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Piecewise-deterministic Markov processes

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Abstract. Poisson driven stochastic differential equations on a separable Banach space are examined. Some sufficient conditions are given for the asymptotic stability of a Markov operator P corresponding to the change of distribution from jump to jump. We also give criteria for the continuous dependence of the invariant measure for P on the intensity of the Poisson process.

1. Introduction. We will consider the stochastic differential equation of the form

$$d\xi(t) = a(\xi(t))dt + \int_{\Theta} \sigma(\xi(t), \theta)\, \mathcal{N}_p(\Lambda_\xi(dt), d\theta) \qquad \text{for } t \geq 0 \tag{1.1}$$

with the initial condition

$$\xi(0) = \xi_0, \tag{1.2}$$

where

$$\Lambda_\xi(t) = \int_0^t \lambda(\xi(s))\, ds \tag{1.3}$$

and $(\xi(t))_{t\geq 0}$ is a stochastic process with values in a separable Banach space X, the functions a and σ are deterministic, and $\mathcal{N}_p$ is a Poisson random counting measure. In (1.3) the function $\lambda\colon X \to \mathbb{R}^+$, called the *intensity* of the Poisson process, is bounded and Lipschitzian. The process Λ_ξ influences the time at which jumps occur and it depends on the solution ξ of the problem (1.1), (1.2). The process $(\mathcal{N}_p(\Lambda_\xi(t), A))_{t\geq 0}$ describes the occurrence of jumps. The fact that $\mathcal{N}_p$ depends on the solution is crucial.

The solution ξ is a Markov process which is piecewise-deterministic. It evolves deterministically until a random time (depending on position) when it jumps to a new random state. Such processes feature significantly in contemporary monographs devoted to Markov processes (see [3]). They have

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been used in models of numerous phenomena, such as the growth of a size-structured population of cells [4, 5, 15], fragmentation processes [23, 24], and in population dynamics [18]. Recently the problem (1.1), (1.2) has appeared in financial investment models [1]. For further examples (short noise, photoconductive detectors, etc.) see [25] .

In the nature of things, the probabilistic description of the solution of (1.1), (1.2) leads us to the examination of a semigroup $(P^t)_{t\geq 0}$ of Markov operators acting on the space of Borel measures on X. This semigroup describes the distribution of the position of a trajectory at any time. Moreover, there exists a discrete semigroup $(P^n)_{n\geq 1}$ defined by the jump operator P. This operator describes the distribution of the position of a trajectory from one perturbation moment to the next.

The aim of this paper is to give criteria for the asymptotic stability of the discrete semigroup $(P^n)_{n\geq 1}$. With this end in view, we will prove the nonexpansiveness of P and its global concentration. Next, we will verify the condition for the local concentration of P by using the asymptotic stability of the associated operator $\overline{P}$ with constant intensity $\overline{\lambda} = \sup_{x\in X} \lambda(x)$. Moreover, we will prove a theorem on continuous dependence of the invariant measure for P on the intensity of the Poisson process. This result strengthens the one of [29] obtained in the case $\lambda = \text{const}$. We will examine the Markov operator P corresponding to the change of distribution of $\xi(t)$ from jump to jump and not the semigroup $(P^t)_{t\geq 0}$ describing the distribution of $\xi(t)$ at any time.

There are many papers devoted to piecewise-deterministic Markov processes, but usually in the case $\lambda = \text{const}$ (see [8, 9, 10, 11, 28]) or in the case $X = \mathbb{R}^d$ (see [2, 16, 30]). Similar problems in the space $L^1(\mathbb{R}^d)$ were considered in [19, 22, 20, 21].

The paper is organized as follows. Sections 2 and 3 have an introductory character. Section 2 presents the notation and some known facts concerning Markov operators and point processes. In Section 3 we define the solution of the problem (1.1), (1.2) and derive a formula for the operator P. In Section 4 we give criteria for the asymptotic stability of P, and in Section 5 we prove continuous dependence.

2. Preliminaries. Let $(X, \|\cdot\|)$ be a separable Banach space. We denote by $\mathcal{B}(X)$ and $\mathcal{B}_b(X)$ the σ-algebra of Borel subsets of X and the algebra of bounded Borel subsets of X, respectively. For $A \in \mathcal{B}(X)$ we denote by $\operatorname{diam} A$ the diameter of A, i.e. $\operatorname{diam} A = \sup\{\|x-y\| : x, y \in X\}$. Let $A \subset X$ and $r > 0$. We denote by $\mathcal{O}(A,r)$ the closed r-neighbourhood of A, i.e.

$$\mathcal{O}(A,r) = \Big\{x \in X : \inf_{y\in A} \|x-y\| \leq r\Big\}.$$

Let $B(X)$ denote the space of all bounded, Borel, real-valued functions on X equipped with the supremum norm, and $C(X)$ the subspace of $B(X)$ which consists of all bounded continuous functions. By $\mathcal{M}_{\mathrm{sig}} \supset \mathcal{M} \supset \mathcal{M}_1$ we denote, respectively, the space of all finite signed Borel measures on X; the subset of all nonnegative finite Borel measures on X; and the subset of all probability measures, called distributions. For any $A \in \mathcal{B}(X)$, we set

$$\mathcal{M}_1^A = \big\{\mu \in \mathcal{M}_1 : \mu(X \setminus A) = 0\big\}.$$

We will use the abbreviation

$$\langle f, \mu\rangle = \int_X f(x)\,\mu(dx) \quad \text{for } f \in B(X),\ \mu \in \mathcal{M}_{\mathrm{sig}}.$$

An operator $P\colon \mathcal{M} \to \mathcal{M}$ is called a *Markov operator* if:

(i) $P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$ for $\lambda_1, \lambda_2 \geq 0$ and $\mu_1, \mu_2 \in \mathcal{M}$,
(ii) $P\mu(X) = \mu(X)$ for $\mu \in \mathcal{M}$.

An operator $U\colon B(X) \to B(X)$ is called *dual* to P if

$$\langle Uf, \mu\rangle = \langle f, P\mu\rangle \quad \text{for } f \in B(X),\ \mu \in \mathcal{M}. \tag{2.1}$$

If there exists a dual operator, it is unique. Setting $\mu = \delta_x$ in (2.1), we obtain

$$Uf(x) = \langle f, P\delta_x\rangle \quad \text{for } f \in B(X),\ x \in X. \tag{2.2}$$

If an operator $U\colon B(X) \to B(X)$ is dual to P then U is a linear operator satisfying the following conditions:

$$\|U\| = 1; \quad U1_X = 1_X; \quad Uf \geq 0 \quad \text{for } f \geq 0;$$
$$Uf_n \downarrow 0 \quad \text{for } f_n \downarrow 0,\ (f_n)_{n\geq 1} \subset B(X).$$

A dual operator U can be extended to the set of all, not necessarily bounded, Borel functions $f\colon X \to \mathbb{R}_+$ in such a way that the resulting operator satisfies (2.1). Namely, we set

$$Uf(x) = \lim_{n\to\infty} Uf_n(x), \quad \text{where} \quad (f_n)_{n\geq 1} \subset B(X) \text{ with } f_n \uparrow f.$$

