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A class of singular fourth-order boundary value problems with nonhomogeneous nonlinearity

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Abstract. We study the existence of positive solutions to a class of singular nonlinear fourth-order boundary value problems in which the nonlinearity may lack homogeneity. By introducing suitable control functions and applying cone expansion and cone compression, we prove three existence theorems. Our main results improve the existence result in [Z. L. Wei, Appl. Math. Comput. 153 (2004), 865–884] where the nonlinearity has a certain homogeneity.

1. Introduction. Let $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, $\delta \geq 0$ and $\rho = \alpha\gamma + \alpha\delta + \beta\gamma > 0$. The purpose of this paper is to study the existence of positive solutions to the following nonlinear fourth-order two-point boundary value problem:

$$\text{(P)}\qquad \begin{cases} u^{(4)}(t) = f(t, u(t), -u''(t)), \quad 0 < t < 1, \\ u(0) = u(1) = 0, \quad \alpha u''(0) - \beta u'''(0) = 0, \quad \gamma u''(1) + \delta u'''(1) = 0. \end{cases}$$

Here, a function $u^* \in C^3[0,1]$ is called a *positive solution* to the problem (P) if $u^*(t)$ satisfies (P) and $u^*(t) > 0$, $0 < t < 1$.

If $\beta = \delta = 0$, the problem (P) is the well-known elastic beam equation with two simply supported ends. When $f : [0,1] \times [0,\infty] \times [0,\infty) \to [0,\infty)$ is continuous, the nonlinear problem (P) has attracted wide attention (see [1, 3, 7, 9, 10, 12, 14, 17, 25] and the references therein).

Throughout this paper, $f : (0,1)\times(0,\infty)\times(0,\infty) \to [0,\infty)$ is continuous. Therefore, $f(t,u,v)$ may be singular at $t = 0$ and/or $t = 1$ for any $(u,v) \in [0,\infty) \times [0,\infty)$, and at $u = 0$ and/or $v = 0$ for any $t \in [0,1]$.

The positive solutions of the problem (P) with singularities have been investigated by many authors (see [4, 8, 11, 15, 16, 22, 24]). For example, Z. L. Wei [22] established the following existence theorem by using the upper and lower solution method.

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THEOREM 1.1 ([22, Theorem 3.1]). *Suppose that*

(a1) $f(t, \xi(t), 1) \not\equiv 0$ *and* $0 < \int_0^1 f(t, \xi(t), \zeta(t))\,dt < \infty$, *where*

$$\xi(t) = t(1-t), \qquad \zeta(t) = \frac{(\alpha t + \beta)[\gamma(1-t)+\delta]}{(\alpha+\beta)(\gamma+\delta)}.$$

(a2) *There exist constants* $\lambda_1, \lambda_2, \mu_1, \mu_2$ $(-\infty < \lambda_1, \lambda_2 \le 0,\ 0 \le \mu_1, \mu_2 < 1,\ \mu_1 + \mu_2 < 1)$ *such that, for any* $0 < c \le 1$ *and* $(t, u, v) \in (0,1) \times (0,\infty) \times (0,\infty)$,

$$c^{\mu_1} f(t,u,v) \le f(t, cu, v) \le c^{\lambda_1} f(t,u,v),$$
$$c^{\mu_2} f(t,u,v) \le f(t, u, cv) \le c^{\lambda_2} f(t,u,v).$$

Then the problem (P) *has a positive solution* $u^* \in C^3[0,1]$.

Theorem 1.1 extends and improves some results of D. O'Regan [16] when $\beta = \delta = 0$ and $f(t,u,v) = f(t,u)$. The theorem has the following advantages:

(1) The nonlinearity $f(t, u(t), -u''(t))$ not only depends on the unknown function $u(t)$ but also on its second derivative $u''(t)$.
(2) The function $f(t,u,v)$ may be singular at $t = 0$, $t = 1$, and at $u = 0$, $v = 0$ if $\lambda_1, \lambda_2 < 0$.
(3) It is easier to verify the homogeneity condition (a2).

For the singular problem (P) satisfying the homogeneity condition (a2), Theorem 1.1 is very effective and convenient. Consequently, the method in [22] has been applied successfully to various singular boundary value problems (see [4, 20, 23, 31, 32]).

However, in Theorem 1.1, the condition (a2) is also rather restrictive. As pointed out by Z. L. Wei [22], typical functions satisfying (a2) are those of the form

$$f(t,u,v) = \sum_{i=1}^{m} \sum_{j=1}^{n} \omega_{j,i} u^{\mu_j} v^{\lambda_i},$$

where $\omega_{j,i} \in C(0,1)$, $\omega_{j,i}(t) > 0$ on $(0,1)$, $\mu_j \in (-\infty,\infty)$, $\lambda_i < 1$, $\mu_j + \lambda_i < 1$, $i = 1, \dots, m$, $j = 1, \dots, n$.

Additionally, Theorem 1.1 only yields the existence of one positive solution, with no information about multiple positive solutions.

The purpose of this paper is to improve Theorem 1.1. We will remove the homogeneity condition (a2) and establish the multiplicity of positive solutions for the singular problem (P).

