$,

$$\rho(\sigma^j(a), \sigma^j(b)) \ge \epsilon.$$

Thus

$$\begin{aligned} F_{a,b}(\epsilon) &= \liminf_{n\to\infty} \frac{1}{n}|\{k : \rho(\sigma^k(a), \sigma^k(b)) < \epsilon, 1 \le k \le n\}| \\ &\le \liminf_{n\to\infty} \frac{1}{\mathscr{L}_{p_n^{q-p}-1} + L_{p_n^{q-p}}/2 - q}|\mathscr{H}_n| \\ &\le \liminf_{n\to\infty} \frac{\mathscr{L}_{p_n^{q-p}-1}}{\mathscr{L}_{p_n^{q-p}-1} + L_{p_n^{q-p}}/2 - q} \\ &= \liminf_{n\to\infty} \frac{\mathscr{L}_{p_n^{q-p}-1}}{\mathscr{L}_{p_n^{q-p}-1} + 2^{\mathscr{L}_{p_n^{q-p}-1}-1} - q} = 0, \end{aligned}$$

where

$$\mathscr{H}_n = \{k : \rho(\sigma^k(a), \sigma^k(b)) < \epsilon,\ 1 \le k \le \mathscr{L}_{p_n^{q-p}-1} + L_{p_n^{q-p}}/2 - q\}.$$

Summing up, by the arbitrariness of a, b, we find that D is a distributionally ϵ-scrambled set for any $0 < \epsilon < \operatorname{diam} \Sigma_2$. The proof of Theorem 3.1 is complete. ■

REMARK 3.2. Noting that $\mathrm{Trans}_\sigma - \mathrm{W}(\sigma) \subset \mathrm{R}(\sigma) - \mathrm{W}(\sigma) \subset \mathrm{R}(\sigma) - \mathrm{A}(\sigma)$, it follows that [O, Theorem 1], [WLCD, Theorem] and [WCL, Theorem A] are direct corollaries of Theorem 3.1.

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Xinxing Wu (corresponding author)
School of Mathematics
University of Electronic Science
and Technology of China
Chengdu, Sichuan, 611731
People's Republic of China
and
Department of Electronic Engineering
City University of Hong Kong
Hong Kong SAR
People's Republic of China
E-mail: wuxinxing5201314@163.com

Peiyong Zhu
School of Mathematics
University of Electronic Science
and Technology of China
Chengdu, Sichuan, 611731
People's Republic of China
E-mail: zpy6940@sina.com.cn

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