Given a dual operator U, its corresponding Markov operator P is of the form

$$P\mu(A) = \langle U1_A, \mu\rangle \quad \text{for } \mu \in \mathcal{M},\ A \in \mathcal{B}(X).$$

A Markov operator P is called a *Feller operator* if there exists an operator U dual to P such that $U(C(X)) \subset C(X)$. In $\mathcal{M}_{\mathrm{sig}}$ we introduce the *Fortet–Mourier norm* (see [17])

$$\|\mu\|_{\mathrm{FM}} = \sup\{|\langle f, \mu\rangle| : f \in \mathcal{F}\},$$

where $\mathcal{F} = \{f \in C(X) : |f(x)| \leq 1,\ |f(x) - f(y)| \leq \|x - y\| \text{ for } x, y \in X\}$. It is well known that $(\mathcal{M}_{\mathrm{sig}}, \|\cdot\|_{\mathrm{FM}})$ is a normed vector space. Furthermore, $(\mathcal{M}_1, \|\cdot\|_{\mathrm{FM}})$ is a complete space, and convergence in the Fortet–Mourier norm on $\mathcal{M}_1$ is equivalent to weak convergence. We say that a sequence $(\mu_n)_{n\geq 1} \subset \mathcal{M}_1$ *converges weakly* to $\mu \in \mathcal{M}_1$ (written $\mu_n \to \mu$) if

$$\lim_{n\to\infty} \langle f, \mu_n\rangle = \langle f, \mu\rangle \quad \text{for } f \in C(X).$$

Apart from the Fortet–Mourier norm, one can introduce on some subset of $\mathcal{M}_1$ another norm called the Hutchinson norm. We set

$$\mathcal{M}_{1,\mathrm{H}} = \Big\{\mu \in \mathcal{M}_1 \colon \int_X \|x\|\,\mu(dx) < \infty\Big\}.$$

The *Hutchinson norm* is defined by the formula

$$\|\mu\|_{\mathrm{H}} = \sup\{|\langle f, \mu\rangle| : f \in \mathcal{H}\} \quad \text{for } \mu \in \mathcal{M}_{1,\mathrm{H}},$$

where $\mathcal{H} = \{f \in C(X) : f \geq 0,\ |f(x) - f(y)| \leq \|x - y\| \text{ for } x, y \in X\}$. Note that

$$\|\mu_1 - \mu_2\|_{\mathrm{FM}} \leq \|\mu_1 - \mu_2\|_{\mathcal{H}} \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_{1,\mathrm{H}}. \tag{2.3}$$

A Markov operator P is called *nonexpansive* with respect to the norm $\|\cdot\|_{\mathrm{FM}}$ if

$$\|P\mu_1 - P\mu_2\|_{\mathrm{FM}} \leq \|\mu_1 - \mu_2\|_{\mathrm{FM}} \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1.$$

A measure $\mu \in \mathcal{M}$ is called *invariant* or *stationary* for a Markov operator P if $P\mu = \mu$. A Markov operator P is called *asymptotically stable* if there is a stationary distribution $\mu_* \in \mathcal{M}_1$ such that

$$\lim_{n\to\infty} \|P^n\mu - \mu_*\|_{\mathrm{FM}} = 0 \quad \text{for } \mu \in \mathcal{M}_1.$$

We now recall some concepts used for point processes [12, pp. 42–43]. Let $(\Omega, \mathcal{E}, \mathbb{P})$ be a complete probability space, and $(\Theta, \mathcal{G})$ be a measurable space. A mapping $\overline{p}\colon \mathcal{D}_{\overline{p}} \to \Theta$, where $\mathcal{D}_{\overline{p}}$ is a countable subset of $(0, \infty)$, is called a *point function* on Θ. Such a function $\overline{p}$ defines a *counting measure* $\mathcal{N}_{\overline{p}}(d\tau, d\theta)$ on the measurable space $(\mathbb{R}_+ \times \Theta, \mathcal{B}(\mathbb{R}_+) \times \mathcal{G})$ by the formula

$$\mathcal{N}_{\overline{p}}([0, t] \times K) = \operatorname{card}\{s \in \mathcal{D}_{\overline{p}}\colon s \leq t,\ \overline{p}(s) \in K\} \quad \text{for } t \geq 0,\ K \in \mathcal{G}.$$

We assume that $\mathcal{N}_{\overline{p}}([0, t] \times K) < \infty$ for all $t \geq 0, K \in \mathcal{G}$. To abbreviate, we write $\mathcal{N}_{\overline{p}}(t, K)$ instead of $\mathcal{N}_{\overline{p}}([0, t] \times K)$. For the point function $\overline{p}$, we also write $\overline{p} = (\tau_n, \theta_n)_{n\geq 1}$, where $\theta_n = \overline{p}(\tau_n)$, $\tau_n \in \mathcal{D}_{\overline{p}}$. Let Π_Θ be the collection of all point functions on Θ, and $\mathcal{B}(\Pi_\Theta)$ be the smallest σ-algebra on Π_Θ with respect to which all mappings $\{\overline{p} \mapsto \mathcal{N}_{\overline{p}}(t, K) : t > 0,\ K \in \mathcal{G}\}$ are measurable. A mapping $p\colon \Omega \to \Pi_\Theta$ which is $\mathcal{F}/\mathcal{B}(\Pi_\Theta)$-measurable is called a *point process.* A point process p is a *Poisson point process* if

- (i) for each $Z \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{G}$, the mapping $\omega \mapsto \mathcal{N}_{p(\omega)}(Z)$ is a Poisson distributed random variable, i.e. $\mathbb{P}(\mathcal{N}_p(Z) = k) = ([n_p(Z)]^k/k!)e^{-n_p(Z)}$, where $n_p(Z) = \mathbb{E}\mathcal{N}_p(Z)$,
- (ii) if $Z_1, \ldots, Z_l \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{G}$ are disjoint sets, then the random variables $\mathcal{N}_p(Z_1), \ldots, \mathcal{N}_p(Z_l)$ are mutually independent.

The process p defines a *Poisson random counting measure* $\mathcal{N}_p$. Let κ be a measure on $(\Theta,\mathcal{G})$ such that $\kappa(\Theta)=1$. The Poisson point process is *stationary* if $\mathbb{E}\mathcal{N}_p(t,K)=t\kappa(K)$ for all $t>0, K\in\mathcal{G}$. The measure κ is called the *characteristic measure* of p.

3. Poisson driven Markov process. In this section we study the solution of the problem (1.1), (1.2). Throughout the paper we assume:

(i) The function $a\colon X\to X$ is Lipschitzian: $\|a(x)-a(y)\|\le l_a\|x-y\|$ for $x,y\in X$.