Let

$$\theta = \frac{\alpha\gamma + \delta\alpha}{2\alpha\gamma + \gamma\beta + \delta\alpha},$$

$$\eta = \frac{3\delta + 2\gamma}{6(\delta+\gamma)}\theta - \frac{\beta + 2\alpha}{2(\beta+\alpha)}\theta^2 + \left[\frac{\delta+2\gamma}{2(\delta+\gamma)} - \frac{3\beta - 2\alpha}{6(\beta+\alpha)}\right]\theta^3 - \left[\frac{\alpha}{3(\beta+\alpha)} + \frac{\gamma}{3(\delta+\gamma)}\right]\theta^4.$$

If $\alpha = \gamma = 1$, $\beta = \delta = 0$, then $\theta = \frac{1}{2}$, $\eta = \frac{1}{24}$. In addition, define

$$q(t) = \min\{t, 1-t\}, \qquad p(t) = \min\left\{\frac{\beta + \alpha t}{\beta + \alpha}, \frac{\delta + \gamma(1-t)}{\delta + \gamma}\right\}.$$

In this paper, we use the following assumption:

(H) For each pair of positive numbers $r_1 < r_2$, there exists a nonnegative function $j_{r_1}^{r_2} \in L^1[0,1] \cap C(0,1)$ such that

$$f(t,u,v) \le j_{r_1}^{r_2}(t), \qquad \forall 0 < t < 1,\ \eta r_1 q(t) \le u \le \tfrac{1}{8} r_2,\ r_1 p(t) \le v \le r_2.$$

For the singular problem (P), we will see that the assumption (H) ensures the complete continuity of the associated integral operator T (see Section 2), and guarantees that the solution $u^*(t)$ belongs to $C^3[0,1] \cap C^4(0,1)$.

In Sections 2 and 3, we will construct a proper cone and introduce two control functions. Applying these new tools, we will prove three theorems on the existence of single and double positive solutions to (P). In Section 4, we will verify that the main results improve Theorem 1.1 in the great majority of cases. In Section 5, we will give two examples.

Recently, a large number of papers on nonlinear singular boundary value problems have appeared (see [2, 6, 13, 15, 19, 21, 26–30]). Motivated by these papers, we will apply the Guo–Krasnosel'skiĭ fixed point theorem of cone expansion-compression type to study the singular problem (P). In this paper, the upper and lower solution method is not applied.

2. Preliminaries. Let $G_1(t,s)$ and $G_2(t,s)$ be the Green functions of the homogeneous linear problems

$$-u''(t) = 0, \qquad 0 \le t \le 1, \qquad u(0) = u(1) = 0,$$

and

$$-u''(t) = 0, \qquad 0 \le t \le 1, \qquad \alpha u(0) - \beta u'(0) = 0, \qquad \gamma u(1) + \delta u'(1) = 0,$$

respectively. Then, for $0 \le t \le s \le 1$,

$$G_1(t,s) = t(1-s), \qquad G_2(t,s) = \frac{(\beta + \alpha t)(\delta + \gamma - \gamma s)}{\rho},$$

while for $0 \le s \le t \le 1$,

$$G_1(t,s) = s(1-t), \qquad G_2(t,s) = \frac{(\beta + \alpha s)(\delta + \gamma - \gamma t)}{\rho}.$$

Obviously, $G_1(t,s) \ge 0$ and $G_2(t,s) \ge 0$ for any $0 \le t, s \le 1$.

Direct computations give that

$$\frac{\beta + \alpha\theta}{\beta + \alpha} = \frac{\delta + \gamma(1-\theta)}{\delta + \gamma}, \qquad \eta = \int_0^1 G_1(\theta, s) p(s)\, ds.$$

This implies $0 < \theta < 1$, $0 < \eta < 1$. Moreover,

$$\max_{0 \le t \le 1} \int_0^1 G_1(t,s)\, ds = \tfrac{1}{2} \max_{0 \le t \le 1} t(1-t) = \tfrac{1}{8}.$$

Consider the Banach space $C^2[0,1]$ equipped with the norm

$$|||u||| = \max\{\|u\|, \|u''\|\}, \quad \text{where} \quad \|u\| = \max_{0 \le t \le 1} |u(t)|.$$

Let

$$C_0^2[0,1] = \{u \in C^2[0,1] : u(0) = u(1) = 0\},$$
$$K = \{u \in C_0^2[0,1] : u(t) \ge \|u\| q(t),\ -u''(t) \ge \|u''\| p(t),\ 0 \le t \le 1\}.$$

Then K is a cone of nonnegative functions in $C^2[0,1]$. Write

$$K(r) = \{u \in K : |||u||| < r\}, \qquad \partial K(r) = \{u \in K : |||u||| = r\}.$$

For $u \in K \setminus \{0\}$, define the associated integral operator T as follows:

$$(Tu)(t) = \int_0^1 \int_0^1 G_1(t,s) G_2(s,\tau) f(\tau, u(\tau), -u''(\tau))\, d\tau\, ds, \qquad 0 \le t \le 1.$$

Lemma 2.1. *If $u \in K$, then $|||u||| = \|u''\|$ and $\eta\|u''\| \le \|u\| \le \frac{1}{8}\|u''\|$.*

