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Decay estimates of solutions of a nonlinearly damped semilinear wave equation

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Abstract. We consider an initial boundary value problem for the equation $u_{tt}-\Delta u-\nabla\phi\cdot\nabla u+f(u)+g(u_t)=0$. We first prove local and global existence results under suitable conditions on f and g. Then we show that weak solutions decay either algebraically or exponentially depending on the rate of growth of g. This result improves and includes earlier decay results established by the authors.

1. Introduction. In [11] Nakao considered the following initial boundary value problem:

$$
(1.1)\qquad
\begin{aligned}
&u_{tt}-\Delta u+\varrho(u_t)+f(u)=0=0, && x\in\Omega,\ t>0,\\
&u(x,t)=0, && x\in\partial\Omega,\ t\geq 0,\\
&u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), && x\in\Omega,
\end{aligned}
$$

where $\varrho(v)=|v|^{\beta}v$, $\beta>-1$, $f(u)=bu|u|^{\alpha}$, $\alpha,b>0$, and Ω is a bounded domain in $\mathbb{R}^n$ $(n\geq 1)$, with a smooth boundary $\partial\Omega$. He showed that (1.1) has a unique global weak solution if $0\leq\alpha\leq 2/(n-2)$, $n\geq 3$, and a unique global strong solution if $\alpha>2/(n-2)$, $n\geq 3$ (of course for $n=1$ or 2 there is no restriction on α). In addition to global existence the issue of the decay rate was addressed. In both cases, it has been shown that the energy of the solution decays algebraically if $\beta>0$ and exponentially if $\beta=0$. This improves an earlier result by the same author [12], where he studied the problem in an abstract setting and established a theorem concerning the decay of the solution energy only for the case $\alpha\leq 2/(n-2)$, $n\geq 3$. Later on, in a joint work with Ono [13], this result was extended to the Cauchy problem for the equation

$$u_{tt}-\Delta u+\lambda^2(x)u+\varrho(u_t)+f(u)=0,\qquad x\in\mathbb{R}^n,\ t>0,$$

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where $\varrho(u_t)$ behaves like $|u_t|^\beta u_t$ and $f(u)$ behaves like $-bu|u|^\alpha$. In this case the authors required that the initial data be small enough in the $H^1(\Omega)\times L^2(\Omega)$ norm and of compact support.

Pucci and Serrin [14] discussed the stability of the following problem:

$$\begin{aligned} &u_{tt}-\Delta u+Q(x,t,u,u_t)+f(x,u)=0, && x\in\Omega,\ t>0,\\ &u(x,t)=0, && x\in\partial\Omega,\ t\ge 0,\\ &u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), && x\in\Omega, \end{aligned} \tag{1.2}$$

and proved that the energy of the solution is a Lyapunov function. Although they did not discuss the decay rate, they did show that in general the energy goes to zero as t approaches infinity. They also considered an important special case of (1.2) when $Q(x,t,u,u_t)=a(t)t^\alpha u_t$ and $f(x,u)=V(x)u$, and showed that the behavior of the solutions depends crucially on the parameter α. If $|\alpha|\le 1$ then the rest field is asymptotically stable. On the other hand, when $\alpha<-1$ or $\alpha>1$ there are solutions that do not approach zero or approach nonzero functions $\phi(x)$ as $t\to\infty$.

Messaoudi [10] discussed an initial boundary value problem for the equation

$$u_{tt}-\Delta u+a(1+|u_t|^{m-2})u_t+bu|u|^{p-2}=0,\qquad x\in\Omega,\ t>0, \tag{1.3}$$

where $a,b>0$, $m\ge 2$, $p>2$, and proved that the energy of the solution decays exponentially. The proof of this result is based on a direct method used in [5] and [6].