(ii) There is a measure space $(\Theta,\mathcal{G},\kappa)$ with $\kappa(\Theta)=1$ such that the perturbation coefficient $\sigma\colon X\times\Theta\to X$ is a $\mathcal{B}(X)\times\mathcal{G}/\mathcal{B}(X)$-measurable function such that $\sigma(z,\cdot)\in L^2(\kappa)$ for each $z\in X$ and

$$\|\sigma(x,\cdot)-\sigma(y,\cdot)\|_{L^2(\kappa)}\le l_\sigma\|x-y\| \quad \text{for } x,y\in X.$$

(iii) The function $\lambda\colon X\to\mathbb{R}^+$ is Lipschitzian:

$$\|\lambda(x)-\lambda(y)\|\le l_\lambda\|x-y\| \quad \text{for } x,y\in X,$$

and

$$0<\underline{\lambda}=\inf_{x\in X}\lambda(x),\quad \overline{\lambda}=\sup_{x\in X}\lambda(x)<\infty.$$

(iv) There are given a probability space $(\Omega,\mathcal{F},\mathbb{P})$, a sequence $(\tau_n)_{n\ge0}$ of nonnegative random variables, and a sequence $(\theta_n)_{n\ge0}$ of random elements with values in Θ. The variables $\varrho_n=\tau_{n+1}-\tau_n$ $(\tau_0=0)$ are nonnegative, independent and have the same distribution with density function e^{-r} for $r\ge0$. The elements θ_n are independent, and have the same distribution κ. Moreover, the sequences $(\tau_n)_{n\ge0}$ and $(\theta_n)_{n\ge0}$ are independent.

By a *solution* of (1.1), (1.2) we mean a process $(\xi(t))_{t\ge0}$ with values in X such that the following two conditions are satisfied with probability one:

(a) the sample path is a right-continuous function such that for every $t>0$ the limit $\xi(t-)=\lim_{s\to t,\, s<t}\xi(s)$ exists;

(b) $\xi(t)=\xi_0+\int_0^t a(\xi(s))\,ds+\int_0^t\int_\Theta \sigma(\xi(s-),\theta)\,\mathcal{N}_p(\Lambda_\xi(ds),d\theta)$ for $t\ge0$.

Assumption (iv) implies that the mapping

$$\Omega\ni\omega\mapsto p(\omega)=(\tau_n(\omega),\theta_n(\omega))_{n\ge1}$$

defines a stationary Poisson point process with characteristic measure κ, and $\mathcal{N}_p$ given by this process is a Poisson random counting measure. The sample path of $\mathcal{N}_p$ has jumps at times $(\tau_n)_{n\ge1}$, whereas $\mathcal{N}_r(t,A)=\mathcal{N}_p(\Lambda_\xi(t),A)$ for

$A \in \mathcal{G}$ have jumps at $(t_n)_{n\geq 1}$ such that

$$\int_0^{t_n} \lambda(\xi(s))\, ds = \tau_n, \qquad n = 1, 2, \ldots.$$

The sequence $(t_n)_{n\geq 1}$ is well defined because $\lambda(\cdot) \geq \underline{\lambda} > 0$. The point process r is of the form $r = (t_n, \theta_n)_{n\geq 1}$. The definition of integral now implies

$$\int_0^t \int_{\Theta} \sigma(\xi(s-), \theta)\mathcal{N}_p(\Lambda_\xi(ds), d\theta) = \sum_{t_n \leq t} \sigma(\xi(t_n-), \theta_n) \qquad \text{for } t \geq 0.$$

We can understand this integral both as a stochastic integral and as an integral on sample paths. For every fixed $\omega \in \Omega$, we can write an explicit formula for the unique solution of (1.1), (1.2). Namely, we consider the Cauchy problem

$$v'(t) = a(v(t)) \qquad \text{for } t \in \mathbb{R}, \qquad v(0) = x, \qquad x \in X. \tag{3.1}$$

We denote the solution of (3.1) by $v(t) = \pi^t x$, $t \in \mathbb{R}$. Then for every fixed value of $r(\omega) = (t_n(\omega), \theta_n(\omega))_{n\geq 1}$ the solution of (1.1), (1.2) is of the form

$$\xi(t_n) = \xi(t_n-) + \sigma(\xi(t_n-), \theta_n), \quad n \in \mathbb{N}, \quad \xi(0) = \xi_0, \tag{3.2}$$

$$\xi(t) = \pi^{t-t_n}\xi(t_n) \qquad \text{for } t \in [t_n, t_{n+1}),\ n \in \mathbb{N}_0, \tag{3.3}$$

where t_{n+1} is such that

$$\int_{t_n}^{t_{n+1}} \lambda(\pi^s(\xi(t_n)))\, ds = \varrho_n. \tag{3.4}$$

Define the mappings L, H by

$$L(t, z) = \int_0^t \lambda(\pi^s z)\, ds, \qquad H(t, z) = L^{-1}(t, z) \qquad \text{for } t \in \mathbb{R}^+,\ z \in X, \tag{3.5}$$

where L^{-1} is the inverse with respect to t. Let

$$q(z, \theta) = z + \sigma(z, \theta) \qquad \text{for } z \in X,\ \theta \in \Theta. \tag{3.6}$$

We denote $\xi_n = \xi(t_n)$. Taking into account (3.4), we obtain

$$L(t_{n+1} - t_n, \xi_n) = \varrho_n, \qquad \triangle t_n = t_{n+1} - t_n = H(\varrho_n, \xi_n).$$

Hence formulae (3.2), (3.3) may be rewritten as

$$\xi_{n+1} = q(\pi^{H(\varrho_n,\, \xi_n)}\xi_n, \theta_{n+1}), \tag{3.7}$$

$$\xi(t) = \sum_{n=0}^{\infty} \pi^{t-t_n}\xi_n 1_{[0, H(\varrho_n,\, \xi_n))}(t - t_n). \tag{3.8}$$

Assumption (i) implies that there exists a constant $\alpha \in \mathbb{R}$ such that the solution $\pi^t x$ of (3.1) satisfies

$$\|\pi^t x - \pi^t y\| \le e^{\alpha t}\|x - y\| \quad \text{for } x, y \in X, t \ge 0. \tag{3.9}$$

Analogously, from assumption (ii) it follows that the function $q\colon X \times \Theta \to X$ given by (3.6) is measurable, $q(x,\cdot) \in L^1(\kappa)$, and there exists a constant $l_q \ge 0$ such that

$$\|q(x,\cdot) - q(y,\cdot)\|_{L^1(\kappa)} \le l_q\|x - y\| \quad \text{for } x, y \in X. \tag{3.10}$$

Now we are going to derive an explicit formula for the operator P which describes the change of the distribution of $\xi(t)$ from a perturbation moment to the next. Denote by μ_k the distribution of ξ_k. Take an arbitrary function $h \in B(X)$. The expectation of $h(\xi_{k+1})$ is given by

$$\mathbb{E}\,(h(\xi_{k+1})) = \int_X h(x)\,\mu_{k+1}(dx). \tag{3.11}$$

Applying (3.7), independence of $\varrho_k, \theta_k, \xi_k$, and (3.5) we obtain

$$\begin{aligned} \mathbb{E}(h(\xi_{k+1})) &= \int_\Omega h(q(\pi^{H(\varrho_k\ \xi_k)}\xi_k, \theta_{k+1}))\,d\mathbb{P} \\ &= \int_X \int_0^\infty \int_\Theta h(q(\pi^{H(t,\,x)}x, \theta))e^{-t}\,\kappa(d\theta)\,dt\,\mu_k(dx) \\ &= \int_X \int_0^\infty \int_\Theta h(q(\pi^t x, \theta))e^{-L(t,x)}\lambda(\pi^t x)\,\kappa(d\theta)\,dt\,\mu_k(dx). \end{aligned} \tag{3.12}$$