Proof. Since $u(0) = u(1) = 0$, one has $u(t) = \int_0^1 G_1(t,s)[-u''(s)]\, ds$. So

$$\|u\| \le \|u''\| \max_{0 \le t \le 1} \int_0^1 G_1(t,s)\, ds = \tfrac{1}{8}\|u''\|.$$

Since $-u''(t) \ge \|u''\| p(t) \ge 0$, one has

$$\begin{aligned} \|u\| &= \max_{0 \le t \le 1} \int_0^1 G_1(t,s)[-u''(s)]\, ds \ge \max_{0 \le t \le 1} \int_0^1 G_1(t,s) \|u''\| p(s)\, ds \\ &\ge \|u''\| \int_0^1 G_1(\theta, s) p(s)\, ds = \eta \|u''\|. \end{aligned}$$

It follows that $\eta\|u''\| \le \|u\| \le \frac{1}{8}\|u''\|$ and $|||u||| = \|u''\|$. ∎

LEMMA 2.2. *Suppose that* (H) *holds. Then:*

(1) $T : \overline{K(r_2)} \setminus K(r_1) \to K$ *is completely continuous for any* $0 < r_1 < r_2$.

(2) *For any* $u \in \overline{K(r_2)} \setminus K(r_1)$,

$$(Tu)''(t) = -\int_0^1 G_2(t,s) f(s, u(s), -u''(s))\, ds, \qquad 0 \le t \le 1.$$

Proof. Define the operators T_1, F, J as follows:

$$\begin{aligned}
(T_1 u)(t) &= \int_0^1 G_2(t,s) f(s, u(s), -u''(s))\, ds, && 0 \le t \le 1,\\
(Fu)(t) &= f(t, u(t), -u''(t)), && 0 < t < 1,\\
(Ju)(t) &= \int_0^1 G_2(t,s) u(s)\, ds, && 0 \le t \le 1.
\end{aligned}$$

STEP I. Let $u \in \overline{K(r_2)} \setminus K(r_1)$. Then $r_1 \le |||u||| \le r_2$. By Lemma 2.1, $\eta r_1 \le \|u\| \le \frac{1}{8} r_2$ and $r_1 \le \|u''\| \le r_2$. So, for any $0 \le t \le 1$,

$$\begin{aligned}
\eta r_1 q(t) \le \|u\| q(t) \le u(t) \le \|u\| \le \tfrac{1}{8} r_2,\\
r_1 p(t) \le \|u''\| p(t) \le -u''(t) \le \|u''\| = r_2.
\end{aligned}$$

Let $j_{r_1}^{r_2}(t)$ be as in (H). Then

$$f(t, u(t), -u''(t)) \le j_{r_1}^{r_2}(t), \qquad \forall 0 < t < 1.$$

STEP II. By Step I,

$$\sup_{u \in \overline{K(r_2)} \setminus K(r_1)} \int_0^1 |(Fu)(t)|\, dt = \int_0^1 f(t, u(t), -u''(t))\, dt \le \int_0^1 j_{r_1}^{r_2}(t)\, dt < \infty.$$

Let $u_n, u_0 \in \overline{K(r_2)} \setminus K(r_1)$, $n = 1, 2, \ldots$, with $|||u_n - u_0||| \to 0$. Then $\max_{0 \le t \le 1} |u_n(t) - u_0(t)| \to 0$ and $\max_{0 \le t \le 1} |u_n''(t) - u_0''(t)| \to 0$. Since $f : (0,1) \times (0,\infty) \times (0,\infty) \to [0,\infty)$ is continuous, one has

$$f(t, u_n(t), u_n''(t)) - f(t, u_0(t), u_0''(t)) \to 0, \qquad \forall 0 < t < 1.$$

By Step I, for any $n = 0, 1, \ldots$ and any $0 < t < 1$,

$$\begin{aligned}
f(t, u_n(t), -u_n''(t)) &\le j_{r_1}^{r_2}(t),\\
|f(t, u_n(t), -u_n''(t)) - f(t, u_0(t), -u_0''(t))| &\le 2 j_{r_1}^{r_2}(t).
\end{aligned}$$

Applying the Lebesgue dominated convergence theorem [5, Theorem 2.1], we get

$$\lim_{n\to\infty}\int_0^1 |(Fu_n)(t)-(Fu_0)(t)|\,dt$$

$$= \lim_{n\to\infty}\int_0^1 |f(t,u_n(t),-u_n''(t))-f(t,u_0(t),-u_0''(t))|\,dt$$

$$= \int_0^1 \lim_{n\to\infty}|f(t,u_n(t),-u_n''(t))-f(t,u_0(t),-u_0''(t))|\,dt = 0.$$

Therefore, $F:\overline{K(r_2)}\setminus K(r_1)\to L^1[0,1]$ is bounded and continuous.

Step III. Obviously, $J: L^1[0,1]\to C[0,1]$ is a bounded linear operator. By the Arzelà–Ascoli theorem and a standard argument, $J: L^1[0,1]\to C[0,1]$ is completely continuous.

By Step II, $T_1 = J\circ F:\overline{K(r_2)}\setminus K(r_1)\to C[0,1]$ is continuous.

If $W\subset \overline{K(r_2)}\setminus K(r_1)$ is a bounded set, then the set $F(W)\subset L^1[0,1]$ is bounded by Step II. So, $T_1(W)=J(F(W))\subset C[0,1]$ is precompact.