In this paper we are concerned with the problem

$$\begin{aligned} &u_{tt}-\Delta u-\nabla\phi\cdot\nabla u+f(u)+g(u_t)=0, && x\in\Omega,\ t>0,\\ &u(x,t)=0, && x\in\partial\Omega,\ t\ge 0,\\ &u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), && x\in\Omega, \end{aligned} \tag{1.4}$$

where ϕ is a function in $W^{1,\infty}(\Omega)$, Ω is a bounded open domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $f,g:\mathbb{R}\to\mathbb{R}$ are two continuous functions satisfying $f(0)=g(0)=0$ and

(H1) $\|f(u)-f(v)\|_2\le a(u,v)\|\nabla(u-v)\|_2$, where $a(u,v)$ is a function depending on the norms of u,v in $H_0^1(\Omega)$,

(H2) g is an increasing function such that

$$c_1\{|s_1-s_2|^r+|s_1-s_2|^p\}\le|g(s_1)-g(s_2)|\le c_2\{|s_1-s_2|+|s_1-s_2|^p\} \tag{1.5}$$

for some constants $c_1,c_2>0$, $1\le r\le p$ with

$$(n-2)p\le n+2. \tag{1.6}$$

2. Local existence. In this section, we establish local and global existence results for (1.4). First we consider, for v given, the linear problem

$$(2.1)\qquad \begin{aligned} &u_{tt} - \Delta u - \nabla\phi\cdot\nabla u + f(v) + g(u_t) = 0, && x\in\Omega,\ t>0\\ &u(x,t) = 0, && x\in\partial\Omega,\ t\ge 0,\\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), && x\in\Omega, \end{aligned}$$

where u is the sought solution.

LEMMA 2.1. *Assume that* (H1) *and* (H2) *hold. Then given any v in $C([0,T];C_0^\infty(\Omega))$ and u_0, u_1 in $C_0^\infty(\Omega)$, the problem* (2.1) *has a unique solution u satisfying*

$$(2.2)\qquad \begin{aligned} u &\in L^\infty((0,T);H^2(\Omega)\cap H_0^1(\Omega)),\\ u_t &\in L^\infty((0,T);H_0^1(\Omega)),\\ u_{tt} &\in L^\infty((0,T);L^2(\Omega)). \end{aligned}$$

This lemma is a direct consequence of [7, Chapter 1, Theorem 3.1] (see also [1]).

LEMMA 2.2. *Assume that* (H1) *and* (H2) *hold. Then given any v in $C([0,T];H_0^1(\Omega))$, u_0 in $H_0^1(\Omega)$, and u_1 in $L^2(\Omega)$, the problem* (2.1) *has a unique weak solution*

$$(2.3)\qquad \begin{aligned} u &\in C([0,T];H_0^1(\Omega)),\\ u_t &\in C([0,T];L^2(\Omega))\cap L^{p+1}(\Omega\times(0,T)). \end{aligned}$$

Moreover,

$$(2.4)\qquad \frac{1}{2}\int_\Omega e^{\phi(x)}[u_t^2 + |\nabla u|^2](x,t)\,dx + \int_0^t\int_\Omega e^{\phi(x)} g(u_t)u_t(x,s)\,dx\,ds$$
$$= \frac{1}{2}\int_\Omega e^{\phi(x)}[u_1^2 + |\nabla u_0|^2](x)\,dx - \int_0^t\int_\Omega e^{\phi(x)} f(v)u_t(x,s)\,dx\,ds, \qquad \forall t\in[0,T].$$

Proof. We approximate u_0, u_1 by sequences (u_0^μ), (u_1^μ) in $C_0^\infty(\Omega)$, and v by a sequence (v^μ) in $C([0,T];C_0^\infty(\Omega))$. We then consider the set of linear problems

$$(2.5)\qquad \begin{aligned} &u_{tt}^\mu - \Delta u^\mu - \nabla\phi\cdot\nabla u^\mu + g(u_t^\mu) + f(v^\mu) = 0, && x\in\Omega,\ t>0,\\ &u^\mu(x,t) = 0, && x\in\partial\Omega,\ t>0,\\ &u^\mu(x,0) = u_0^\mu(x), \quad u_t^\mu(x,0) = u_1^\mu(x), && x\in\Omega. \end{aligned}$$

