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A pair of linear functional inequalities and a characterization of L^p-norm

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Abstract. It is shown that, under some general algebraic conditions on fixed real numbers a, b, α, β, every solution $f : \mathbb{R} \to \mathbb{R}$ of the system of functional inequalities $f(x+a) \le f(x)+\alpha$, $f(x+b) \le f(x)+\beta$ that is continuous at some point must be a linear function (up to an additive constant). Analogous results for three other similar simultaneous systems are presented. An application to a characterization of L^p-norm is given.

1. Introduction. Every subadditive function $f : \mathbb{R} \to \mathbb{R}$, that is, such that

$$f(x+y) \le f(x) + f(y), \qquad x, y \in \mathbb{R},$$

where $\mathbb{R}$ stands for the set of reals, satisfies the simultaneous system of functional inequalities of *additive type*:

$$f(a+x) \le \alpha + f(x), \qquad f(b+x) \le \beta + f(x), \qquad x \in \mathbb{R},$$

where $a, b \in \mathbb{R}$ are arbitrarily fixed and $\alpha = f(a)$, $\beta = f(b)$. In Section 2 we present some algebraic conditions on a, b, α, β under which the only function satisfying this pair of functional inequalities and continuous at some point is $f(x) = \frac{\alpha}{a}x + f(0)$.

In Sections 3, 4 and 5, respectively, we also present analogous conditions for pairs of functional inequalities

$$f(a+x) \le \alpha f(x), \qquad f(b+x) \le \beta f(x);$$
$$f(ax) \le \alpha + f(x), \qquad f(bx) \le \beta + f(x);$$
$$f(ax) \le \alpha f(x), \qquad f(bx) \le \beta f(x).$$

The theorems of Sections 2–5 generalize the results of [4], where the corresponding pairs of functional equations were considered (Remark 1). They allow us, in particular, to derive some classical theorems on the Cauchy type functional equation (cf. J. Aczél [1] and M. Kuczma [3]).

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For a measure space (Ω, Σ, μ) denote by $\mathbb{S} = \mathbb{S}(\Omega, \Sigma, \mu)$ the linear space of all μ-integrable simple functions $x : \Omega \to \mathbb{R}$. Let $\phi : (0, \infty) \to (0, \infty)$ be an arbitrary bijection. As an application, in Section 6, we give a new characterization of the L^p-norm with the aid of a rather weak subhomogeneity condition on the L^p-norm-like functional $\mathbf{p}_\phi$,

$$\mathbf{p}_\phi(x) := \begin{cases} \phi^{-1}\Big(\int\limits_{\Omega_x} \phi \circ |x| \, d\mu\Big), & \mu(\Omega_x) > 0, \\ 0, & \mu(\Omega_x) = 0, \end{cases} \qquad x \in \mathbb{S},$$

where $\Omega_x := \{\omega \in \Omega : x(\omega) \neq 0\}$. Let us mention that in A. C. Zaanen [8], W. Wnuk [7], and J. Matkowski [5], the functional $\mathbf{p}_\phi$ is assumed to be positively homogeneous.

By $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ we denote, respectively, the sets of natural, integer, and rational numbers.

2. Inequalities of additive type

THEOREM 1. *Let $a, b, \alpha, \beta \in \mathbb{R}$ be fixed numbers. Suppose that*

$$a < 0 < b, \qquad \frac{b}{a} \notin \mathbb{Q}, \qquad \frac{\alpha}{a} \geq \frac{\beta}{b},$$

and a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at least at one point.