Therefore, $T_1:\overline{K(r_2)}\setminus K(r_1)\to C[0,1]$ is completely continuous.

Step IV. Let $u\in\overline{K(r_2)}\setminus K(r_1)$. By Step III, $T_1u\in C[0,1]$ and

$$(Tu)(t)=\int_0^1 G_1(t,s)(T_1u)(s)\,ds,\qquad \forall 0\le t\le 1.$$

Since $G(0,s)=G(1,s)=0$ for any $0\le s\le 1$, one has $(Tu)(0)=(Tu)(1)=0$. By the definition of $G_1(t,s)$, then

$$(Tu)(t)=(1-t)\int_0^t s(T_1u)(s)\,ds + t\int_t^1 (1-s)(T_1u)(s)\,ds,\qquad \forall 0\le t\le 1.$$

Differentiating the above equality twice, we get

$$(Tu)''(t) = -(T_1u)(t) = -\int_0^1 G_2(t,s)f(s,u(s),-u''(s))\,ds,\qquad \forall 0\le t\le 1.$$

Consequently, $Tu\in C_0^2[0,1]$.

By the complete continuity of T_1, we see that $T:\overline{K(r_2)}\backslash K(r_1)\to C_0^2[0,1]$ is completely continuous.

Step V. Simple computations give, for $0\le t,s\le 1$,

$$q(t)G_1(s,s)\le G_1(t,s)\le G_1(s,s),\qquad p(t)G_2(s,s)\le G_2(t,s)\le G_2(s,s).$$

So, for any $u \in \overline{K(r_2)} \setminus K(r_1)$ and $0 \le t \le 1$,

$$\begin{aligned}(Tu)(t) &\ge q(t) \int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)f(\tau,u(\tau),-u''(\tau))\,d\tau\,ds \\ &\ge q(t) \max_{0\le t\le 1} \int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f(\tau,u(\tau),-u''(\tau))\,d\tau\,ds \\ &= \|Tu\|q(t),\end{aligned}$$

and

$$\begin{aligned}-(Tu)''(t) &\ge p(t)\int_0^1 G_2(s,s)f(s,u(s),-u''(s))\,ds \\ &\ge p(t)\max_{0\le t\le 1}\int_0^1 G_2(t,s)f(s,u(s),-u''(s))\,ds = \|(Tu)''\|p(t).\end{aligned}$$

It follows that $T : \overline{K(r_2)} \setminus K(r_1) \to K$.

Our study is based on the following Guo–Krasnosel'skiĭ fixed point theorem of cone expansion-compression type.

LEMMA 2.3. *Let X be a Banach space, let K be a cone in X, and let Ω_1, Ω_2 be bounded open subsets of K with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Assume that $T : \overline{\Omega}_2 \setminus \Omega_1 \to K$ is a completely continuous operator such that either*

- $\|Tx\| \le \|x\|$, $x \in \partial\Omega_1$, *and* $\|Tx\| \ge \|x\|$, $x \in \partial\Omega_2$, *or*
- $\|Tx\| \ge \|x\|$, $x \in \partial\Omega_1$, *and* $\|Tx\| \le \|x\|$, $x \in \partial\Omega_2$.

Then T has a fixed point in $\overline{\Omega}_2 \setminus \Omega_1$.

3. Main results. In order to state the main results, we need the following control functions and constants:

$$\varphi(r) = \int_0^1 \max\big\{f(t,u,v) : \eta r q(t) \le u \le \tfrac{1}{8}r,\ rp(t) \le v \le r\big\}\,dt,$$

$$\psi(r) = \int_\sigma^{1-\sigma} \min\big\{f(t,u,v) : \eta r q(t) \le u \le \tfrac{1}{8}r,\ rp(t) \le v \le r\big\}\,dt,$$

$$\underline{\varphi}_0 = \liminf_{r\to+0} \varphi(r)/r, \qquad \underline{\varphi}_\infty = \liminf_{r\to\infty} \varphi(r)/r,$$

$$\overline{\psi}_0 = \limsup_{r\to+0} \psi(r)/r, \qquad \overline{\psi}_\infty = \limsup_{r\to\infty} \psi(r)/r,$$

$$A = \max_{0\le t,s\le 1} G_2(t,s), \qquad B = \min_{\sigma\le t,s\le 1-\sigma} G_2(t,s),$$

where $0 < \sigma < \frac{1}{2}$ is a constant. In real problems, we can choose σ depending on the properties of $f(t,u,v)$.

If the assumption (H) is satisfied, then the control functions $\varphi(r)$, $\psi(r)$ are well defined for any $r > 0$.

Direct computations show that $B = \frac{1}{\rho}(\beta + \sigma\alpha)(\delta + \sigma\gamma)$ and

$$A = \begin{cases} \dfrac{\alpha\delta + \beta\delta}{\rho} & \text{if } -\alpha\gamma + \alpha\delta \geq \gamma\beta; \\ \dfrac{\rho}{4\alpha\gamma} & \text{if } -\alpha\gamma + \alpha\delta \leq \gamma\beta \leq \alpha\gamma + \alpha\delta; \\ \dfrac{\gamma\beta + \beta\delta}{\rho} & \text{if } \alpha\gamma + \alpha\delta \leq \gamma\beta. \end{cases}$$

In particular, if $\alpha = \gamma = 1$, $\beta = \delta = 0$, then $A = 1/4$, $B = \sigma^2$.