Lemma 2.1 guarantees the existence of a sequence of unique solutions (u^μ) satisfying (2.3). Now we proceed to show that (u^μ, u_t^μ) is a Cauchy sequence in

$$\mathbf{Y} := \{w : w\in C([0,T];H_0^1(\Omega)),\ w_t\in C([0,T];L^2(\Omega))\cap L^p(\Omega\times(0,T))\}.$$

For this purpose we set

$$U := u^\mu - u^\nu, \qquad V := v^\mu - v^\nu.$$

It is straightforward to see that U satisfies

$$(2.6)\qquad \begin{aligned} &U_{tt} - \Delta U - \nabla\phi \cdot \nabla U + g(u_t^\mu) - g(u_t^\nu) + f(v^\mu) - f(v^\nu) = 0,\\ &U(x,t) = 0, \quad x \in \partial\Omega,\ t > 0,\\ &U(x,0) = U_0(x) = u_0^\mu(x) - u_0^\nu(x),\\ &U_t(x,0) = U_1(x) = u_1^\mu(x) - u_1^\nu(x). \end{aligned}$$

We multiply the first equation of (2.6) by $e^{\phi(x)}U_t$ and integrate over $\Omega\times(0,t)$ to get

$$(2.7)\qquad \frac{1}{2}\int_\Omega e^{\phi(x)}[U_t^2 + |\nabla U|^2](x,t)\,dx + \int_0^t\int_\Omega e^{\phi(x)}(g(u_t^\mu) - g(u_t^\nu))U_t(x,s)\,dx\,ds$$
$$= \frac{1}{2}\int_\Omega e^{\phi(x)}[U_1^2 + |\nabla U_0|^2](x)\,dx + \int_0^t\int_\Omega e^{\phi(x)}[f(v^\mu) - f(v^\nu)]U_t(x,s)\,dx\,ds.$$

By using (H1) and the fact that g is increasing, (2.7) yields

$$\frac{1}{2}\int_\Omega [U_t^2 + |\nabla U|^2](x,t)\,dx \le \int_\Omega [U_1^2 + |\nabla U_0|^2](x)\,dx + \Gamma\int_0^t \|U_t(\cdot,s)\|_2\|\nabla V(\cdot,s)\|_2\,ds,$$

where Γ is a generic positive constant depending on C, the supremum and the infimum of $e^{\phi(x)}$, and the radius of the ball in $C([0,T];H_0^1(\Omega))$ containing v^μ and v^ν. Young's inequality then gives

$$\max_{0\le t\le T}\int_\Omega [U_t^2 + |\nabla U|^2](x,t)\,dx \le \Gamma\int_\Omega [U_1^2 + |\nabla U_0|^2](x)\,dx + \Gamma T\max_{0\le t\le T}\int_\Omega [V_t^2 + |\nabla V|^2](x,t)\,dx.$$

Since (ϕ^μ) is Cauchy in $H_0^1(\Omega)$ and in $L^2(\Omega)$, and (v^μ) is Cauchy in $C([0,T];H_0^1(\Omega))$ we conclude that (u^μ, u_t^μ) is Cauchy in $C([0,T];H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$. To show that u_t is Cauchy in $L^{p+1}(\Omega\times(0,T))$ we use (H2) to obtain

$$(2.8)\qquad \|U_t\|_{L^{p+1}(\Omega\times(0,T))}^{p+1} \le C\int_0^t\int_\Omega e^{\phi(x)}(g(u_t^\mu) - g(u_t^\nu))U_t(x,s)\,dx\,ds,$$

which yields, by virtue of (2.7),

$$\|U_t\|_{L^{p+1}(\Omega\times(0,T))}^{p+1} \le \Gamma\int_\Omega [U_1^2 + |\nabla U_0|^2](x)\,dx + \Gamma\int_0^T \|U_t(\cdot,s)\|_2\|\nabla V(\cdot,s)\|_2\,ds.$$