If f satisfies the pair of functional inequalities

$$f(a + x) \leq \alpha + f(x), \qquad f(b + x) \leq \beta + f(x), \qquad x \in \mathbb{R}, \tag{1}$$

then

$$f(x) = \frac{\alpha}{a} x + f(0), \qquad x \in \mathbb{R}.$$

Proof. From (1), by induction, we obtain

$$f(ma + x) \leq m\alpha + f(x), \qquad f(nb + x) \leq n\beta + f(x), \qquad m, n \in \mathbb{N},\ x \in \mathbb{R}.$$

Replacing x by $nb + x$ in the first of these inequalities we hence get

$$f(ma + nb + x) \leq m\alpha + n\beta + f(x), \qquad m, n \in \mathbb{N},\ x \in \mathbb{R}. \tag{2}$$

Since $b/a \notin \mathbb{Q}$, and $ab < 0$, the Kronecker theorem (cf. [6]) implies that the set

$$A = \{ma + nb : m, n \in \mathbb{N}\}$$

is dense in $\mathbb{R}$. Thus there exist two sequences (m_k), (n_k) of positive integers such that

$$\lim_{k \to \infty} (m_k a + n_k b) = 0.$$

Note that

$$\lim_{k \to \infty} m_k = \lim_{k \to \infty} n_k = \infty \tag{3}$$

(otherwise b/a would be rational). Obviously,

$$\lim_{k\to\infty} \frac{m_k a + n_k b}{m_k} = 0,$$

and, consequently,

$$\lim_{k\to\infty} \frac{n_k}{m_k} = -\frac{a}{b}. \tag{4}$$

Let $x_0 \in \mathbb{R}$ be a point of continuity of f. From (2) we get

$$f(m_k a + n_k b + x_0) \le m_k \alpha + n_k \beta + f(x_0), \qquad k \in \mathbb{N},$$

or, equivalently,

$$\frac{f(m_k a + n_k b + x_0)}{m_k} \le \alpha + \frac{n_k}{m_k}\beta + \frac{f(x_0)}{m_k}, \qquad k \in \mathbb{N}.$$

Letting $k \to \infty$, and making use of (3), (4), and the continuity of f at x_0, we hence get $0 \le \alpha - \frac{a}{b}\beta$, i.e.

$$\frac{\beta}{b} \ge \frac{\alpha}{a}.$$

As, by the assumption, the reverse inequality holds true, we have shown that

$$\frac{\alpha}{a} = \frac{\beta}{b}.$$

Now, setting

$$p := \frac{\alpha}{a} = \frac{\beta}{b},$$

we can write inequality (2) in the form

$$f(t + x) \le pt + f(x), \qquad t \in A,\ x \in \mathbb{R}. \tag{5}$$

Take an arbitrary $x \in \mathbb{R}$. By the density of A there is a sequence (t_n) such that

$$t_n \in A \quad (n \in \mathbb{N}), \qquad \lim_{n\to\infty} t_n = x_0 - x.$$

From (5) we have

$$f(t_n + x) \le pt_n + f(x), \qquad n \in \mathbb{N}.$$

Letting $n \to \infty$, and making use of the continuity of f at x_0, we obtain

$$f(x_0) \le p(x_0 - x) + f(x), \quad x \in \mathbb{R}.$$

To prove the opposite inequality note that replacing x by $x - t$ in (5) we get

$$f(x) \le pt + f(x - t), \qquad t \in A,\ x \in \mathbb{R}.$$

Taking an $x \in \mathbb{R}$, and, by the density of A, a sequence (t_n) such that

$$t_n \in A \quad (n \in \mathbb{N}), \qquad \lim_{n\to\infty} t_n = x - x_0,$$

we hence get

$$f(x) \leq pt_n + f(x - t_n), \qquad n \in \mathbb{N}.$$

Letting $n \to \infty$, and again making use of the continuity of f at x_0, we obtain

$$f(x) \leq p(x - x_0) + f(x_0), \qquad x \in \mathbb{R}.$$

Thus

$$f(x) = px + (f(x_0) - px_0), \qquad x \in \mathbb{R},$$

which was to be shown.

Remark 1. Let $a, b, \alpha, \beta \in \mathbb{R}$, $ab \neq 0$, be such that $\beta/b = \alpha/a$.

