A pair of linear functional inequalities and a characterization of $L^p$-norm

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Abstract. It is shown that, under some general algebraic conditions on fixed real numbers $a, b, \alpha, \beta$, every solution $f : \mathbb{R} \to \mathbb{R}$ of the system of functional inequalities $f(x + a) \leq f(x) + \alpha$, $f(x + b) \leq f(x) + \beta$ that is continuous at some point must be a linear function (up to an additive constant). Analogous results for three other similar simultaneous systems are presented. An application to a characterization of $L^p$-norm is given.

1. Introduction. Every subadditive function $f : \mathbb{R} \to \mathbb{R}$, that is, such that

$$f(x + y) \leq f(x) + f(y), \quad x, y \in \mathbb{R},$$

where $\mathbb{R}$ stands for the set of reals, satisfies the simultaneous system of functional inequalities of additive type:

$$f(a + x) \leq \alpha + f(x), \quad f(b + x) \leq \beta + f(x), \quad x \in \mathbb{R},$$

where $a, b \in \mathbb{R}$ are arbitrarily fixed and $\alpha = f(a)$, $\beta = f(b)$. In Section 2 we present some algebraic conditions on $a, b, \alpha, \beta$ under which the only function satisfying this pair of functional inequalities and continuous at some point is $f(x) = \frac{\alpha}{a} x + f(0)$.

In Sections 3, 4 and 5, respectively, we also present analogous conditions for pairs of functional inequalities

$$f(a + x) \leq \alpha f(x), \quad f(b + x) \leq \beta f(x);$$

$$f(ax) \leq \alpha + f(x), \quad f(bx) \leq \beta + f(x);$$

$$f(ax) \leq \alpha f(x), \quad f(bx) \leq \beta f(x).$$

The theorems of Sections 2–5 generalize the results of [4], where the corresponding pairs of functional equations were considered (Remark 1). They allow us, in particular, to derive some classical theorems on the Cauchy type functional equation (cf. J. Aczél [1] and M. Kuczma [3]).

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[1]
For a measure space \((\Omega, \Sigma, \mu)\) denote by \(S = S(\Omega, \Sigma, \mu)\) the linear space of all \(\mu\)-integrable simple functions \(x : \Omega \rightarrow \mathbb{R}\). Let \(\phi : (0, \infty) \rightarrow (0, \infty)\) be an arbitrary bijection. As an application, in Section 6, we give a new characterization of the \(L^p\)-norm with the aid of a rather weak subhomogeneity condition on the \(L^p\)-norm-like functional \(p_\phi\),

\[
p_\phi(x) := \begin{cases} 
\phi^{-1}\left(\int_{\Omega_x} \phi \circ |x| \, d\mu\right), & \mu(\Omega_x) > 0, \\
0, & \mu(\Omega_x) = 0,
\end{cases} 
\]

where \(\Omega_x := \{\omega \in \Omega : x(\omega) \neq 0\}\). Let us mention that in A. C. Zaanen [8], W. Wnuk [7], and J. Matkowski [5], the functional \(p_\phi\) is assumed to be positively homogeneous.

By \(\mathbb{N}, \mathbb{Z},\) and \(\mathbb{Q}\) we denote, respectively, the sets of natural, integer, and rational numbers.

2. Inequalities of additive type

**Theorem 1.** Let \(a, b, \alpha, \beta \in \mathbb{R}\) be fixed numbers. Suppose that

\[a < 0 < b, \quad \frac{b}{a} \notin \mathbb{Q}, \quad \frac{\alpha}{a} \geq \frac{\beta}{b},\]

and a function \(f : \mathbb{R} \rightarrow \mathbb{R}\) is continuous at least at one point.