If we pick $h = 1_D$, where 1_D denotes the indicator function of D, and equate (3.11) with (3.12), we have

$$\mu_{k+1}(D) = \int_X \int_0^\infty \int_\Theta 1_D(q(\pi^t x, \theta))e^{-L(t,x)}\lambda(\pi^t x)\,\kappa(d\theta)\,dt\,\mu_k(dx) \tag{3.13}$$

for $D \in \mathcal{B}(X)$. Define the operator P by

$$P\mu(D) = \int_X \int_0^\infty \int_\Theta 1_D(q(\pi^t x, \theta))e^{-L(t,x)}\lambda(\pi^t x)\,\kappa(d\theta)\,dt\,\mu(dx). \tag{3.14}$$

Then (3.13) may be rewritten as $\mu_{k+1} = P\mu_k$.

The operator P is called the *jump operator*. It is a linear operator in the space $\mathcal{M}$, and it maps every probability measure to a probability measure, so it is a Markov operator.

A straightforward calculation by applying (2.2) shows that the operator U dual to P is of the form

$$Uf(x) = \int_0^\infty \int_\Theta f(q(\pi^t x, \theta))e^{-L(t,x)}\lambda(\pi^t x)\,\kappa(d\theta)\,dt \quad \text{for } f \in C(X). \tag{3.15}$$

4. The asymptotic stability of the Markov operator. In this section we present theorems which give conditions for the asymptotic stability of the Markov operator P. The proof is based on the criterion developed by Szarek [26, Theorem 3.1]. Let us recall the notions which appear in this criterion.

A Markov operator P is called *globally concentrating* if for every $\varepsilon > 0$ and every $A \in \mathcal{B}_b(X)$ there exist $B \in \mathcal{B}_b(X)$ and $n_0 \in \mathbb{N}$ such that

$$P^n\mu(B) \geq 1-\varepsilon \quad \text{for } n \geq n_0,\ \mu \in \mathcal{M}_1^A. \tag{4.1}$$

A Markov operator P is called *locally concentrating* if for every $\varepsilon > 0$ there exists $\gamma > 0$ such that for every $A \in \mathcal{B}_b(X)$ there exist $C \in \mathcal{B}_b(X)$ with $\operatorname{diam} C < \varepsilon$ and $n_0 \in \mathbb{N}$ such that

$$P^n\mu(C) \geq \gamma \quad \text{for } n \geq n_0,\ \mu \in \mathcal{M}_1^A. \tag{4.2}$$

THEOREM 4.1. *If a nonexpansive Markov operator is globally and locally concentrating then it is asymptotically stable.*

We introduce a new norm on the space X:

$$X \ni x \mapsto c\|x\| \in [0,\infty),$$

where c is an arbitrary constant satisfying

$$c \geq \frac{l_\lambda(\underline{\lambda}+\overline{\lambda})}{\underline{\lambda}(\underline{\lambda}-\alpha-\overline{\lambda}l_q)}.$$

This norm gives the same topology and the same class of bounded sets. Denote by $\|\cdot\|_c$ the Fortet–Mourier norm given by the formula

$$\|\mu\|_c = \sup\{|\langle f,\mu\rangle| : f \in \mathcal{F}_c\} \quad \text{for } \mu \in \mathcal{M}_{\mathrm{sig}},$$

where $\mathcal{F}_c = \{f \in C(X) : |f(x)| \leq 1,\ |f(x)-f(y)| \leq c\|x-y\| \text{ for } x,y \in X\}$. For all measures $\mu_n, \mu \in \mathcal{M}_1$, we have

$$\lim_{n\to\infty} \|\mu_n-\mu\|_c = 0 \ \Leftrightarrow\ \lim_{n\to\infty} \|\mu_n-\mu\|_{\mathrm{FM}} = 0.$$

We now prove a theorem on the nonexpansiveness of P.

THEOREM 4.2. *Assume that conditions* (3.9) *and* (3.10) *are satisfied. If additionally*

$$\overline{\lambda}l_q + \alpha < \underline{\lambda} \tag{4.3}$$

then P given by (3.14) *is nonexpansive with respect to the norm $\|\cdot\|_c$.*

Proof. We will show that $Uf \in \mathcal{F}_c$ for every $f \in \mathcal{F}_c$. Then

$$\|P\mu_1 - P\mu_2\|_c := \sup_{f\in\mathcal{F}_c} |\langle f, P\mu_1 - P\mu_2\rangle| = \sup_{f\in\mathcal{F}_c} |\langle Uf, \mu_1-\mu_2\rangle| \leq \|\mu_1-\mu_2\|_c$$

implies the nonexpansiveness of P with respect to the norm $\|\cdot\|_c$.

Fix $f \in \mathcal{F}_c$. From the definition of U it follows that $Uf \in C(X)$ and $|Uf| \le 1$. Moreover, using (3.15) we obtain

$$\begin{aligned}|Uf(x) - Uf(y)| &\le \int_0^\infty \int_\Theta |f(q(\pi^t x,\theta)) - f(q(\pi^t y,\theta))|\lambda(\pi^t y)e^{-L(t,y)}\,\kappa(d\theta)\,dt \\ &+ \int_0^\infty \int_\Theta |f(q(\pi^t x,\theta))|\,|\lambda(\pi^t x)e^{-L(t,x)} - \lambda(\pi^t y)e^{-L(t,y)}|\,\kappa(d\theta)\,dt \\ &= I_1 + I_2.\end{aligned}$$

Taking into consideration $f \in \mathcal{F}_c$, (3.10), (3.9), the boundedness of λ, and the inequality $\underline{\lambda} > \alpha$, we obtain

$$I_1 \le c\frac{\overline{\lambda} l_q}{\underline{\lambda} - \alpha}\|x - y\|. \tag{4.4}$$

Now we will estimate the integral I_2. In view of $|f| \le 1$, we have

$$I_2 \le \int_0^\infty |\lambda(\pi^t x) - \lambda(\pi^t y)|e^{-L(t,x)}\,dt + \int_0^\infty |e^{-L(t,x)} - e^{-L(t,y)}|\lambda(\pi^t y)\,dt.$$

Since

$$|e^{-\beta} - e^{-\gamma}| \le e^{-c}|\beta - \gamma| \quad \text{for } \beta,\gamma \ge c > 0, \tag{4.5}$$

we have $|e^{-L(t,x)} - e^{-L(t,y)}| \le e^{-\underline{\lambda} t}|L(t,x) - L(t,y)|$. Definition (3.5), assumption (iii), and condition (3.9) now imply that

$$|e^{-L(t,x)} - e^{-L(t,y)}| \le \left(\frac{l_\lambda}{\alpha}e^{-(\underline{\lambda}-\alpha)t} - \frac{l_\lambda}{\alpha}e^{-\underline{\lambda} t}\right)\|x - y\|. \tag{4.6}$$