If $b/a \notin \mathbb{Q}$ and a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at least at one point and satisfies the simultaneous system of functional equations

$$f(a + x) = \alpha + f(x), \qquad f(b + x) = \beta + f(x), \qquad x \in \mathbb{R},$$

then $f(x) = px + q$ for some $p, q \in \mathbb{R}$, $x \in \mathbb{R}$ (cf. [4]).

If $b/a \in \mathbb{Q}$ then this system of functional equations reduces to the single functional equation

$$f(d + x) = \frac{\alpha}{a} + f(x), \qquad x \in \mathbb{R},$$

where $d := \min\{ma + nb > 0 : m, n \in \mathbb{N}\}$.

Since the continuous and monotonic solution of this equation depends on an arbitrary function (cf. M. Kuczma [2]), the assumption that $b/a \notin \mathbb{Q}$ in Theorem 1 is essential.

Remark 2. The assumption $\alpha/a \geq \beta/b$ is essential for the uniqueness of the solution of system (1) in Theorem 1. Indeed, if $\alpha/a < \beta/b$ the set of solutions of (1) is large; for instance the function $f := \sin$ satisfies (1) for all $a, b \in \mathbb{R}$ and $\alpha, \beta \geq 2$. Moreover every affine function of the form $f(x) = Ax + B$ where $B \in \mathbb{R}$ is arbitrary and $\alpha/a \leq A \leq \beta/b$ is a solution of (1).

3. Inequalities of additive-multiplicative type

Lemma 1. *Let a, b, α, β be fixed real numbers such that*

$$a < 0 < b, \qquad \frac{b}{a} \notin \mathbb{Q}, \qquad \alpha, \beta > 0.$$

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at least at one point and satisfies the system of functional inequalities

$$f(a + x) \leq \alpha f(x), \quad f(b + x) \leq \beta f(x), \qquad x \in \mathbb{R}, \tag{6}$$

or

$$f(a + x) \geq \alpha f(x), \qquad f(b + x) \geq \beta f(x), \qquad x \in \mathbb{R}. \tag{7}$$

Then f is either positive in $\mathbb{R}$, negative in $\mathbb{R}$, or identically zero.

Proof. Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies (6), x_0 is a point of continuity of f and $f(x_0) > 0$. From (6), by induction, we get

$$f(ma + nb + x) \leq \alpha^m \beta^n f(x), \qquad m, n \in \mathbb{N},\ x \in \mathbb{R}. \tag{8}$$

Take an arbitrary $x \in \mathbb{R}$. By the density of the set $A = \{ma+nb : m, n \in \mathbb{N}\}$ in $\mathbb{R}$ there exists a sequence $(m_k a + n_k b : k, m_k, n_k \in \mathbb{N})$ such that

$$\lim_{k\to\infty} (m_k a + n_k b) = x_0 - x.$$

From (8) we have

$$f(m_k a + n_k b + x) \leq \alpha^{m_k} \beta^{n_k} f(x), \qquad m_k, n_k \in \mathbb{N},\ x \in \mathbb{R}.$$

For k large enough, by the continuity of f at x_0, the left-hand side of this inequality is positive. It follows that f is positive.

Suppose now that $f(x_0) < 0$. Replacing x by $x-(ma+nb)$ in (8) we get

$$f(x) \leq \alpha^m b^n f(x - (ma + nb)), \qquad m, n \in \mathbb{N},\ x \in \mathbb{R}. \tag{9}$$

Now, similarly to the previous case, fix $x \in \mathbb{R}$ and take a sequence $(m_k a + n_k b : k, m_k, n_k \in \mathbb{N})$ such that

$$\lim_{k\to\infty} (m_k a + n_k b) = x - x_0.$$

Again by the continuity of f at x_0, for k large enough, the right-hand side of inequality (9) is negative and hence so is $f(x)$.

If $f(x_0) = 0$ an argument analogous to the first step shows that $f(x) \geq 0$ for all $x \in \mathbb{R}$, and a slight modification of the argument of the second step gives the inequality $f(x) \leq 0$ for all $x \in \mathbb{R}$, and, consequently, $f = 0$ in $\mathbb{R}$.