If \(f\) satisfies the pair of functional inequalities

\[f(a + x) \leq \alpha + f(x), \quad f(b + x) \leq \beta + f(x), \quad x \in \mathbb{R},\]

then

\[f(x) = \frac{\alpha}{a} x + f(0), \quad x \in \mathbb{R}.\]

**Proof.** From (1), by induction, we obtain

\[f(ma + x) \leq m\alpha + f(x), \quad f(nb + x) \leq n\beta + f(x), \quad m, n \in \mathbb{N}, \quad x \in \mathbb{R}.
\]

Replacing \(x\) by \(nb + x\) in the first of these inequalities we hence get

\[f(ma + nb + x) \leq m\alpha + n\beta + f(x), \quad m, n \in \mathbb{N}, \quad x \in \mathbb{R}.
\]

Since \(b/a \notin \mathbb{Q}\), and \(ab < 0\), the Kronecker theorem (cf. [6]) implies that the set

\[A = \{ma + nb : m, n \in \mathbb{N}\}
\]

is dense in \(\mathbb{R}\). Thus there exist two sequences \((m_k), (n_k)\) of positive integers such that

\[
\lim_{k \to \infty} (m_k a + n_k b) = 0.
\]

Note that

\[\lim_{k \to \infty} m_k = \lim_{k \to \infty} n_k = \infty\]
(otherwise $b/a$ would be rational). Obviously,
\[
\lim_{k \to \infty} \frac{m_k a + n_k b}{m_k} = 0,
\]
and, consequently,
\[
(4) \quad \lim_{k \to \infty} \frac{n_k}{m_k} = -\frac{a}{b}.
\]
Let $x_0 \in \mathbb{R}$ be a point of continuity of $f$. From (2) we get
\[
f(m_k a + n_k b + x_0) \leq m_k \alpha + n_k \beta + f(x_0), \quad k \in \mathbb{N},
\]
or, equivalently,
\[
\frac{f(m_k a + n_k b + x_0)}{m_k} \leq \alpha + \frac{n_k}{m_k} \beta + \frac{f(x_0)}{m_k}, \quad k \in \mathbb{N}.
\]
Letting $k \to \infty$, and making use of (3), (4), and the continuity of $f$ at $x_0$, we hence get $0 \leq \alpha - \frac{a}{b} \beta$, i.e.
\[
\frac{\beta}{b} \geq \frac{\alpha}{a}.
\]
As, by the assumption, the reverse inequality holds true, we have shown that
\[
\frac{\alpha}{a} = \frac{\beta}{b}.
\]
Now, setting
\[
p := \frac{\alpha}{a} = \frac{\beta}{b},
\]
we can write inequality (2) in the form
\[
(5) \quad f(t + x) \leq pt + f(x), \quad t \in A, \ x \in \mathbb{R}.
\]
Take an arbitrary $x \in \mathbb{R}$. By the density of $A$ there is a sequence $(t_n)$ such that
\[
t_n \in A \ (n \in \mathbb{N}), \quad \lim_{n \to \infty} t_n = x_0 - x.
\]
From (5) we have
\[
f(t_n + x) \leq pt_n + f(x), \quad n \in \mathbb{N}.
\]
Letting $n \to \infty$, and making use of the continuity of $f$ at $x_0$, we obtain
\[
f(x_0) \leq p(x_0 - x) + f(x), \quad x \in \mathbb{R}.
\]
To prove the opposite inequality note that replacing $x$ by $x - t$ in (5) we get
\[
f(x) \leq pt + f(x - t), \quad t \in A, \ x \in \mathbb{R}.
\]
Taking an $x \in \mathbb{R}$, and, by the density of $A$, a sequence $(t_n)$ such that
\[
t_n \in A \ (n \in \mathbb{N}), \quad \lim_{n \to \infty} t_n = x - x_0,
\]
we hence get
\[ f(x) \leq pt_n + f(x - t_n), \quad n \in \mathbb{N}. \]
Letting \( n \to \infty \), and again making use of the continuity of \( f \) at \( x_0 \), we obtain
\[ f(x) \leq p(x - x_0) + f(x_0), \quad x \in \mathbb{R}. \]
Thus
\[ f(x) = px + (f(x_0) - px_0), \quad x \in \mathbb{R}, \]
which was to be shown.