Therefore (u_t^μ) is Cauchy in $L^{p+1}(\Omega\times(0,T))$, hence (u^μ,u_t^μ) is Cauchy in $\mathbf{Y}$. We now show that the limit (u,u_t) is a weak solution of (2.1) in the sense of [6]. That is, for each θ in $H_0^1(\Omega)$ we must show that

$$\frac{d}{dt}\int_\Omega u_t(x,t)\theta(x)\,dx+\int_\Omega \nabla u(x,t)\cdot\nabla\theta(x)\,dx \\ -\int_\Omega(\nabla\phi\cdot\nabla u)\theta(x)\,dx+\int_\Omega[f(u)+g(u_t)]\theta(x)\,dx=0, \tag{2.9}$$

for almost all t in $(0,\,T)$. To establish this we multiply equation (2.5) by θ and integrate over Ω to obtain

$$\frac{d}{dt}\int_\Omega u_t^\mu(x,t)\theta(x)\,dx+\int_\Omega \nabla u^\mu(x,t)\cdot\nabla\theta(x)\,dx \\ -\int_\Omega(\nabla\phi\cdot\nabla u^\mu)\theta(x)\,dx+\int_\Omega[f(u^\mu)+g(u_t^\mu)]\theta(x)\,dx=0. \tag{2.10}$$

As $\mu\to\infty$, we see that

$$\int_\Omega \nabla u^\mu(x,t)\cdot\nabla\theta(x)\,dx\to\int_\Omega \nabla u(x,t)\cdot\nabla\theta(x)\,dx, \\ \int_\Omega f(u^\mu)\theta(x)\,dx\to\int_\Omega f(u)\theta(x)\,dx \quad \text{in } C([0,T])$$

and $\int_\Omega g(u_t^\mu)\theta(x)\,dx\to\int_\Omega g(u_t)\theta(x)\,dx$ in $L^1((0,T))$. Thus $\int_\Omega u_t(x,t)\theta(x)\,dx$ $[=\lim\int_\Omega u_t^\mu(x,t)\theta(x)\,dx]$ is an absolutely continuous function on $[0,\,T]$, so (2.9) holds for almost all t in $[0,T]$. For the energy equality (2.4), we start from the energy equality for u^μ and proceed in the same way to establish it for u. To prove uniqueness we take v^μ and v^ν and let u^μ and u^ν be the corresponding solutions of (2.1). It is clear that $U=u^\mu-u^\nu$ satisfies

$$\frac{1}{2}\int_\Omega e^{\phi(x)}[U_t^2+|\nabla U|^2](x,t)\,dx+\int_0^t\int_\Omega e^{\phi(x)}(g(u_t^\mu)-g(u_t^\nu))U_t(x,s)\,dx\,ds \\ +\int_0^t\int_\Omega e^{\phi(x)}[f(v^\mu)-f(v^\nu)]U_t(x,s)\,dx\,ds=0. \tag{2.11}$$

If $v^\mu=v^\nu$ then (2.11) shows that $U=0$, which implies uniqueness. This completes the proof.

Remark 2.1. Note that condition (1.6) on p is needed for $\int_\Omega g(u_t^\mu)\theta(x)\,dx$ to make sense.