To complete the proof it is enough to repeat the same reasoning for system (7).

THEOREM 2. *Let $a, b \in \mathbb{R}$ and $\alpha, \beta > 0$ be fixed numbers such that*

$$a < 0 < b, \qquad \frac{b}{a} \notin \mathbb{Q}, \qquad \frac{\log\alpha}{a} \geq \frac{\log\beta}{b}.$$

Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at least at one point and such that $f(\mathbb{R}) \nsubseteq (-\infty, 0)$. If f satisfies the pair of functional inequalities (6), *i.e.*

$$f(a + x) \leq \alpha f(x), \qquad f(b + x) \leq \beta f(x), \qquad x \in \mathbb{R},$$

then either f is identically zero in $\mathbb{R}$, or

$$f(x) = f(0)e^{\frac{\log\alpha}{a}x}, \qquad x \in \mathbb{R}.$$

Proof. Assume that f satisfies (6). By the assumptions and Lemma 1 the function f is either identically zero in $\mathbb{R}$ or positive in $\mathbb{R}$. In the first case there is nothing to prove. In the second case f is positive and the function $g := \log\circ f$ satisfies the inequalities

$$g(a + x) \leq \log\alpha + \log f(x), \qquad g(b + x) \leq \log\beta + \log f(x), \qquad x \in \mathbb{R},$$

and our theorem results from Theorem 1.

Obviously, for inequalities (7) an analogous result holds true.

REMARK 3. The assumption $f(\mathbb{R}) \nsubseteq (-\infty, 0)$ in Theorem 2 is essential. To see this, take arbitrary $a, b \in \mathbb{R}$ such that $a < 0 < b$, $b/a \notin \mathbb{Q}$, $\alpha, \beta \in (0, 1/2)$, and an arbitrary function $f : \mathbb{R} \to [-2, -1]$. Then for all $x \in \mathbb{R}$,

$$f(a+x) \le -1 = \frac{1}{2} \cdot (-2) \le \frac{1}{2} f(x) \le \alpha f(x)$$

and similarly, for all $x \in \mathbb{R}$,

$$f(b+x) \le \beta f(x),$$

which proves that f satisfies (6). Since $\log \alpha < 0$ and $\log \beta < 0$ and $a < 0 < b$, we have

$$\frac{\log \alpha}{a} < 0 < \frac{\log \beta}{b}.$$

Thus all assumptions of Theorem 2 except the condition $f(\mathbb{R}) \nsubseteq (-\infty, 0)$ are satisfied.

4. Inequalities of multiplicative-additive type. As an easy consequence of Theorem 1 we have

THEOREM 3. *Let a, b, α, β be fixed real numbers such that*

$$0 < a < 1 < b, \qquad \frac{\log b}{\log a} \notin \mathbb{Q}, \qquad \frac{\alpha}{\log a} \ge \frac{\beta}{\log b}.$$

Suppose that a function $f : I \to \mathbb{R}$ is continuous at least at one point and satisfies the pair of functional inequalities

$$f(ax) \le \alpha + f(x), \qquad f(bx) \le \beta + f(x), \qquad x \in I, \tag{10}$$

where either $I = (0, \infty)$ or $I = (-\infty, 0)$.