**Remark 1.** Let \( a, b, \alpha, \beta \in \mathbb{R}, ab \neq 0 \), be such that \( \beta/b = \alpha/a \).

If \( b/a \notin \mathbb{Q} \) and a function \( f : \mathbb{R} \to \mathbb{R} \) is continuous at least at one point and satisfies the simultaneous system of functional equations
\[
\begin{align*}
f(a + x) &= \alpha + f(x), \quad x \in \mathbb{R}, \\
f(b + x) &= \beta + f(x), \quad x \in \mathbb{R},
\end{align*}
\]
then \( f(x) = px + q \) for some \( p, q \in \mathbb{R}, x \in \mathbb{R} \) (cf. [4]).

If \( b/a \in \mathbb{Q} \) then this system of functional equations reduces to the single functional equation
\[
f(d + x) = \frac{\alpha}{a} + f(x), \quad x \in \mathbb{R},
\]
where \( d := \min\{ma + nb > 0 : m, n \in \mathbb{N}\} \).

Since the continuous and monotonic solution of this equation depends on an arbitrary function (cf. M. Kuczma [2]), the assumption that \( b/a \notin \mathbb{Q} \) in Theorem 1 is essential.

**Remark 2.** The assumption \( \alpha/a \geq \beta/b \) is essential for the uniqueness of the solution of system (1) in Theorem 1. Indeed, if \( \alpha/a < \beta/b \) the set of solutions of (1) is large; for instance the function \( f := \sin \) satisfies (1) for all \( a, b \in \mathbb{R} \) and \( \alpha, \beta \geq 2 \). Moreover every affine function of the form \( f(x) = Ax + B \) where \( B \in \mathbb{R} \) is arbitrary and \( \alpha/a \leq A \leq \beta/b \) is a solution of (1).

3. Inequalities of additive-multiplicative type

**Lemma 1.** Let \( a, b, \alpha, \beta \) be fixed real numbers such that
\[
a < 0 < b, \quad \frac{b}{a} \notin \mathbb{Q}, \quad \alpha, \beta > 0.
\]
Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is continuous at least at one point and satisfies the system of functional inequalities
\[
\begin{align*}
f(a + x) &\leq \alpha f(x), \quad f(b + x) \leq \beta f(x), \quad x \in \mathbb{R},
\end{align*}
\]
or
\[
\begin{align*}
f(a + x) &\geq \alpha f(x), \quad f(b + x) \geq \beta f(x), \quad x \in \mathbb{R}.
\end{align*}
\]
Then \( f \) is either positive in \( \mathbb{R} \), negative in \( \mathbb{R} \), or identically zero.
Proof. Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies (6), $x_0$ is a point of continuity of $f$ and $f(x_0) > 0$. From (6), by induction, we get
\[(8) f(ma + nb + x) \leq \alpha^m \beta^n f(x), \quad m, n \in \mathbb{N}, \ x \in \mathbb{R}.
\]
Take an arbitrary $x \in \mathbb{R}$. By the density of the set $A = \{ma + nb : m, n \in \mathbb{N}\}$ in $\mathbb{R}$ there exists a sequence $(m_k a + n_k b : k, m_k, n_k \in \mathbb{N})$ such that
\[
\lim_{k \to \infty} (m_k a + n_k b) = x_0 - x.
\]
From (8) we have
\[f(m_k a + n_k b + x) \leq \alpha^{m_k} \beta^{n_k} f(x), \quad m_k, n_k \in \mathbb{N}, \ x \in \mathbb{R}.
\]
For $k$ large enough, by the continuity of $f$ at $x_0$, the left-hand side of this inequality is positive. It follows that $f$ is positive.

Suppose now that $f(x_0) < 0$. Replacing $x$ by $x - (ma + nb)$ in (8) we get
\[(9) f(x) \leq \alpha^m b^n f(x - (ma + nb)), \quad m, n \in \mathbb{N}, \ x \in \mathbb{R}.
\]
Now, similarly to the previous case, fix $x \in \mathbb{R}$ and take a sequence $(m_k a + n_k b : k, m_k, n_k \in \mathbb{N})$ such that
\[
\lim_{k \to \infty} (m_k a + n_k b) = x - x_0.
\]
Again by the continuity of $f$ at $x_0$, for $k$ large enough, the right-hand side of inequality (9) is negative and hence so is $f(x)$.