By the properties of λ, (3.9), (4.6), and since $\underline{\lambda} > \alpha$, $\underline{\lambda} > 0$, we have

$$I_2 \le \frac{l_\lambda(\underline{\lambda} + \overline{\lambda})}{\underline{\lambda}(\underline{\lambda} - \alpha)}\|x - y\|. \tag{4.7}$$

Combining (4.4) with (4.7) we get

$$|Uf(x) - Uf(y)| \le c\frac{\overline{\lambda} l_q}{\underline{\lambda} - \alpha}\|x - y\| + \frac{l_\lambda(\underline{\lambda} + \overline{\lambda})}{\underline{\lambda}(\underline{\lambda} - \alpha)}\|x - y\|.$$

From the choice of the constant c it follows that

$$c\frac{\overline{\lambda} l_q}{\underline{\lambda} - \alpha} + \frac{l_\lambda(\underline{\lambda} + \overline{\lambda})}{\underline{\lambda}(\underline{\lambda} - \alpha)} \le c.$$

Therefore $|Uf(x) - Uf(y)| \le \|x - y\|_c$, which concludes the proof. ∎

We now show global concentration for the jump operator P. A condition which guarantees this property can be formulated by applying the Lyapunov function. Recall that a continuous function $V\colon X \to [0,\infty)$ is called a *Lyapunov function* if

$$\lim_{\|x\|\to\infty} V(x) = \infty.$$

The following lemma connects the existence of a Lyapunov function satisfying a certain inequality with a condition which implies global concentration.

LEMMA 4.3. *Let P be a Feller operator and U its dual. Assume that there exists a Lyapunov function V, bounded on bounded sets and such that*

$$UV(x) \le aV(x) + b \quad \text{for } x \in X, \tag{4.8}$$

where a, b are nonnegative constants and $a < 1$. Then for every $\varepsilon > 0$ there exists a set $B \in \mathcal{B}_b(X)$ (depending only on a, b and V) such that for every set $A \in \mathcal{B}_b(X)$ there is $n_0 \in \mathbb{N}$ satisfying

$$P^n\mu(B) \ge 1 - \varepsilon \quad \text{for } n \ge n_0,\ \mu \in \mathcal{M}_1^A. \ \blacksquare \tag{4.9}$$

The proof of this lemma is the same as the proof of [26, Lemma 4.1]. Analysing that reasoning we can see that the set B does not depend on A. Clearly, a Markov operator satisfying (4.9) is globally concentrating.

The operator U given by (3.15) can be extended to the set of all Borel nonnegative functions, not necessarily bounded, in such a way that condition (2.1) is satisfied. Let $V\colon X \to [0,\infty)$ be given by

$$V(x) = \|x\| \quad \text{for } x \in X.$$

THEOREM 4.4. *Assume that* (3.9), (3.10), *and* (4.3) *hold, and*

$$\underline{\lambda} > l_a. \tag{4.10}$$

Then for any nonnegative constants d_1, d_2 such that

$$\frac{\overline{\lambda} l_q}{\underline{\lambda} - \alpha} \le d_1 < 1, \tag{4.11}$$

$$d_2 \ge \frac{\overline{\lambda} l_q \|a(0)\|}{\underline{\lambda}(\underline{\lambda} - l_a)} + \|q(0,\cdot)\|_{L^1(\kappa)}, \tag{4.12}$$

the following inequality is satisfied:

$$UV(x) \le d_1 V(x) + d_2 \quad \text{for } x \in X.$$

Proof. By (3.10), (3.9), the boundedness of λ, and (4.3) we obtain

$$\begin{aligned} UV(x) &\le \int_0^\infty \int_\Theta \|q(\pi^t x,\theta) - q(\pi^t 0,\theta)\| \lambda(\pi^t x) e^{-L(t,x)}\, \kappa(d\theta)\, dt \\ &\quad + \int_0^\infty \int_\Theta \|q(\pi^t 0,\theta)\| \lambda(\pi^t x) e^{-L(t,x)}\, \kappa(d\theta)\, dt \\ &\le \frac{\overline{\lambda} l_q}{\underline{\lambda} - \alpha} V(x) + \int_0^\infty \int_\Theta \|q(\pi^t 0,\theta)\| \lambda(\pi^t x) e^{-L(t,x)}\, \kappa(d\theta)\, dt. \end{aligned}$$

From the fact that $\int_0^\infty e^{-L(t,x)}\lambda(\pi^t x)\,dt = 1$ for every $x \in X$, and the properties of λ and q, it follows that

$$\int_0^\infty \int_\Theta \|q(\pi^t 0,\theta)\|\lambda(\pi^t x)e^{-L(t,x)}\,\kappa(d\theta)\,dt \le \overline{\lambda} l_q \int_0^\infty \|\pi^t 0\| e^{-\underline{\lambda} t}\,dt + \|q(0,\cdot)\|_{L^1(\kappa)}.$$

We now estimate $\|\pi^t 0\|$. We have

$$\|\pi^t 0\| \le \int_0^t \|a(\pi^s 0) - a(0)\|\,ds + \|a(0)\|t \le l_a \int_0^t \|\pi^s 0\|\,ds + \|a(0)\|t.$$

An application of Gronwall's inequality gives

$$\|\pi^t 0\| \le \frac{\|a(0)\|}{l_a}(e^{l_a t} - 1).$$

Inequality (4.10) now implies

$$\int_0^\infty \int_\Theta \|q(\pi^t 0,\theta)\|\lambda(\pi^t x)e^{-L(t,x)}\,\kappa(d\theta)\,dt \le \frac{\overline{\lambda} l_q \|a(0)\|}{\underline{\lambda}(\underline{\lambda} - l_a)} + \|q(0,\cdot)\|_{L^1(\kappa)}.$$

Taking into consideration (4.11) and (4.12), we obtain

$$UV(x) \le d_1 V(x) + d_2 \quad \text{ for } x \in X. \ ■$$

From Theorem 4.4 and Lemma 4.3 we obtain:

Remark 4.5. If conditions (3.9), (3.10), (4.3), and (4.10) hold, then the jump operator P is globally concentrating.

In the proof of local concentration we will apply

Lemma 4.6 ([27, Lemma 3.1]). *Let $\mu_1, \mu_2 \in \mathcal{M}_1$ and $\varepsilon > 0$. If $\|\mu_1 - \mu_2\|_{\mathrm{FM}} \le \varepsilon^2$ then $\mu_1(\mathcal{O}(A,\varepsilon)) \ge \mu_2(A) - \varepsilon$ for $A \in \mathcal{B}(X)$.* ■

Now, we will prove that an operator $\overline{P}$ associated with P is asymptotically stable. The operator $\overline{P}$ is derived from P by substituting the constant $\overline{\lambda}$ for the function λ. Thus the Markov operator $\overline{P}$ is given by the formula

$$\overline{P}\mu(A) = \int_X \int_0^\infty \int_\Theta 1_A(q(\pi^t x_0,\theta))\overline{\lambda} e^{-\overline{\lambda} t}\,\kappa(d\theta)\,dt\,\mu(dx).$$

We will deduce the asymptotic stability of $\overline{P}$ from the following theorem of A. Lasota.