We obtain the following existence theorems.

Theorem 3.1. *Suppose that* (H) *holds and there exist* $0 < a < b$ *such that one of the following conditions is satisfied:*

(b1) $\varphi(a) \leq A^{-1}a$, $\psi(b) \geq B^{-1}b$.

(b2) $\psi(a) \geq B^{-1}a$, $\varphi(b) \leq A^{-1}b$.

Then the problem (P) *has a positive solution* $u^* \in K$ *such that* $u^* \in C^3[0,1] \cap C^4(0,1)$ *and* $a \leq \|\!|u^*|\!\| \leq b$.

Theorem 3.2. *Suppose that* (H) *holds and there exist* $0 < a < b < c$ *such that one of the following conditions is satisfied:*

(c1) $\varphi(a) \leq A^{-1}a$, $\psi(b) > B^{-1}b$ *and* $\varphi(c) \leq A^{-1}c$.

(c2) $\psi(a) \geq B^{-1}a$, $\varphi(b) < A^{-1}b$ *and* $\psi(c) \geq B^{-1}c$.

Then the problem (P) *has two positive solutions* $u_1^*, u_2^* \in K$ *such that* $u_1^*, u_2^* \in C^3[0,1] \cap C^4(0,1)$ *and* $a \leq \|\!|u_1^*|\!\| < b < \|\!|u_2^*|\!\| \leq c$.

Theorem 3.3. *Suppose that* (H) *holds and one of the following conditions is satisfied:*

(d1) $\underline{\varphi}_0 < A^{-1}$, $\overline{\psi}_\infty > B^{-1}$.

(d2) $\overline{\psi}_0 > B^{-1}$, $\underline{\varphi}_\infty < A^{-1}$.

Then the problem (P) *has a positive solution* $u^* \in C^3[0,1] \cap C^4(0,1)$.

Proof of Theorem 3.1. We only prove the case (b2).

By Lemma 2.2(1), $T : \overline{K(b)} \setminus K(a) \to K$ is completely continuous.

If $u \in \partial K(b)$, then $\|\!|u|\!\| = b$. By Lemma 2.1, $\eta b \leq \|u\| \leq \frac{1}{8}b$, $\|u''\| = b$. This implies that

$$\eta b q(t) \leq u(t) \leq \tfrac{1}{8}b, \qquad bp(t) \leq -u''(t) \leq b, \qquad \forall 0 \leq t \leq 1.$$

So, $\int_0^1 f(t, u(t), -u''(t))\,dt \leq \varphi(b)$. Since $Tu \in K$, one has $\|\!|Tu|\!\| = \|(Tu)''\|$

by Lemma 2.1. It follows that

$$\|\!|Tu|\!\|=\|(Tu)''\|=\max_{0\le t\le 1}\int_0^1 G_2(t,s)f(s,u(s),-u''(s))\,ds$$
$$\le \max_{0\le t,s\le 1} G_2(t,s)\int_0^1 f(s,u(s),-u''(s))\,ds\le AA^{-1}b=b=\|\!|u|\!\|.$$

If $u\in\partial K(a)$, then $\|\!|u|\!\|=a$. By Lemma 2.1,

$$\eta a q(t)\le u(t)\le \tfrac18 a,\qquad ap(t)\le -u''(t)\le a,\qquad \forall 0\le t\le 1.$$

Hence, $\int_\sigma^{1-\sigma} f(t,u(t),-u''(t))\,dt\ge \psi(a)$. It follows that

$$\|\!|Tu|\!\|=\|(Tu)''\|\ge \min_{\sigma\le t\le 1-\sigma}\int_\sigma^{1-\sigma} G_2(t,s)f(s,u(s),-u''(s))\,ds$$
$$\ge \min_{\sigma\le t,s\le 1-\sigma} G_2(t,s)\int_\sigma^{1-\sigma} f(s,u(s),-u''(s))\,ds\ge BB^{-1}a=a=\|\!|u|\!\|.$$

According to Lemma 2.3, the operator T has a fixed point $u^*\in\overline{K(b)}\setminus K(a)$. So, $u^*\in K$, $a\le\|\!|u^*|\!\|\le b$ and $u^*(t)\ge\|u^*\|q(t)\ge \eta a q(t)>0$, $0<t<1$.

Applying $u^*=Tu^*$ and Lemma 2.2(1), one has, for $0\le t\le 1$,

$$u^*(t)=\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f(\tau,u^*(\tau),-(u^*)''(\tau))\,d\tau\,ds,$$
$$(u^*)''(t)=-\int_0^1 G_2(t,s)f(s,u^*(s),-(u^*)''(s))\,ds.$$

Since $f(t,u^*(t),-(u^*)''(t))$ is integrable on $[0,1]$, successively differentiating the second equality, we get

$$(u^*)'''(t)=-\int_0^1\frac{\partial}{\partial t}G_2(t,s)f(s,u^*(s),-(u^*)''(s))\,ds,\qquad \forall 0\le t\le 1,$$
$$(u^*)^{(4)}(t)=f(t,u^*(t),-(u^*)''(t)),\qquad \forall 0<t<1.$$

These equalities show $u^*\in C^3[0,1]\cap C^4(0,1)$.