Theorem 2.3. *Assume that* (H1) *and* (H2) *hold. Then given any* u_0 *in* $H_0^1(\Omega)$ *and any* u_1 *in* $L^2(\Omega)$, *the problem* (1.4) *has a unique weak solution* u *satisfying* (2.3) *for* T *small enough.*

Proof. For $M > 0$ large and $T > 0$, we define $Z(M,T)$ to be the class of all functions w in $\mathbf{Y}$ satisfying the initial conditions of (1.4) and

$$\max_{0\le t\le T}\int_\Omega [w_t^2+|\nabla w|^2](x,t)\,dx+\int_0^T\int_\Omega |w_t(x,s)|^{p+1}\,dx\,ds\le M^2. \tag{2.12}$$

$Z(M,T)$ is nonempty if M is large enough. This follows from the trace theorem (see [8]). We also define the map h from $Z(M,T)$ into $\mathbf{Y}$ by $u := h(v)$, where u is the unique solution of the linear problem (2.1). We would like to show, for M sufficiently large and T sufficiently small, that h is a contraction from $Z(M,T)$ into itself.

By using the energy equality (2.4), (H1) and (H2) we get

$$\begin{aligned}\int_\Omega [u_t^2+|\nabla u|^2](x,t)\,dx&+\int_0^t\int_\Omega |u_t(x,s)|^{p+1}\,dx\,ds\\ &\le C\int_\Omega [u_1^2+|\nabla u_0|^2](x)\,dx+C\int_0^t\int_\Omega |f(v)|\,|u_t|(x,s)\,dx\,ds\\ &\le C\int_\Omega [u_1^2+|\nabla u_0|^2](x)\,dx+C\int_0^t a(u,0)\|\nabla v\|_2\|u_t\|_2,\qquad \forall t\in[0,T],\end{aligned}\tag{2.13}$$

and consequently

$$\|u\|_{\mathbf{Y}}^2\le C\int_\Omega [u_1^2+|\nabla u_0|^2](x)\,dx+CKT\|u\|_{\mathbf{Y}},$$

where K is a constant depending on M. By choosing M large enough and T sufficiently small, (2.12) is satisfied; hence $u\in Z(M,T)$. This shows that h maps $Z(M,T)$ into itself.

Next we verify that h is a contraction. Set $U = u-\overline{u}$ and $V = v-\overline{v}$, where $u = h(v)$ and $\overline{u} = h(\overline{v})$. It is straightforward to see that U satisfies

$$\begin{aligned}&U_{tt}-\Delta U-\nabla\phi\cdot\nabla U+g(u_t)u_t-g(\overline{u}_t)\overline{u}_t+f(v)-f(\overline{v})=0,\\ &U(x,t)=0, && x\in\partial\Omega,\ t>0,\\ &U(x,0)=U_t(x,0)=0, && x\in\Omega.\end{aligned}\tag{2.14}$$

By multiplying the first equation of (2.14) by $e^{\phi(x)}U_t$ and integrating over $\Omega\times(0,t)$, we arrive at

$$\begin{aligned}\int_\Omega [U_t^2+|\nabla U|^2](x,t)\,dx&+\int_0^t\int_\Omega [g(u_t)u_t-g(\overline{u}_t)\overline{u}_t]U_t(x,s)\,dx\,ds\\ &\le C\int_0^t\int_\Omega |f(v)-f(\overline{v})|\,|U_t|(x,s)\,dx\,ds.\end{aligned}\tag{2.15}$$

By using (H1) and (H2) we obtain

$$\int_\Omega [U_t^2 + |\nabla U|^2](x,t)\,dx + \int_0^t\int_\Omega |U_t(x,s)|^{p+1}\,dx\,ds \le C\int_0^t a(v,\overline{v})\|U_t\|_2\|\nabla V\|_2(\cdot,s)\,ds.$$

Thus we have

$$\|U\|_{\mathbf{Y}}^2 \le CTK\|V\|_{\mathbf{Y}}^2. \tag{2.16}$$

By choosing T so small that $CTK < 1$, (2.16) shows that h is a contraction. The contraction mapping theorem then guarantees the existence of a unique u satisfying $u = h(u)$. Obviously it is a solution of (1.4). The uniqueness of this solution follows from inequality (2.15). The proof is complete.