Theorem 4.7 ([13, Theorem 3.2]). *Let $P\colon \mathcal{M} \to \mathcal{M}$ be a Markov Feller operator and U its dual. Assume that there is a constant $b < 1$ such that*

$$(4.13) \qquad |Uf(x) - Uf(y)| \le b\|x - y\| \quad \text{for } x, y \in X,\ f \in \mathcal{H}.$$

Moreover, assume that

$$Ug(0) < \infty, \quad \text{where} \quad g(x) = \|x\|.$$

Then P is asymptotically stable.

Theorem 4.8. *Assume that* (3.9) *and* (3.10) *hold, and*

$$\overline{\lambda} l_q + \alpha < \overline{\lambda}, \quad l_a < \overline{\lambda}. \tag{4.14}$$

Then $\overline{P}$ *is asymptotically stable.*

Proof. We will show that the assumptions of Theorem 4.7 are satisfied. Fix $f \in \mathcal{H}$. Using (3.9), (3.10), and $\overline{\lambda} - \alpha > 0$ we obtain

$$\begin{aligned}|\overline{U} f(x) - \overline{U} f(y)| &\le \int_0^\infty \int_\Theta |f(q(\pi^t x, \theta)) - f(q(\pi^t y, \theta))| \overline{\lambda} e^{-\overline{\lambda} t} \, \kappa(d\theta)\, dt \\ &\le \frac{\overline{\lambda} l_q}{\overline{\lambda} - \alpha} \|x - y\|.\end{aligned}$$

Condition (4.14) implies that inequality (4.13) holds with $b = \overline{\lambda} l_q / (\overline{\lambda} - \alpha) < 1$. Now we check that $\overline{U} g(0) < \infty$. Estimating as in Theorem 4.4 we obtain

$$\overline{U} g(0) = \int_0^\infty \int_\Theta \|q(\pi^t 0, \theta)\| \overline{\lambda} e^{-\overline{\lambda} t} \, \kappa(d\theta)\, dt \le \frac{l_q \|a(0)\|}{\overline{\lambda} - l_a} + \|q(0, \cdot)\|_{L^1(\kappa)}.$$

Now Theorem 4.7 yields the asymptotic stability of $\overline{P}$. ■

Observe that if the assumptions of Theorem 4.4 are satisfied, then the operator $\overline{P}$ is asymptotically stable.

From Theorems 4.7 and 4.8 it follows that $\overline{P}$ has the following properties:

$$\overline{P}(\mathcal{M}_{1,\mathrm{H}}) \subset \mathcal{M}_{1,\mathrm{H}}, \tag{4.15}$$

$$\|\overline{P}\mu_1 - \overline{P}\mu_2\|_{\mathrm{H}} \le b \|\mu_1 - \mu_2\|_{\mathrm{H}} \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_{1,\mathrm{H}}. \ ■ \tag{4.16}$$

We now show the local concentration of P.

Theorem 4.9. *Assume that all assumptions* (3.9), (3.10), (4.3), *and* (4.10) *hold. Then* P *is locally concentrating.*

Proof. From Theorem 4.4 and Lemma 4.3 it follows that there exists $B \in \mathcal{B}_b(X)$ such that for every $A \in \mathcal{B}_b(X)$ one can find $n_0 \in \mathbb{N}$ such that

$$P^n \mu(B) \ge 1/2 \quad \text{for } n \ge n_0,\ \mu \in \mathcal{M}_1^A. \tag{4.17}$$

Fix $\varepsilon > 0$. Fix $A \in \mathcal{B}(X)$ and choose $n_0 \in \mathbb{N}$ such that (4.17) is satisfied. Fix $\mu \in \mathcal{M}_1^A$. We claim that

$$U f(x) \ge \frac{\lambda}{\overline{\lambda}} \overline{U} f(x) \quad \text{for } f \in B(X).$$

Indeed, applying the boundedness of λ, we obtain

$$\begin{aligned}Uf(x) &\geq \int_0^\infty \int_\Theta f(q(\pi^t x,\theta))\underline{\lambda} e^{-\overline{\lambda} t}\,\kappa(d\theta)\,dt \\ &= \frac{\underline{\lambda}}{\overline{\lambda}} \int_0^\infty \int_\Theta f(q(\pi^t x,\theta))\overline{\lambda} e^{-\overline{\lambda} t}\,\kappa(d\theta)\,dt = \frac{\underline{\lambda}}{\overline{\lambda}}\,\overline{U} f(x).\end{aligned}$$

Then for any $D \in \mathcal{B}(X), n \in \mathbb{N}$, we have

$$P^n\mu(D) = \int_X U1_D(x)P^{n-1}\mu(dx) \geq \frac{\underline{\lambda}}{\overline{\lambda}} \int_X \overline{U}1_D(x)P^{n-1}\,\mu(dx).$$

Thus by an induction argument,

$$P^n\mu(D) \geq \left(\frac{\underline{\lambda}}{\overline{\lambda}}\right)^m \int_X \overline{U}^m 1_D(x)P^{n-m}\mu(dx) \quad \text{for } D \in \mathcal{B}(X),\ n,m \in \mathbb{N},\ m \leq n.$$

The operator $\overline{P}$ is asymptotically stable. Denote by $\overline{\mu}_*$ an invariant measure for $\overline{P}$. Take any $y \in \operatorname{supp} \overline{\mu}_*$. Let $B_1 = B(y, \varepsilon/4)$. Set $a := \overline{\mu}_*(B_1)$. Define $C := \mathcal{O}(B_1, \delta)$, where $\delta < \varepsilon/4$ and $\delta < a$. Then $\operatorname{diam} C \leq \varepsilon$. From the asymptotic stability of $\overline{P}$ it follows that

$$\overline{P}^n\delta_x \to \overline{\mu}_* \quad \text{for } x \in X. \tag{4.18}$$

We will show

$$\lim_{n\to\infty} \|\overline{P}^n\delta_x - \overline{\mu}_*\|_{\mathrm{FM}} = 0 \quad \text{uniformly in } x \in B. \tag{4.19}$$

Fix $x_0 \in B$. Take an arbitrary $x \in B$. For $n \in \mathbb{N}$ we have

$$\|\overline{P}^n\delta_x - \overline{\mu}_*\|_{\mathrm{FM}} \leq \|\overline{P}^n\delta_x - \overline{P}^n\delta_{x_0}\|_{\mathrm{FM}} + \|\overline{P}^n\delta_{x_0} - \overline{\mu}_*\|_{\mathrm{FM}}.$$

Applying (2.3), (4.16), and (4.15) we obtain

$$\begin{aligned}\|\overline{P}^n\delta_x - \overline{P}^n\delta_{x_0}\|_{\mathrm{FM}} &\leq \|\overline{P}^n\delta_x - \overline{P}^n\delta_{x_0}\|_{\mathcal{H}} \\ &\leq b^n\|\delta_x - \delta_{x_0}\|_{\mathcal{H}} \leq b^n\|x - x_0\| \leq b^n \operatorname{diam} B.\end{aligned}$$