Since $u^*=Tu^*\in C_0^2[0,1]$, one has $u^*(0)=u^*(1)=0$. By the expressions of $u^*(t)$ and $(u^*)''(t)$, it is easy to prove that

$$\alpha(u^*)''(0)-\beta(u^*)'''(0)=0,\qquad \gamma(u^*)''(1)+\delta(u^*)'''(1)=0.$$

Therefore, $u^*\in K$ is a positive solution of the problem (P).

Proof of Theorem 3.2. We only prove the case (c1).

Applying the conditions $\varphi(a) \le A^{-1}a$, $\psi(b) > B^{-1}b$ and imitating the proof of Theorem 3.1, we can prove that the problem (P) has a positive solution $u_1^* \in K$ such that $u_1^* \in C^3[0,1] \cap C^4(0,1)$ and $a \le \|\!|u_1^*|\!\| < b$. Similarly, since $\psi(b) > B^{-1}b$, $\varphi(c) \le A^{-1}c$, (P) has another positive solution $u_2^* \in K$ such that $u_2^* \in C^3[0,1] \cap C^4(0,1)$ and $b < \|\!|u_2^*|\!\| \le c$.

Proof of Theorem 3.3. The proof is direct from Theorem 3.1.

4. On Theorem 1.1. Proposition 4.4 below shows that Theorem 1.1 is a special case of Theorem 3.3 under some stronger conditions, and Theorem 3.3 improves Theorem 1.1 in the great majority of cases because the homogeneity condition (a2) is canceled.

Remark 4.1. By Remark 1 in [22], if (a2) is satisfied, then the following inequalities hold for any $1 \le c < \infty$ and $(t,u,v) \in (0,1) \times (0,\infty) \times (0,\infty)$:

$$\begin{aligned} c^{\lambda_1} f(t,u,v) &\le f(t,cu,v) \le c^{\mu_1} f(t,u,v), \\ c^{\lambda_2} f(t,u,v) &\le f(t,u,cv) \le c^{\mu_2} f(t,u,v). \end{aligned}$$

Remark 4.2. If (a2) is satisfied, then

$$f(t,q(t),p(t)) \le \left[\frac{q(t)}{\xi(t)}\right]^{\mu_1} \left[\frac{p(t)}{\zeta(t)}\right]^{\mu_2} f(t,\xi(t),\zeta(t)).$$

This can be derived from Remark 4.1 and the simple facts that $\xi(t) \le q(t)$ and $\zeta(t) \le p(t)$ for any $0 \le t \le 1$.

Remark 4.3. If (a2) holds and

$$\int_0^1 \frac{f(t,\xi(t),\zeta(t))\,dt}{\xi^{\mu_1}(t)\zeta^{\mu_2}(t)} < \infty,$$

then (H) is satisfied. Indeed, let $0 < r_1 < r_2$. By Remarks 4.1 and 4.2, for $0 < t < 1$,

$$\begin{aligned} &\max\{f(t,u,v) : \eta r_1 q(t) \le u \le \tfrac{1}{8} r_2, r_1 p(t) \le v \le r_2\} \\ &\le \max\left\{\left(\frac{u}{\eta r_1 q(t)}\right)^{\mu_1} \left(\frac{v}{r_1 p(t)}\right)^{\mu_2} f(t,\eta r_1 q(t), r_1 p(t)) : \begin{matrix} \eta r_1 q(t) \le u \le \frac{1}{8} r_2, \\ r_1 p(t) \le v \le r_2 \end{matrix}\right\} \\ &\le \max\left\{\eta^{\lambda_1}\left(\frac{u}{\eta r_1 q(t)}\right)^{\mu_1} \left(\frac{v}{r_1 p(t)}\right)^{\mu_2} f(t, r_1 q(t), r_1 p(t)) : \begin{matrix} \eta r_1 q(t) \le u \le \frac{1}{8} r_2, \\ r_1 p(t) \le v \le r_2 \end{matrix}\right\} \\ &= \eta^{\lambda_1}\left(\frac{r_2}{8\eta r_1 q(t)}\right)^{\mu_1} \left(\frac{r_2}{r_1 p(t)}\right)^{\mu_2} f(t, r_1 q(t), r_1 p(t)) \end{aligned}$$

$$\le \eta^{\lambda_1}\left(\frac{r_2}{8\eta r_1 q(t)}\right)^{\mu_1}\left(\frac{r_2}{r_1 p(t)}\right)^{\mu_2}\max\{r_1^{\lambda_1}, r_1^{\mu_1}\}\max\{r_1^{\lambda_2}, r_1^{\mu_2}\}f(t,q(t),p(t))$$
$$\le \eta^{\lambda_1}\left(\frac{r_2}{8\eta r_1 \xi(t)}\right)^{\mu_1}\left(\frac{r_2}{r_1 \zeta(t)}\right)^{\mu_2}\max\{r_1^{\lambda_1}, r_1^{\mu_1}\}\max\{r_1^{\lambda_2}, r_1^{\mu_2}\}f(t,\xi(t),\zeta(t)).$$

Hence the function

$$j_{r_1}^{r_2}(t) = \eta^{\lambda_1}\left(\frac{r_2}{8\eta r_1 \xi(t)}\right)^{\mu_1}\left(\frac{r_2}{r_1 \zeta(t)}\right)^{\mu_2}\max\{r_1^{\lambda_1}, r_1^{\mu_1}\}\max\{r_1^{\lambda_2}, r_1^{\mu_2}\}f(t,\xi(t),\zeta(t))$$

satisfies the assumption (H).