If $f(x_0) = 0$ an argument analogous to the first step shows that $f(x) \geq 0$ for all $x \in \mathbb{R}$, and a slight modification of the argument of the second step gives the inequality $f(x) \leq 0$ for all $x \in \mathbb{R}$, and, consequently, $f = 0$ in $\mathbb{R}$.

To complete the proof it is enough to repeat the same reasoning for system (7).

**Theorem 2.** Let $a, b \in \mathbb{R}$ and $\alpha, \beta > 0$ be fixed numbers such that
\[a < 0 < b, \quad \frac{b}{a} \notin \mathbb{Q}, \quad \frac{\log \alpha}{a} \geq \frac{\log \beta}{b}.
\]
Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at least at one point and such that $f(\mathbb{R}) \not\subset (-\infty, 0)$. If $f$ satisfies the pair of functional inequalities (6), i.e.
\[f(a + x) \leq \alpha f(x), \quad f(b + x) \leq \beta f(x), \quad x \in \mathbb{R},
\]
then either $f$ is identically zero in $\mathbb{R}$, or
\[f(x) = f(0) e^{\log \frac{\alpha}{a} x}, \quad x \in \mathbb{R}.
\]

**Proof.** Assume that $f$ satisfies (6). By the assumptions and Lemma 1 the function $f$ is either identically zero in $\mathbb{R}$ or positive in $\mathbb{R}$. In the first case there is nothing to prove. In the second case $f$ is positive and the function $g := \log f$ satisfies the inequalities
\[g(a + x) \leq \log \alpha + g f(x), \quad g(b + x) \leq \log \beta + g f(x), \quad x \in \mathbb{R},
\]
and our theorem results from Theorem 1.
Obviously, for inequalities (7) an analogous result holds true.

**Remark 3.** The assumption \( f(\mathbb{R}) \notin (\infty, 0) \) in Theorem 2 is essential. To see this, take arbitrary \( a, b \in \mathbb{R} \) such that \( a < 0 < b, \) \( b/a \notin \mathbb{Q}, \) \( \alpha, \beta \in (0, 1/2), \) and an arbitrary function \( f : \mathbb{R} \to [-2, -1]. \) Then for all \( x \in \mathbb{R}, \)
\[
f(a + x) \leq -1 = \frac{1}{2} \cdot (-2) \leq \frac{1}{2} f(x) \leq \alpha f(x)
\]
and similarly, for all \( x \in \mathbb{R}, \)
\[
f(b + x) \leq \beta f(x),
\]
which proves that \( f \) satisfies (6). Since \( \log \alpha < 0 \) and \( \log \beta < 0 \) and \( a < 0 < b, \) we have
\[
\frac{\log \alpha}{a} < 0 < \frac{\log \beta}{b}.
\]
Thus all assumptions of Theorem 2 except the condition \( f(\mathbb{R}) \notin (\infty, 0) \) are satisfied.

**4. Inequalities of multiplicative-additive type.** As an easy consequence of Theorem 1 we have

**Theorem 3.** Let \( a, b, \alpha, \beta \) be fixed real numbers such that
\[
0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad \frac{\alpha}{\log a} \geq \frac{\beta}{\log b}.
\]
Suppose that a function \( f : I \to \mathbb{R} \) is continuous at least at one point and satisfies the pair of functional inequalities
\[
(10) \quad f(ax) \leq \alpha + f(x), \quad f(bx) \leq \beta + f(x), \quad x \in I,
\]
where either \( I = (0, \infty) \) or \( I = (-\infty, 0). \)

(i) If \( I = (0, \infty), \) then
\[
f(x) = \frac{\alpha}{\log a} \log x + f(1), \quad x > 0.
\]

(ii) If \( I = (-\infty, 0), \) then
\[
f(x) = \frac{\alpha}{\log a} \log(-x) + f(-1), \quad x < 0.
\]