Taking into account $b < 1$ and $\operatorname{diam} B < \infty$, we have

$$\lim_{n\to\infty} \sup_{x\in B} \|\overline{P}^n\delta_x - \overline{P}^n\delta_{x_0}\|_{\mathrm{FM}} = 0. \tag{4.20}$$

According to (4.18), we obtain

$$\lim_{n\to\infty} \|\overline{P}^n\delta_{x_0} - \overline{\mu}_*\|_{\mathrm{FM}} = 0. \tag{4.21}$$

Combining (4.20) and (4.21) immediately yields (4.19). Let $m \in \mathbb{N}$ be such that

$$\|\overline{P}^m\delta_x - \overline{\mu}_*\|_{\mathrm{FM}} \leq \delta^2 \quad \text{for } x \in B.$$

Using Lemma 4.6, we obtain

$$\overline{P}^m\delta_x(C) \geq \overline{\mu}_*(B_1) - \delta = a - \delta \quad \text{for } x \in B. \tag{4.22}$$

Take $n \geq m + n_0$. Then

$$P^n\mu(C) \geq \left(\frac{\underline{\lambda}}{\overline{\lambda}}\right)^m \int_X \overline{U}^m 1_C(x)\, P^{n-m}\mu(dx).$$

From this, and from (4.22) and (4.17), it follows that

$$P^n\mu(C) \geq \left(\frac{\underline{\lambda}}{\overline{\lambda}}\right)^m \int_B \overline{P}^m \delta_x(C)\, P^{n-m}\mu(dx) \geq \left(\frac{\underline{\lambda}}{\overline{\lambda}}\right)^m (a-\delta) P^{n-m}\mu(B)$$
$$\geq \frac{1}{2}\left(\frac{\underline{\lambda}}{\overline{\lambda}}\right)^m (a-\delta),$$

completing the proof. ■

Combining Theorems 4.2 and 4.9 and Corollary 4.5 we obtain

THEOREM 4.10. *Assume that all hypotheses of Theorem* 4.4 *are satisfied. Then the operator P is asymptotically stable.*

5. Continuous dependence. In this section we prove the continuous dependence of the invariant measure for P_λ on the function λ. Denote by P_λ the jump operator defined by (3.14), which varies with λ. Similarly denote by U_λ the dual operator for P_λ.

In the proof we are going to use

LEMMA 5.1 ([26, Theorem 3.1, Step 3]). *If a nonexpansive Markov operator P is locally and globally concentrating, then for every $A \in \mathcal{B}_b(X)$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$\|P^N\mu_1 - P^N\mu_2\|_{\mathrm{FM}} \leq \varepsilon \quad \textit{for } \mu_1, \mu_2 \in \mathcal{M}_1^A. \ ■$$

To formulate the main theorem of this section we define the family of functions

$$\mathcal{A} = \{\lambda(\cdot) \colon \overline{\lambda} l_q + \alpha < \underline{\lambda},\ l_a < \underline{\lambda}\},$$

where α, l_q, and l_a are determined by conditions (3.9), (3.10), and assumption (i) respectively. Moreover, $\overline{\lambda}$, $\underline{\lambda}$ depend on $\lambda(\cdot)$. By the previous section, for each $\lambda \in \mathcal{A}$ there exists a unique invariant distribution for P_λ. Denote it by μ_λ. Thus we can define function $\Lambda \colon \mathcal{A} \to \mathcal{M}_1$ by

$$\Lambda(\lambda(\cdot)) = \mu_\lambda \quad \text{for } \lambda \in \mathcal{A}. \tag{5.1}$$

We will show the continuity of Λ, where the convergence on $\mathcal{M}_1$ is in the Fortet–Mourier norm, and in $\mathcal{A}$ we have uniform convergence, denoted by $\lambda_n \rightrightarrows \lambda_0$.

LEMMA 5.2. *Let $\lambda_0 \in \mathcal{A}$, suppose $(\lambda_n)_{n\in\mathbb{N}}$ converges uniformly to λ_0. Then for every $\varepsilon > 0$ there exists $Z_0 \in \mathcal{B}_b(X)$ such that*

$$\mu_{\lambda_n}(Z_0) \geq 1 - \varepsilon \quad \textit{for } n \in \mathbb{N}.$$

Proof. From the assumption it follows that there exist $k_1, k_2 > 0$ and $n_0 \in \mathbb{N}$ such that

$$k_1 < \underline{\lambda}_n, \quad \overline{\lambda}_n < k_2, \quad \lambda_n \in \mathcal{A} \quad \text{for } n \ge n_0. \tag{5.2}$$

The proof of Theorem 4.4 now implies that there exist a Lyapunov function $V : X \to [0, \infty)$, bounded on bounded sets, and nonnegative constants d_1, d_2 such that

$$\frac{\overline{\lambda}_n l_q}{\underline{\lambda}_n - \alpha} \le d_1 < 1, \qquad d_2 \ge \frac{\overline{\lambda}_n l_q \|a(0)\|}{\underline{\lambda}_n(\underline{\lambda}_n - l_a)} + \|q(0, \cdot)\|_{L^1(\kappa)},$$
$$U_{\lambda_n} V(x) \le d_1 V(x) + d_2 \quad \text{for } x \in X, n \ge n_0.$$

Thus Lemma 4.3 shows that for every $\varepsilon > 0$ there exists $Z \in \mathcal{B}_b(X)$ such that

$$\liminf_{m\to\infty} P_{\lambda_n}^m \delta_x(Z) \ge 1 - \varepsilon \quad \text{for } n \ge n_0,\ x \in X.$$

Without loss of generality, we may assume that Z is closed. The asymptotic stability of P_{λ_n} for $n \ge n_0$ and the Aleksandrov theorem yield

$$\mu_{\lambda_n}(Z) \ge \liminf_{m\to\infty} P_{\lambda_n}^m \delta_x(Z) \ge 1 - \varepsilon \quad \text{for } n \ge n_0.$$

The Ulam theorem implies that there exists a compact set $K \subset X$ such that

$$\mu_i(K) \ge 1 - \varepsilon \quad \text{for } i \in \{1, \ldots, n_0 - 1\}.$$

Setting $Z_0 = Z \cup K$ we obtain the conclusion of the theorem. ■

Now, we prove the continuous dependence of the invariant measure for P_λ on the function λ. The first part of the proof will be analogous to the argument of Szarek and Wędrychowicz ([29, Theorem 4.5]).