(i) *If $I = (0, \infty)$, then*

$$f(x) = \frac{\alpha}{\log a} \log x + f(1), \qquad x > 0.$$

(ii) *If $I = (-\infty, 0)$, then*

$$f(x) = \frac{\alpha}{\log a} \log(-x) + f(-1), \qquad x < 0.$$

COROLLARY 1. *Let a, b, α, β satisfy the assumptions of Theorem* 3. *If a function $f : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$ satisfies the pair of inequalities* (10), *and in each of the intervals* $(-\infty, 0)$ *and* $(0, \infty)$ *there is at least one point of continuity of f, then*

$$f(x) = \begin{cases} \dfrac{\alpha}{\log a} \log x + f(1) & \text{for } x \in (0, \infty), \\ \dfrac{\alpha}{\log a} \log(-x) + f(-1) & \text{for } x \in (-\infty, 0). \end{cases}$$

REMARK 4. Suppose that a, b, α, β are fixed real numbers such that $0 < a < 1 < b$ and $\alpha/\log a = \beta/\log b$. Note that if $0 \in I$ then there is no function satisfying (10). Indeed, putting $x = 0$ into (10) we get $0 \le \alpha,\ 0 \le \beta$, which contradicts the assumptions.

5. Inequalities of multiplicative type. The following counterpart of Lemma 1 is easy to verify.

LEMMA 2. *Let a, b, α, β be fixed positive real numbers such that*

$$a < 1 < b, \qquad \frac{\log b}{\log a} \notin \mathbb{Q},$$

and $I = (0, \infty)$ or $I = (-\infty, 0)$. Suppose that $f : I \to \mathbb{R}$ is continuous at least at one point and satisfies the system of functional inequalities

$$f(ax) \le \alpha f(x), \qquad f(bx) \le \beta f(x), \qquad x \in I, \tag{11}$$

or

$$f(ax) \ge \alpha f(x), \qquad f(bx) \ge \beta f(x), \qquad x \in I. \tag{12}$$

Then f is either positive in I, negative in I, or identically zero.

Applying Lemma 2 and Theorem 1 we obtain

THEOREM 4. *Let a, b, α, β be fixed positive real numbers such that*

$$a < 1 < b, \qquad \frac{\log b}{\log a} \notin \mathbb{Q}, \qquad \frac{\log \alpha}{\log a} \ge \frac{\log \beta}{\log b},$$

and $I = (0, \infty)$ or $I = (-\infty, 0)$. Suppose that $f : I \to \mathbb{R}$ is continuous at least at one point, satisfies the pair of functional inequalities

$$f(ax) \le \alpha f(x), \qquad f(bx) \le \beta f(x), \qquad x \in I, \tag{13}$$

and $f(I) \nsubseteq (-\infty, 0)$. Then either f is identically zero in I or

(i) *in the case $I = (0, \infty)$,*

$$f(x) = f(1) x^{\frac{\log \alpha}{\log a}}, \qquad x > 0,$$

(ii) *in the case $I = (-\infty, 0)$,*

$$f(x) = f(-1)(-x)^{\frac{\log \alpha}{\log a}}, \qquad x < 0.$$

We omit the formulation of the corresponding result for inequalities (12).

REMARK 5. Suppose that a, b, α, β are fixed positive real numbers such that $a < 1 < b$ and $\frac{\log \alpha}{\log a} = \frac{\log \beta}{\log b}$. Note that if $I = \mathbb{R}$ or $I = [0, \infty)$ or $I = (-\infty, 0]$ and $f : I \to \mathbb{R}$ satisfies (13), then $f(0) = 0$. Indeed, by assumptions either $\alpha < 1 < \beta$ or $\beta < 1 < \alpha$ and, moreover, $f(0)(1 - \alpha) \le 0$ and $f(0)(1 - \beta) \le 0$. Thus $f(0) = 0$.

Hence we get

Remark 6. (i) Suppose that $f : [0,\infty) \to \mathbb{R}$ satisfies (13). If $f|_{(0,\infty)}$ and a, b, α, β satisfy the assumptions of Theorem 4, then

$$f(x) = \begin{cases} f(1)x^{\frac{\log\alpha}{\log a}} & \text{for } x \in (0,\infty), \\ 0 & \text{for } x = 0. \end{cases}$$

(ii) Suppose that $f : (-\infty, 0] \to \mathbb{R}$ satisfies (13). If $f|_{(-\infty,0)}$ and a, b, α, β satisfy the assumptions of Theorem 4, then

$$f(x) = \begin{cases} f(-1)(-x)^{\frac{\log\alpha}{\log a}} & \text{for } x \in (-\infty,0), \\ 0 & \text{for } x = 0. \end{cases}$$

Finally, let us record the following

Remark 7. For obvious reasons the counterparts of Theorems 1–4 for the reverse inequalities remain true.