3. Global existence and decay. In this section, we are interested in the precise decay rate of an equivalent energy of the solution of (1.4). We define the equivalent energy of the solution by the formula

$$E(t) = \int_\Omega e^{\phi(x)}[u_t^2 + |\nabla u|^2 + 2F(u)]\,dx, \qquad t \in \mathbb{R}^+, \tag{3.1}$$

where

$$F(s) = \int_0^s f(\sigma)\,d\sigma, \qquad \forall s \in \mathbb{R}. \tag{3.2}$$

We suppose that

$$F(s) \ge -a|s|^2, \quad \forall s \in \mathbb{R}, \tag{3.3}$$

for some

$$0 \le a < \frac{1}{2c_0}, \tag{3.4}$$

where c_0 is the positive constant satisfying (Sobolev embedding)

$$\int_\Omega |u|^2\,dx \le c_0\int_\Omega |\nabla u|^2\,dx, \quad \forall u \in H_0^1(\Omega). \tag{3.5}$$

REMARK 3.1. Conditions (3.3) and (3.4) ensure the following inequality:

$$\|(u,u_t)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \le kE(t), \quad \forall t \in \mathbb{R}^+, \tag{3.6}$$

where $k = 1/(m(1-2ac_0)) > 0$ and $m = \inf_\Omega e^{\phi(x)}$. Indeed, (3.3) and (3.5) imply that

$$E(t) \ge \int_\Omega e^{\phi(x)}[u_t^2 + |\nabla u|^2 - 2a|u|^2]\,dx \ge \int_\Omega e^{\phi(x)}[u_t^2 + (1-2ac_0)|\nabla u|^2]\,dx$$
$$\ge m(1-2ac_0)\int_\Omega [u_t^2 + |\nabla u|^2]\,dx = m(1-2ac_0)\|(u,u_t)\|^2_{H_0^1(\Omega)\times L^2(\Omega)},$$

which gives (3.6).

Using the first equation of (1.4) and the boundary condition, we can easily prove that the energy E satisfies

$$E'(t) = -2\int_\Omega e^{\phi(x)} u_t g(u_t)\,dx \le 0, \quad t \in \mathbb{R}^+, \tag{3.7}$$

since g is increasing; hence the energy is nonincreasing. We take $0 \le S < T < \infty$ and integrate (3.7) over $[S,T]$ to get

$$\int_S^T\int_\Omega e^{\phi(x)} u_t g(u_t)\,dx = \frac{1}{2}\,[E(S) - E(T)]. \tag{3.8}$$

Theorem 3.1. *Assume that* (H1), (H2), (3.2)–(3.4) *hold. Then given any* u_0 *in* $H_0^1(\Omega)$ *and any* u_1 *in* $L^2(\Omega)$, *the solution of problem* (1.4) *is global.*

Proof. It suffices to show that

$$\int_\Omega (u_t^2 + |\nabla u|^2)(x,t)\,dx$$

remains bounded independently of t. To achieve this, we multiply (1.4) by $e^\phi u_t$, integrate over $\Omega \times (0,t)$ and use the boundary conditions to obtain

$$\frac{1}{2}\int_\Omega e^{\phi(x)}[u_t^2 + |\nabla u|^2 + 2F(u)](x,t)\,dx + \int_0^t\int_\Omega e^{\phi(x)} g(u_t)u_t(x,s)\,dx\,ds$$
$$= \frac{1}{2}\int_\Omega e^{\phi(x)}[u_1^2 + |\nabla u_0|^2 + 2F(u_0)](x)\,dx, \qquad \forall t \in [0,T].$$

By using (3.6), we arrive at

$$\int_\Omega (u_t^2 + |\nabla u|^2)(x,t)\,dx$$
$$\le kE(t) \le k\int_\Omega e^{\phi(x)}[u_1^2 + |\nabla u_0|^2 + 2F(u_0)](x)\,dx, \quad \forall t \ge 0.$$

This completes the proof.