THEOREM 5.3. *The function $\Lambda : \mathcal{A} \to \mathcal{M}_1$ defined by* (5.1) *is continuous.*

Proof. Fix $\varepsilon > 0$ and $\lambda_0 \in \mathcal{A}$. By Theorem 4.2, Corollary 4.5, and Theorem 4.9 the operator P_{λ_0} is nonexpansive, globally and locally concentrating. Suppose $\lambda_n \rightrightarrows \lambda_0$. From Lemma 5.2 it follows that there exists $Z_0 \in \mathcal{B}_b(X)$ satisfying

$$\mu_{\lambda_n}(Z_0) \ge 1 - \varepsilon/6 \quad \text{for } n \in \mathbb{N}_0.$$

Define $\mu_{\lambda_n}^{Z_0}, \nu_{\lambda_n}^{Z_0} \in \mathcal{M}_1^{Z_0}$ for $n \in \mathbb{N}_0$ by

$$\mu_{\lambda_n}^{Z_0}(B) = \frac{\mu_{\lambda_n}(B \cap Z_0)}{\mu_{\lambda_n}(Z_0)},$$
$$\nu_{\lambda_n}^{Z_0}(B) = \frac{6}{\varepsilon}[\mu_{\lambda_n}(B) - (1 - \varepsilon/6)\mu_{\lambda_n}^{Z_0}(B)] \quad \text{for } B \in \mathcal{B}(X),\ n \in \mathbb{N}_0.$$

Then

$$\mu_{\lambda_n} = (1 - \varepsilon/6)\mu_{\lambda_n}^{Z_0} + (\varepsilon/6)\nu_{\lambda_n}^{Z_0}$$

and

$$\begin{aligned}
&\|P^m_{\lambda_0}\mu_{\lambda_n} - P^m_{\lambda_0}\mu_{\lambda_0}\|_{\mathrm{FM}} \\
&\quad \le (1-\varepsilon/6)\|P^m_{\lambda_0}\mu^{Z_0}_{\lambda_n} - P^m_{\lambda_0}\mu^{Z_0}_{\lambda_0}\|_{\mathrm{FM}} + (\varepsilon/6)\|P^m_{\lambda_0}\nu^{Z_0}_{\lambda_n}\|_{\mathrm{FM}} + (\varepsilon/6)\|P^m_{\lambda_0}\nu^{Z_0}_{\lambda_0}\|_{\mathrm{FM}} \\
&\quad \le (1-\varepsilon/6)\|P^m_{\lambda_0}\mu^{Z_0}_{\lambda_n} - P^m_{\lambda_0}\mu^{Z_0}_{\lambda_0}\|_{\mathrm{FM}} + \varepsilon/3 \quad \text{for } m,n\in\mathbb{N}_0.
\end{aligned}$$

From Lemma 5.1 it follows that there exists $N\in\mathbb{N}$ such that

$$\|P^N_{\lambda_0}\mu^{Z_0}_{\lambda_n} - P^N_{\lambda_0}\mu^{Z_0}_{\lambda_0}\|_{\mathrm{FM}} \le \varepsilon/3 \quad \text{for } n\in\mathbb{N}_0.$$

Hence

$$\|P^N_{\lambda_0}\mu_{\lambda_n} - P^N_{\lambda_0}\mu_{\lambda_0}\|_{\mathrm{FM}} \le 2\varepsilon/3 \quad \text{for } n\in\mathbb{N}_0.$$

For $n\in\mathbb{N}_0$ we have

$$\begin{aligned}
(5.3)\quad \|\mu_{\lambda_n} - \mu_{\lambda_0}\|_{\mathrm{FM}} &= \|P^N_{\lambda_n}\mu_{\lambda_n} - P^N_{\lambda_0}\mu_{\lambda_0}\|_{\mathrm{FM}} \\
&\le \|P^N_{\lambda_n}\mu_{\lambda_n} - P^N_{\lambda_0}\mu_{\lambda_n}\|_{\mathrm{FM}} + \|P^N_{\lambda_0}\mu_{\lambda_n} - P^N_{\lambda_0}\mu_{\lambda_0}\|_{\mathrm{FM}} \\
&\le \sup_{f\in\mathcal{F}}\sup_{x\in X}|U^N_{\lambda_n}f(x) - U^N_{\lambda_0}f(x)| + 2\varepsilon/3 \\
&\le \sup_{\|f\|\le 1}\sup_{x\in X}|U^N_{\lambda_n}f(x) - U^N_{\lambda_0}f(x)| + 2\varepsilon/3 \\
&= \|U^N_{\lambda_n} - U^N_{\lambda_0}\| + 2\varepsilon/3.
\end{aligned}$$

In the second part of the proof, we will estimate $\|U^N_{\lambda_n} - U^N_{\lambda_0}\|$. Applying $\|U_{\lambda_n}\| = 1$ for $n\in\mathbb{N}_0$ we obtain

$$\begin{aligned}
(5.4)\quad &\|U^N_{\lambda_n} - U^N_{\lambda_0}\| \\
&\quad = \|(U_{\lambda_n} - U_{\lambda_0})U^{N-1}_{\lambda_n} + U_{\lambda_0}(U_{\lambda_n} - U_{\lambda_0})U^{N-2}_{\lambda_n} + \cdots + U^{N-1}_{\lambda_0}(U_{\lambda_n} - U_{\lambda_0})\| \\
&\quad \le N\|U_{\lambda_n} - U_{\lambda_0}\|.
\end{aligned}$$

Take any $f\in C(X)$ such that $\|f\|\le 1$. According to (3.15), we obtain

$$|U_{\lambda_n}f(x) - U_{\lambda_0}f(x)| \le \int_0^\infty |\lambda_n(\pi^t x)e^{-L_{\lambda_n}(t,x)} - \lambda_0(\pi^t x)e^{-L_{\lambda_0}(t,x)}|\,dt = h_n(x),$$

where L_λ given by (3.5) depends on λ. We will show that $h_n \underset{X}{\rightrightarrows} 0$. The convergence $\lambda_n \rightrightarrows \lambda_0$ implies that there exists $k>0$ such that $\underline{\lambda}_n \ge k$ for $n\in\mathbb{N}_0$, and there exists $n_0\in\mathbb{N}$ such that $|\lambda_n(x) - \lambda_0(x)| < \varepsilon k^2/(k+\overline{\lambda}_0)$ for $n\ge n_0, x\in X$. Hence inequality (4.5) yields

$$\begin{aligned}h_n(x) &\le \int_0^\infty e^{-L_{\lambda_n}(t,x)}|\lambda_n(\pi^t x)-\lambda_0(\pi^t x)|\,dt\\&\quad+\int_0^\infty \lambda_0(\pi^t x)|e^{-L_{\lambda_n}(t,x)}-e^{-L_{\lambda_0}(t,x)}|\,dt\\&< \int_0^\infty \frac{\varepsilon k^2}{k+\overline{\lambda}_0}e^{-kt}\,dt+\overline{\lambda}_0\int_0^\infty e^{-kt}|L_{\lambda_n}(t,x)-L_{\lambda_0}(t,x)|\,dt\\&< \frac{\varepsilon k}{k+\overline{\lambda}_0}+\overline{\lambda}_0\int_0^\infty \frac{\varepsilon k^2}{k+\overline{\lambda}_0}te^{-kt}\,dt=\varepsilon \quad \text{for } n\ge n_0.\end{aligned}$$

This estimate depends neither on x nor on f. Therefore,

$$\lim_{n\to\infty}\|U_{\lambda_n}-U_{\lambda_0}\|=0.$$

Hence there exists $n_1\in\mathbb{N}$ such that

$$\|U_{\lambda_n}-U_{\lambda_0}\|<\frac{\varepsilon}{3N} \quad \text{for } n\ge n_1. \tag{5.5}$$

Combining (5.5) and (5.4), we conclude that condition (5.3) is satisfied, and the proof of the theorem is complete. ∎

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