6. A characterization of L^p-norm. Recall that A. C. Zaanen [8], for the counting measure space, W. Wnuk [7], assuming the continuity of the function ϕ, and J. Matkowski [5], assuming much weaker regularity conditions, characterized the L^p-norm with the aid of the homogeneity of the functional $\mathbf{p}_\phi$ (cf. the definition in the Introduction).

As an application of Theorem 4 we present a far-reaching generalization of these results. It turns out that the homogeneity condition can be replaced by an inequality assumed to be satisfied only for two characteristic functions χ_A, χ_B of suitably chosen measurable sets A and B.

Theorem 5. *Let (Ω, Σ, μ) be a measure space with two sets $A, B \in \Sigma$ such that*

$$0 < \mu(A) < 1 < \mu(B) < \infty, \qquad \frac{\log\mu(B)}{\log\mu(A)} \notin \mathbb{Q}.$$

Suppose that $\phi : (0,\infty) \to (0,\infty)$ is a bijection such that ϕ^{-1} is continuous at least at one point and

$$\frac{\log\phi^{-1}(\mu(A)\phi(1))}{\log\mu(A)} \geq \frac{\log\phi^{-1}(\mu(B)\phi(1))}{\log\mu(B)}. \tag{14}$$

If $\mathbf{p}_\phi$ satisfies the condition

$$\mathbf{p}_\phi(tx) \leq t\mathbf{p}_\phi(x), \qquad t > 0,\ x \in \{\chi_A, \chi_B\}, \tag{15}$$

then

$$\phi(t) = \phi(1)t^p, \qquad t > 0,$$

where

$$p := \frac{\log\phi^{-1}(\mu(A)\phi(1))}{\log\mu(A)}.$$

Moreover, if $p \geq 1$ then $\mathbf{p}_\phi$ coincides with the L^p-norm.

Proof. Let $a = \mu(A)$ and $b = \mu(B)$. From (15) we obtain

$$\phi^{-1}(a\phi(t)) \le t\phi^{-1}(a\phi(1)), \quad \phi^{-1}(b\phi(t)) \le t\phi^{-1}(b\phi(1)), \quad t > 0,$$

which with $\alpha := \phi^{-1}(a\phi(1))$ and $\beta := \phi^{-1}(b\phi(1))$ reduces to the pair of functional inequalities

$$\phi^{-1}(a\phi(t)) \le \alpha t, \quad \phi^{-1}(b\phi(t)) \le \beta t, \quad t > 0.$$

From the bijectivity of ϕ, replacing here t by $\phi^{-1}(t)$, we get the equivalent system of inequalities

$$\phi^{-1}(at) \le \alpha\phi^{-1}(t), \quad \phi^{-1}(bt) \le \beta\phi^{-1}(t), \qquad t > 0,$$

which, with $f := \phi^{-1}$ and $I = (0, \infty)$, takes the form (13). Now our result follows from Theorem 4.

Remark 8. Note that (15) is a very weak substitute of the homogeneity of the functional $\mathbf{p}_\phi$.

Discussing the assumptions in Theorem 5, note that the condition: $\frac{\log b}{\log a} \notin \mathbb{Q}$ or $\frac{\log(a+b)}{\log a} \notin \mathbb{Q}$ is not too demanding.