We now establish some decay estimates of the energy under hypotheses (H1), (H2), (3.2)–(3.4), and

(H3) There exists a constant $0 < b < 1$ such that

$$2bF(s) \le sf(s), \qquad \forall s \in \mathbb{R}^+. \tag{3.9}$$

Remark 3.2. If f is increasing then (3.2) and (3.9) are satisfied with $a = 0$ and $b = 1/2$.

We also note that (H2) and the fact that $g(0) = 0$ yield

$$c_1\{|s|^r + |s|^p\} \le |g(s)| \le c_2\{|s| + |s|^p\}. \tag{3.10}$$

THEOREM 3.2. *Under hypotheses* (H1)–(H3) *and* (3.2)–(3.4) *there exist constants* $\omega, c > 0$ *such that*

$$E(t) \le E(0)e^{1-\omega t}, \qquad \forall t \in \mathbb{R}^+, \tag{3.11}$$

if $r = 1$, *and*

$$E(t) \le c(1+t)^{-2/(r-1)}, \qquad \forall t \in \mathbb{R}^+, \tag{3.12}$$

if $r > 1$.

REMARK 3.3. If $\phi \equiv 0$ and $g(s) = \alpha s$ for all $s \in \mathbb{R}$ with $\alpha > 0$ (that is, $r = p = 1$), then we find the results obtained in [9]. On the other hand, if $g(s) = \alpha(1 + |s|^{m-2})s$ for all $s \in \mathbb{R}^+$ with $m > 2$ (that is, $p = m - 1$ and $r = 1$) then we obtain the results of [10].

REMARK 3.4. It is possible to weaken the growth assumption (3.10) as was done for elasticity systems in [2], and for the Petrovsky system in [3]. In any case, the proof of our estimates (3.11) and (3.12) is similar to those in the two papers.

Proof of Theorem 3.2. We are going to prove that the energy E satisfies, for any $0 \le S < T < \infty$,

$$\int_S^T E^{(r+1)/2}(t)\, dt \le cE(S). \tag{3.13}$$

Here and in what follows we shall denote by c various positive constants, by ε various positive constants small enough, and by c_ε various positive constants depending on ε. The inequality (3.13) gives (3.11) and (3.12) (see [2, Proposition 3.7]).

We multiply the first equation of (1.4) by $E^{(r-1)/2}(t)e^{\phi(x)}u$ and integrate over $\Omega \times [S,T]$ to get

$$\begin{aligned} &\int_S^T\int_\Omega E^{(r-1)/2}(t)e^{\phi(x)}[u_t^2 + |\nabla u|^2 + uf(u)]\, dx\, dt \\ &\quad = \int_S^T\int_\Omega E^{(r-1)/2}(t)e^{\phi(x)}[2u_t^2 - ug(u_t)]\, dx\, dt \\ &\quad + \frac{r-1}{2}\int_S^T\int_\Omega E^{(r-3)/2}(t)E'(t)e^{\phi(x)}uu_t\, dx\, dt - \Big[\int_\Omega E^{(r-1)/2}(t)e^{\phi(x)}uu_t\, dx\, dt\Big]_S^T. \end{aligned} \tag{3.14}$$

The last two terms of (3.14) can be easily majorized by $cE^{(r+1)/2}(S)$ (see [2] and [3]). We now follow the proof given in [4]. We set $1/q = 1 - p/(p+1)$, $\Omega^+ = \{x \in \Omega : |u_t| > 1\}$ and $\Omega^- = \Omega \setminus \Omega^+$. We apply the Schwarz and Young inequalities and the embedding $H_0^1(\Omega) \subset L^q(\Omega)$ to get