**Corollary 1.** Let \( a, b, \alpha, \beta \) satisfy the assumptions of Theorem 3. If a function \( f : (-\infty, 0) \cup (0, \infty) \to \mathbb{R} \) satisfies the pair of inequalities (10), and in each of the intervals \( (-\infty, 0) \) and \( (0, \infty) \) there is at least one point of continuity of \( f, \) then
\[
f(x) = \begin{cases} 
\frac{\alpha}{\log a} \log x + f(1) & \text{for } x \in (0, \infty), \\
\frac{\alpha}{\log a} \log(-x) + f(-1) & \text{for } x \in (-\infty, 0).
\end{cases}
\]
Remark 4. Suppose that $a, b, \alpha, \beta$ are fixed real numbers such that $0 < a < 1 < b$ and $\alpha / \log a = \beta / \log b$. Note that if $0 \in I$ then there is no function satisfying (10). Indeed, putting $x = 0$ into (10) we get $0 \leq \alpha$, $0 \leq \beta$, which contradicts the assumptions.

5. Inequalities of multiplicative type. The following counterpart of Lemma 1 is easy to verify.

Lemma 2. Let $a, b, \alpha, \beta$ be fixed positive real numbers such that

$$a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q},$$

and $I = (0, \infty)$ or $I = (-\infty, 0)$. Suppose that $f : I \to \mathbb{R}$ is continuous at least at one point and satisfies the system of functional inequalities

(11) \quad $f(ax) \leq \alpha f(x), \quad f(bx) \leq \beta f(x), \quad x \in I,$

or

(12) \quad $f(ax) \geq \alpha f(x), \quad f(bx) \geq \beta f(x), \quad x \in I.$

Then $f$ is either positive in $I$, negative in $I$, or identically zero.

Applying Lemma 2 and Theorem 1 we obtain

Theorem 4. Let $a, b, \alpha, \beta$ be fixed positive real numbers such that

$$a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad \frac{\log \alpha}{\log a} \geq \frac{\log \beta}{\log b},$$

and $I = (0, \infty)$ or $I = (-\infty, 0)$. Suppose that $f : I \to \mathbb{R}$ is continuous at least at one point, satisfies the pair of functional inequalities

(13) \quad $f(ax) \leq \alpha f(x), \quad f(bx) \leq \beta f(x), \quad x \in I,$

and $f(I) \notin (-\infty, 0)$. Then either $f$ is identically zero in $I$ or

(i) in the case $I = (0, \infty)$,

$$f(x) = f(1)x^{\frac{\log \alpha}{\log a}}, \quad x > 0,$$

(ii) in the case $I = (-\infty, 0)$,

$$f(x) = f(-1)(-x)^{\frac{\log \alpha}{\log a}}, \quad x < 0.$$

We omit the formulation of the corresponding result for inequalities (12).

Remark 5. Suppose that $a, b, \alpha, \beta$ are fixed positive real numbers such that $a < 1 < b$ and $\frac{\log \alpha}{\log a} = \frac{\log \beta}{\log b}$. Note that if $I = \mathbb{R}$ or $I = [0, \infty)$ or $I = (-\infty, 0]$ and $f : I \to \mathbb{R}$ satisfies (13), then $f(0) = 0$. Indeed, by assumptions either $\alpha < 1 < \beta$ or $\beta < 1 < \alpha$ and, moreover, $f(0)(1 - \alpha) \leq 0$ and $f(0)(1 - \beta) \leq 0$. Thus $f(0) = 0$. 
Hence we get

**Remark 6.** (i) Suppose that \( f : [0, \infty) \to \mathbb{R} \) satisfies (13). If \( f|_{(0, \infty)} \) and \( a, b, \alpha, \beta \) satisfy the assumptions of Theorem 4, then

\[
f(x) = \begin{cases} 
  f(1)x^{\frac{\log \alpha}{\log a}} & \text{for } x \in (0, \infty), \\
  0 & \text{for } x = 0.
\end{cases}
\]