To show that the assumption of the existence of sets A and B with $0 < \mu(A) < 1 < \mu(B) < \infty$ is essential, we indicate some wide classes of non-power functions ϕ for which the functional $\mathbf{p}_\phi$ satisfies the condition (15), in each of the cases

$$\begin{aligned} &\mu(A) \le 1 \text{ or } \mu(A) = \infty \qquad \text{for every } A \in \Sigma; \\ &\mu(A) \ge 1 \text{ or } \mu(A) = 0 \qquad \text{for every } A \in \Sigma. \end{aligned}$$

Example 1. Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) \le 1$. Put $\delta := \inf\{\mu(A) : A \in \Sigma \ \wedge \ \mu(A) > 0\}$. Let $\phi : (0, \infty) \to (0, \infty)$ be an increasing bijection such that the function $(0, \infty) \ni t \mapsto \phi(t)/t$ is non-increasing and $\phi(\delta) = \delta$, $\phi(1) = 1$. Then $\phi(t) = t$ for all $t \in [\delta, 1]$, the function $(0, \infty) \ni t \mapsto \phi^{-1}(t)/t$ is non-decreasing and, therefore, for each $A \in \Sigma$ with $a := \mu(A) > 0$, we have

$$\begin{aligned} \mathbf{p}_\phi(t\chi_A) &= \phi^{-1}(a\phi(t)) = \frac{\phi^{-1}(a\phi(t))}{a\phi(t)}\, a\phi(t) \\ &\le \frac{\phi^{-1}(\phi(t))}{\phi(t)}\, a\phi(t) = ta = t\phi^{-1}(a\phi(1)) \\ &= t\mathbf{p}_\phi(\chi_A), \qquad t > 0. \end{aligned}$$

Thus $\mathbf{p}_\phi$ satisfies (15) and ϕ is not a power function.

Example 2. Let (Ω, Σ, μ) be a measure space for which $\mu(A) \ge 1$ for every set $A \in \Sigma$ such that $\mu(A) > 0$, and there exists $B \in \Sigma$ such that $1 < \mu(B) < \infty$. Then $\delta := \inf\{\mu(A) : A \in \Sigma \wedge \mu(A) > 0\} \ge 1$. Let $\phi : (0, \infty) \to (0, \infty)$ be a bijection such that the function $(0, \infty) \ni t \mapsto \phi(t)/t$

is non-decreasing and $\phi(1) = 1$, $\phi(\delta) = \delta$. Then ϕ is strictly increasing, $\phi(t) = t$ for all $t \in [1, \delta]$, the function $(0, \infty) \ni t \mapsto \phi^{-1}(t)/t$ is non-increasing, and therefore, in the same way as in the previous example, for all $B \in \Sigma$ such that $0 < \mu(B) < \infty$, we have

$$\mathbf{p}_\phi(t\chi_B) \leq t\mathbf{p}_\phi(\chi_B), \qquad t > 0.$$

We end our discussion with an example showing that the assumption (14) is indispensable.

EXAMPLE 3. Let (Ω, Σ, μ) be an arbitrary measure space, and $f : \mathbb{R} \to \mathbb{R}$ a bijection such that $f(0) = 0$ and f^{-1} is subadditive (for instance, for f one can take the inverse function to $x \mapsto x + |\sin x|$). Define $\phi : (0, \infty) \to (0, \infty)$ by $\phi(t) = e^{f(\log t)}$. Then, making use of the definition of $\mathbf{p}_\phi$, the subadditivity of f^{-1}, and the monotonicity of the exponential function, for all $A \in \Sigma$ with $a := \mu(A) > 0$, we have

$$\begin{aligned}\mathbf{p}_\phi(t\chi_A) &= \phi^{-1}(a\phi(t)) = e^{f^{-1}(\log a + f(\log t))} \\ &\leq e^{f^{-1}(\log a)} e^{f^{-1}(f(\log t))} \\ &= t\phi^{-1}(a) = t\mathbf{p}_\phi(\chi_A), \qquad t > 0.\end{aligned}$$

Thus $\mathbf{p}_\phi$ satisfies the subhomogeneity condition (15) for all functions χ_A (here $\mu(A)$ can be smaller or greater than 1). This shows that in Theorem 5 condition (14) cannot be omitted.

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