$$
\begin{aligned}
-\int_S^T \int_{\Omega^+} & E^{(r-1)/2}(t) e^{\phi(x)} u g(u_t)\, dx \\
& \le c \int_S^T E^{(r-1)/2}(t) \Big(\int_{\Omega^+} |u|^q\, dx \Big)^{1/q} \Big(\int_{\Omega^+} |g(u_t)|^{1+1/p}\, dx \Big)^{p/(p+1)} dt \\
& \le c \int_S^T E^{(r-1)/2}(t) \Big[\varepsilon \int_{\Omega^+} |u|^q\, dx + c_\varepsilon \int_{\Omega^+} |g(u_t)|^{1+1/p}\, dx \Big] dt \\
& \le \varepsilon c \int_S^T E^{(r+q-1)/2}(t)\, dt + c_\varepsilon E^{(r-1)/2}(S) \int_S^T \int_{\Omega^+} e^{\phi(x)} u_t g(u_t)\, dx\, dt \\
& \le \varepsilon c \int_S^T E^{(r+1)/2}(t)\, dt + c_\varepsilon [E^{(r+1)/2}(S) - E^{(r+1)/2}(T)].
\end{aligned}
$$

On the other hand, using the growth assumption (3.10), we have

$$
\begin{aligned}
& -\int_S^T \int_{\Omega^-} E^{(r-1)/2}(t) e^{\phi(x)} u g(u_t)\, dx \\
& \le c \int_S^T E^{(r-1)/2}(t) \Big[\varepsilon \int_{\Omega^-} u^2\, dx + c_\varepsilon \int_{\Omega^-} g^2(u_t)\, dx \Big] dt \\
& \le \varepsilon c \int_S^T E^{(r-1)/2}(t) \int_{\Omega^-} e^{\phi(x)} |\nabla u|^2\, dx\, dt + c_\varepsilon E^{(r-1)/2}(S) \int_S^T \int_{\Omega^-} e^{\phi(x)} u_t g(u_t)\, dx\, dt \\
& \le \varepsilon c \int_S^T E^{(r+1)/2}(t)\, dt + c_\varepsilon [E^{(r+1)/2}(S) - E^{(r+1)/2}(T)].
\end{aligned}
$$

Adding the last two inequalities and substituting the result into the right-hand side of (3.14) and using (3.9), we obtain

$$
\begin{aligned}
(b - \varepsilon c) \int_S^T & E^{(r+1)/2}(t)\, dt \\
& \le c E^{(r+1)/2}(S) + 2 \int_S^T \int_{\Omega} E^{(r-1)/2}(t) e^{\phi(x)} u_t^2\, dx\, dt.
\end{aligned} \tag{3.15}
$$

Using Young's inequality once again we have, by (3.8) and (3.10),

$$
\begin{aligned}
2 \int_S^T \int_{\Omega^+} E^{(r-1)/2}(t) e^{\phi(x)} u_t^2\, dx\, dt & \le c E^{(r-1)/2}(S) \int_S^T \int_{\Omega^+} e^{\phi(x)} u_t g(u_t)\, dx\, dt \\
& \le c [E^{(r+1)/2}(S) - E^{(r+1)/2}(T)].
\end{aligned}
$$

In the same way, we get

$$2\int_S^T \int_{\Omega^-} E^{(r-1)/2}(t) e^{\phi(x)} u_t^2 \, dx \, dt \le c \int_S^T \int_{\Omega^-} E^{(r-1)/2}(t) (e^{\phi(x)} u_t g(u_t))^{2/(r+1)} \, dx \, dt$$

$$\le \varepsilon \int_S^T \int_{\Omega^-} E^{(r+1)/2}(t) \, dt + c_\varepsilon \int_S^T \int_{\Omega^-} e^{\phi(x)} u_t g(u_t) \, dx \, dt$$

$$\le \varepsilon \int_S^T \int_{\Omega} E^{(r+1)/2}(t) \, dt + c_\varepsilon [E(S) - E(T)].$$

Substituting the sum of these two estimates into the right-hand side of (3.15) and choosing ε small enough we obtain

$$\int_S^T \int_{\Omega} E^{(r+1)/2}(t) \, dt \le c[1 + E^{(r-1)/2}(0)] E(S) \le cE(S),$$

and (3.13) follows.

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