(ii) Suppose that \( f : (-\infty, 0] \to \mathbb{R} \) satisfies (13). If \( f|_{(-\infty, 0)} \) and \( a, b, \alpha, \beta \) satisfy the assumptions of Theorem 4, then

\[
f(x) = \begin{cases} 
  f(-1)(-x)^{\frac{\log \alpha}{\log a}} & \text{for } x \in (-\infty, 0), \\
  0 & \text{for } x = 0.
\end{cases}
\]

Finally, let us record the following

**Remark 7.** For obvious reasons the counterparts of Theorems 1–4 for the reverse inequalities remain true.

6. A characterization of \( L^p \)-norm. Recall that A. C. Zaanen [8], for the counting measure space, W. Wnuk [7], assuming the continuity of the function \( \phi \), and J. Matkowski [5], assuming much weaker regularity conditions, characterized the \( L^p \)-norm with the aid of the homogeneity of the functional \( p_\phi \) (cf. the definition in the Introduction).

As an application of Theorem 4 we present a far-reaching generalization of these results. It turns out that the homogeneity condition can be replaced by an inequality assumed to be satisfied only for two characteristic functions \( \chi_A, \chi_B \) of suitably chosen measurable sets \( A \) and \( B \).

**Theorem 5.** Let \((\Omega, \Sigma, \mu)\) be a measure space with two sets \( A, B \in \Sigma \) such that

\[
0 < \mu(A) < 1 < \mu(B) < \infty, \quad \frac{\log \mu(B)}{\log \mu(A)} \notin \mathbb{Q}.
\]

Suppose that \( \phi : (0, \infty) \to (0, \infty) \) is a bijection such that \( \phi^{-1} \) is continuous at least at one point and

\[
\frac{\log \phi^{-1}(\mu(A)\phi(1))}{\log \mu(A)} \geq \frac{\log \phi^{-1}(\mu(B)\phi(1))}{\log \mu(B)}.
\]  \((14)\)

If \( p_\phi \) satisfies the condition

\[
p_\phi(tx) \leq tp_\phi(x), \quad t > 0, \quad x \in \{\chi_A, \chi_B\},
\]  \((15)\)

then

\[
\phi(t) = \phi(1)t^p, \quad t > 0,
\]

where

\[
p := \frac{\log \phi^{-1}(\mu(A)\phi(1))}{\log \mu(A)}.
\]

Moreover, if \( p \geq 1 \) then \( p_\phi \) coincides with the \( L^p \)-norm.
Proof. Let $a = \mu(A)$ and $b = \mu(B)$. From (15) we obtain
\[
\phi^{-1}(a\phi(t)) \leq t\phi^{-1}(a\phi(1)), \quad \phi^{-1}(b\phi(t)) \leq t\phi^{-1}(b\phi(1)), \quad t > 0,
\]
which with $\alpha := \phi^{-1}(a\phi(1))$ and $\beta := \phi^{-1}(b\phi(1))$ reduces to the pair of functional inequalities
\[
\phi^{-1}(a\phi(t)) \leq \alpha t, \quad \phi^{-1}(b\phi(t)) \leq \beta t, \quad t > 0.
\]
From the bijectivity of $\phi$, replacing here $t$ by $\phi^{-1}(t)$, we get the equivalent system of inequalities
\[
\phi^{-1}(at) \leq \alpha\phi^{-1}(t), \quad \phi^{-1}(bt) \leq \beta\phi^{-1}(t), \quad t > 0,
\]
which, with $f := \phi^{-1}$ and $I = (0, \infty)$, takes the form (13). Now our result follows from Theorem 4.

Remark 8. Note that (15) is a very weak substitute of the homogeneity of the functional $p_\phi$.

Discussing the assumptions in Theorem 5, note that the condition: $\frac{\log b}{\log a} \notin \mathbb{Q}$ or $\frac{\log(a+b)}{\log a} \notin \mathbb{Q}$ is not too demanding.