PROPOSITION 4.4. *Theorem* 1.1 *is a special case of Theorem* 3.3 *if*

(1) $f(t,\xi(t),1)\not\equiv 0$ *is replaced by* $f(t,1,1)\not\equiv 0$.
(2) $0<\int_0^1 f(t,\xi(t),\zeta(t))\,dt<\infty$ *is replaced by* $\int_0^1 \frac{f(t,\xi(t),\zeta(t))}{\xi^{\mu_1}(t)\zeta^{\mu_2}(t)}\,dt<\infty$.

Proof. By Remark 4.3, the assumption (H) holds.

For $r\ge 1/\eta>1$, one has

$$\begin{aligned}
&\max\{f(t,u,v):\eta r q(t)\le u\le \tfrac18 r,\ rp(t)\le v\le r\}\\
&\quad\le \max\left\{\left(\frac{u}{\eta r q(t)}\right)^{\mu_1}\left(\frac{v}{rp(t)}\right)^{\mu_2} f(t,\eta r q(t), rp(t)) : \begin{array}{l}\eta r q(t)\le u\le \frac18 r,\\ rp(t)\le v\le r\end{array}\right\}\\
&\quad=\left(\frac{1}{8\eta}\right)^{\mu_1}\frac{f(t,\eta r q(t),rp(t))}{q^{\mu_1}(t)p^{\mu_2}(t)}\le \frac{r^{\mu_1+\mu_2}}{8^{\mu_1}\eta^{\mu_1-\lambda_1}}\,\frac{f(t,q(t),p(t))}{q^{\mu_1}(t)p^{\mu_2}(t)}\\
&\quad\le \frac{r^{\mu_1+\mu_2}}{8^{\mu_1}\eta^{\mu_1-\lambda_1}}\,\frac{f(t,\xi(t),\zeta(t))}{\xi^{\mu_1}(t)\zeta^{\mu_2}(t)}.
\end{aligned}$$

For $0<r\le 1$, one has

$$\begin{aligned}
&\min\{f(t,u,v):\eta r q(t)\le u\le \tfrac18 r,\ rp(t)\le v\le r\}\\
&\quad\ge \min\left\{\left(\frac{8u}{r}\right)^{\mu_1}\left(\frac{v}{r}\right)^{\mu_2} f\left(t,\frac18 r,r\right) : \begin{array}{l}\eta r q(t)\le u\le \frac18 r,\\ rp(t)\le v\le r\end{array}\right\}\\
&\quad=\eta^{\mu_1}q^{\mu_1}(t)p^{\mu_2}(t)f\big(t,\tfrac18 r,r\big)\ge (\eta/8)^{\mu_1}r^{\mu_1+\mu_2}q^{\mu_1}(t)p^{\mu_2}(t)f(t,1,1).
\end{aligned}$$

Since $f(t,1,1)\not\equiv 0$, there exists $0<t_0<1$ such that $f(t_0,1,1)>0$. Let $0<\sigma<\frac12\min\{t_0,1-t_0\}$. Then $\int_\sigma^{1-\sigma} q^{\mu_1}(t)p^{\mu_2}(t)f(t,1,1)\,dt>0$.

Additionally, by (a2), $0 \le \mu_1 + \mu_2 < 1$. It follows that

$$\begin{aligned}
\underline{\varphi}_\infty &= \liminf_{r\to\infty} \frac{\int_0^1 \max\{f(t,u,v) : \eta r q(t) \le u \le r/8,\, rp(t) \le v \le r\}\, dt}{r} \\
&\le \frac{1}{8^{\mu_1}\eta^{\mu_1-\lambda_1}} \int_0^1 \frac{f(t,\xi(t),\zeta(t))}{q^{\mu_1}(t)p^{\mu_2}(t)}\, dt \lim_{r\to\infty} \frac{1}{r^{1-\mu_1-\mu_2}} = 0, \\
\overline{\psi}_0 &= \limsup_{r\to+0} \frac{\int_\sigma^{1-\sigma} \min\{f(t,u,v) : \eta r q(t) \le u \le 1/8r,\, rp(t) \le v \le r\}\, dt}{r} \\
&\ge (\eta/8)^{\mu_1} \lim_{r\to+0} \frac{\int_\sigma^{1-\sigma} q^{\mu_1}(t)p^{\mu_2}(t) f(t,1,1)\, dt}{r^{1-\mu_1-\mu_2}} = \infty.
\end{aligned}$$

By Theorem 3.3(d2), (P) has a positive solution $u^* \in C^3[0,1] \cap C^4(0,1)$.

REMARK 4.5. Theorem 1.1 is not a corollary of Theorem 3.3.

For example, consider the problem

$$u^{(4)}(t) = \frac{\sqrt{u(t)}}{t(1-t)}, \qquad 0 < t < 1, \qquad u(0) = u(1) = u''(0) = u''(1) = 0.$$

Here $f(t,u,v) = f(t,u) = \frac{\sqrt{u}}{t(1-t)}$ and $\zeta(t) = \xi(t) = t(1-t)$. So,

$$\int_0^1 f(t,\xi(t))\, dt = \int_0^1 \frac{dt}{\sqrt{t(1-t)}} < \infty.$$

Obviously, the other conditions of Theorem 1.1 are satisfied. This implies that the problem has a positive solution $u^* \in C^3[0,1]$.

However, for any $r_2 > r_1 > 0$,

$$\int_0^1 \max\left\{ \frac{\sqrt{u}}{t(1-t)} : \eta r_1 q(t) \le u \le \frac{1}{8} r_2 \right\} dt = \sqrt{\frac{r_2}{8}} \int_0^1 \frac{dt}{t(1-t)} = \infty.$$

This shows that the assumption (H) does not hold. Consequently, the above existence conclusion cannot be derived from Theorem 3.3.

5. Two examples. Examples 5.1 and 5.2 below illustrate our improvements.

EXAMPLE 5.1. Let $\sigma = 1/4$. Consider the boundary value problem

$$\begin{cases} u^{(4)}(t) = \dfrac{5}{8}\left(1 + \dfrac{1}{16\sqrt{u(t)}} \sin^2 \dfrac{1}{u(t)}\right) e^{u(t)-u''(t)}, \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

In this problem, $\alpha = \gamma = 1$, $\beta = \delta = 0$, $\eta = 1/24$, $A = 1/4$, $B = 1/16$,

$p(t) = q(t) = \min\{t, 1-t\}$ and

$$f(t,u,v) = f(u,v) = \frac{5}{8}\left(1 + \frac{1}{16\sqrt{u}} \sin^2 \frac{1}{u}\right) e^{u+v}.$$

So, $f(u,v)$ is singular at $u = 0$ for any $v \in [0,\infty)$. Obviously, $f(u,v)$ satisfies the assumption (H).

Direct computations give

$$\begin{aligned}\varphi(1) &\le \frac{5}{8}\int_0^1 \max\left\{\left(1 + \frac{1}{16\sqrt{u}}\right)e^{u+v} : \frac{1}{24}q(t) \le u \le \frac{1}{8},\ q(t) \le v \le 1\right\} dt \\ &\le \frac{5}{8}e^{9/8}\int_0^1 \left[1 + \frac{\sqrt{6}}{8\sqrt{q(t)}}\right] dt = \frac{5}{8}e^{9/8}[1 + \sqrt{3}/2] \approx 3.5923 < 4 = A^{-1}.\end{aligned}$$

Since $\min_{1/4 \le t \le 3/4} q(t) = 1/4$, we obtain

$$\begin{aligned}\psi(30) &\ge \frac{5}{8}\int_{1/4}^{3/4} \min\{e^v : 30q(t) \le v \le 33\}\, dt \\ &\ge \frac{5}{16}e^{7.5} \approx 565.01 > 512 = 32B^{-1}.\end{aligned}$$

By Theorem 3.1(b1), the problem has a positive solution $u^* \in C^3[0,1]$ and $1 \le \|u^*\| \le 30$. Since the function $f(u,v)$ does not satisfy (a2), the conclusion cannot be derived from Theorem 1.1.

EXAMPLE 5.2. Consider the fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = \sin^2(u(t) - t(1-t)) + \max\left\{0, \dfrac{1}{\sqrt{-u''(t)}} - 1\right\}, & 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

In this problem $\alpha = \gamma = 1$, $\beta = \delta = 0$, $\eta = 1/24$, $A = 1/4$, $B = 1/16$, $p(t) = q(t) = \min\{t, 1-t\}$ and

$$f(t,u,v) = \sin^2(u - t(1-t)) + \max\left\{0, \frac{1}{\sqrt{v}} - 1\right\}.$$

So $f(t,\xi(t),1) \equiv 0$. Obviously, $f(t,u,v)$ satisfies (H).

Let $\sigma = 1/4$. Direct computations give

$$\begin{aligned}\underline{\varphi}_\infty &= \liminf_{r\to\infty} \frac{1}{r}\int_0^1 \max\{1 + 1/\sqrt{v} : rq(t) \le v \le r\}\, dt \\ &\le \lim_{r\to\infty} \frac{1}{r}\int_0^1 \left[1 + \frac{1}{\sqrt{rq(t)}}\right] dt = 0,\end{aligned}$$

$$\overline{\psi}_0 \geq \limsup_{r\to+0} \frac{1}{r} \int_{1/4}^{3/4} \min\{\max\{0, 1/\sqrt{v} - 1\} : rp(t) \leq v \leq r\}\, dt$$

$$= \lim_{r\to+0} \frac{1}{r} \left[\frac{1}{\sqrt{r}} \int_{1/4}^{3/4} \frac{dt}{\sqrt{q(t)}} - \frac{1}{2} \right] = \lim_{r\to+0} \frac{1}{r} \left[\frac{2(\sqrt{2}-1)}{\sqrt{r}} - \frac{1}{2} \right] = \infty.$$

By Theorem 3.3(d2), the problem has a positive solution $u^* \in C^3[0,1]$. Since $f(t, \xi(t), 1) \equiv 0$, the conclusion cannot be derived from Theorem 1.1.

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