To show that the assumption of the existence of sets $A$ and $B$ with $0 < \mu(A) < 1 < \mu(B) < \infty$ is essential, we indicate some wide classes of non-power functions $\phi$ for which the functional $p_\phi$ satisfies the condition (15), in each of the cases
\[
\mu(A) \leq 1 \text{ or } \mu(A) = \infty \quad \text{for every } A \in \Sigma;
\]
\[
\mu(A) \geq 1 \text{ or } \mu(A) = 0 \quad \text{for every } A \in \Sigma.
\]

Example 1. Let $(\Omega, \Sigma, \mu)$ be a measure space such that $\mu(\Omega) \leq 1$. Put $\delta := \inf\{\mu(A) : A \in \Sigma \land \mu(A) > 0\}$. Let $\phi : (0, \infty) \to (0, \infty)$ be an increasing bijection such that the function $(0, \infty) \ni t \mapsto \phi(t)/t$ is non-increasing and $\phi(\delta) = \delta, \phi(1) = 1$. Then $\phi(t) = t$ for all $t \in [\delta, 1]$, the function $(0, \infty) \ni t \mapsto \phi^{-1}(t)/t$ is non-decreasing and, therefore, for each $A \in \Sigma$ with $a := \mu(A) > 0$, we have
\[
p_\phi(t\chi_A) = \phi^{-1}(a\phi(t)) = \frac{\phi^{-1}(a\phi(t))}{a\phi(t)} a\phi(t)
\]
\[
\leq \frac{\phi^{-1}(\phi(t))}{\phi(t)} a\phi(t) = ta = t\phi^{-1}(a\phi(1))
\]
\[
= tp_\phi(\chi_A), \quad t > 0.
\]
Thus $p_\phi$ satisfies (15) and $\phi$ is not a power function.

Example 2. Let $(\Omega, \Sigma, \mu)$ be a measure space for which $\mu(A) \geq 1$ for every set $A \in \Sigma$ such that $\mu(A) > 0$, and there exists $B \in \Sigma$ such that $1 < \mu(B) < \infty$. Then $\delta := \inf\{\mu(A) : A \in \Sigma \land \mu(A) > 0\} \geq 1$. Let $\phi : (0, \infty) \to (0, \infty)$ be a bijection such that the function $(0, \infty) \ni t \mapsto \phi(t)/t$
is non-decreasing and $\phi(1) = 1$, $\phi(\delta) = \delta$. Then $\phi$ is strictly increasing, $\phi(t) = t$ for all $t \in [1, \delta]$, the function $(0, \infty) \ni t \mapsto \phi^{-1}(t)/t$ is non-
increasing, and therefore, in the same way as in the previous example, for all $B \in \Sigma$ such that $0 < \mu(B) < \infty$, we have

$$p_\phi(t\chi_B) \leq tp_\phi(\chi_B), \quad t > 0.$$  

We end our discussion with an example showing that the assumption

(14) is indispensable.

**Example 3.** Let $(\Omega, \Sigma, \mu)$ be an arbitrary measure space, and $f : \mathbb{R} \to \mathbb{R}$ a bijection such that $f(0) = 0$ and $f^{-1}$ is subadditive (for instance, for $f$ one can take the inverse function to $x \mapsto x + |\sin x|$). Define $\phi : (0, \infty) \to (0, \infty)$ by $\phi(t) = e^{f(\log t)}$. Then, making use of the definition of $p_\phi$, the subadditivity

of $f^{-1}$, and the monotonicity of the exponential function, for all $A \in \Sigma$ with $a := \mu(A) > 0$, we have

$$p_\phi(t\chi_A) = e^{f^{-1}(a\phi(t))} = e^{f^{-1}(\log a + f(\log t))}$$

$$\leq e^{f^{-1}(\log a)}e^{f^{-1}(f(\log t))}$$

$$= t\phi^{-1}(a) = tp_\phi(\chi_A), \quad t > 0.$$  

Thus $p_\phi$ satisfies the subhomogeneity condition (15) for all functions $\chi_A$ (here $\mu(A)$ can be smaller or greater than 1). This shows that in Theorem 5

condition (14) cannot be omitted.

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**References**


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