ANNALES
POLONICI MATHEMATICI
85.1 (2005)

Forced oscillation of certain hyperbolic equations with continuous distributed deviating arguments

by SATOSHI TANAKA (Hachinohe) and NORIO YOSHIDA (Toyama)

Abstract. Certain hyperbolic equations with continuous distributed deviating arguments are studied, and sufficient conditions are obtained for every solution of some boundary value problems to be oscillatory in a cylindrical domain. Our approach is to reduce the multi-dimensional oscillation problems to one-dimensional oscillation problems for functional differential inequalities by using some integral means of solutions.

1. Introduction. We are concerned with the oscillatory properties of solutions of the hyperbolic equation with continuous distributed deviating arguments

$$\begin{aligned}(1)\qquad &\frac{\partial}{\partial t}\left[p(t)\,\frac{\partial}{\partial t}\Big(u(x,t)+\int_{\alpha}^{\beta} h(t,\xi)u(x,\varrho(t,\xi))d\eta(\xi)\Big)\right]-a(t)\Delta u(x,t)\\ &\quad-\sum_{i=1}^{k} b_i(t)\Delta u(x,\tau_i(t))-q_0(x,t)u(x,t)-\sum_{i=1}^{k} q_i(x,t)u(x,\tau_i(t))\\ &\quad-\int_{\gamma}^{\delta} q(x,t,\zeta)\varphi(u(x,\sigma(t,\zeta)))\,d\omega(\zeta)\\ &= f(x,t),\qquad (x,t)\in\Omega\equiv G\times(0,\infty),\end{aligned}$$

where G is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G.

It is assumed that:

(A$_1$) $p(t)\in C([0,\infty);(0,\infty))$, $a(t)\in C([0,\infty);[0,\infty))$,
$b_i(t)\in C([0,\infty);[0,\infty))$ $(i=1,\ldots,k)$,
$h(t,\xi)\in C([0,\infty)\times[\alpha,\beta];[0,\infty))$, $q(x,t,\zeta)\in C(\overline{\Omega}\times[\gamma,\delta];[0,\infty))$,
$q_i(x,t)\in C(\overline{\Omega};[0,\infty))$ $(i=0,1,\ldots,k)$ and $f(x,t)\in C(\overline{\Omega};\mathbb{R})$;

2000 *Mathematics Subject Classification*: 35B05, 35R10.
Key words and phrases: forced oscillation, hyperbolic equations, continuous distributed deviating arguments.

(A$_2$) $\tau_i(t) \in C([0,\infty);\mathbb{R})$ $(i = 1,\dots,k)$, $\varrho(t,\xi) \in C([0,\infty) \times [\alpha,\beta];\mathbb{R})$, $\sigma(t,\zeta) \in C([0,\infty) \times [\gamma,\delta];\mathbb{R})$ such that $\lim_{t\to\infty} \tau_i(t) = \infty$, $\lim_{t\to\infty} \min_{\xi\in[\alpha,\beta]} \varrho(t,\xi) = \infty$ and $\lim_{t\to\infty} \min_{\zeta\in[\gamma,\delta]} \sigma(t,\zeta) = \infty$;

(A$_3$) $\eta(\xi) \in C([\alpha,\beta];\mathbb{R})$ and $\omega(\zeta) \in C([\gamma,\delta];\mathbb{R})$ are increasing functions on $[\alpha,\beta]$ and $[\gamma,\delta]$, respectively, and the integrals appearing in (1) are Stieltjes integrals;

(A$_4$) $\varphi(s) \in C(\mathbb{R};\mathbb{R})$, $\varphi(-s) = -\varphi(s)$, $\varphi(s) > 0$ for $s > 0$, and $\varphi(s)$ is nondecreasing and convex in $(0,\infty)$.

We consider the following two kinds of boundary conditions:

(B$_1$) $u = \psi$ on $\partial G \times (0,\infty)$,

(B$_2$) $\dfrac{\partial u}{\partial \nu} - \mu u = \widetilde{\psi}$ on $\partial G \times (0,\infty)$,

where $\psi, \widetilde{\psi} \in C(\partial G \times (0,\infty);\mathbb{R})$, $\mu \in C(\partial G \times (0,\infty);[0,\infty))$ and ν denotes the unit exterior normal vector to ∂G.

Definition 1. By a *solution* of equation (1) we mean a function $u(x,t) \in C^2(\overline{G} \times [t_{-1},\infty);\mathbb{R}) \cap C(\overline{G} \times [\widetilde{t}_{-1},\infty);\mathbb{R})$ which satisfies (1), where

$$t_{-1} = \min\{0, \min_{1\le i\le k}\{\inf_{t\ge0} \tau_i(t)\}, \min_{\xi\in[\alpha,\beta]}\{\inf_{t\ge0} \varrho(t,\xi)\}\},$$
$$\widetilde{t}_{-1} = \min\{0, \min_{\zeta\in[\gamma,\delta]}\{\inf_{t\ge0} \sigma(t,\zeta)\}\}.$$

Definition 2. A solution $u(x,t)$ of equation (1) is said to be *oscillatory* in Ω if $u(x,t)$ has a zero in $G \times (t,\infty)$ for any $t > 0$.

The oscillations of hyperbolic equations without functional arguments were studied by Kreith, Kusano and Yoshida [5] and Yoshida [12] by using the averaging techniques (cf. [13] dealing with parabolic equations). In 1984 Mishev and Bainov [7] first established oscillation results for hyperbolic equations with delay. Recently there is much interest in studying oscillations of hyperbolic equations with continuous distributed deviating arguments. We refer the reader to [3, 4, 9, 10] for linear hyperbolic equations with continuous distributed deviating arguments, and to [2, 6, 11] for nonlinear hyperbolic equations with continuous distributed deviating arguments. However, all of them pertain to the hyperbolic equations of the form

$$\frac{\partial}{\partial t}\Big[p(t)\,\frac{\partial}{\partial t}\Big(u(x,t) + \sum_{i=1}^{l} h_i(t)u(x,\varrho_i(t))\Big)\Big] - a(t)\Delta u(x,t)$$
$$- \sum_{i=1}^{k} b_i(t)\Delta u(x,\tau_i(t)) + \int_{\gamma}^{\delta} q(x,t,\zeta)\varphi(u(x,\sigma(t,\zeta)))\,d\omega(\zeta) = f(x,t),$$

where $h_i(t) \geq 0$ and $q(x,t,\zeta) \geq 0$. It seems that no work has been done on the case where $q(x,t,\zeta) \leq 0$ and $\sum_{i=1}^{l} h_i(t)u(x,\varrho_i(t)))$ is extended to

$$\int_{\alpha}^{\beta} h(x,t,\xi)u(x,\varrho(t,\xi))\,d\eta(\xi)$$

(cf. Shoukaku [8] dealing with parabolic equations).

The purpose of this paper is to derive sufficient conditions for every solution of certain boundary value problems for (1) to be oscillatory in a cylindrical domain.

In Section 2 we reduce the multi-dimensional oscillation problems to the nonexistence problems of eventually positive solutions of functional differential inequalities. In Section 3 we present sufficient conditions for functional differential inequalities to have no eventually positive solutions. Oscillation results for the boundary value problems (1), (B_i) ($i = 1,2$) are derived in Section 4.

2. Reduction to one-dimensional oscillation problems. In this section we reduce the multi-dimensional oscillation problems for (1) to the nonexistence problems of eventually positive solutions of functional differential inequalities.

It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$\begin{aligned} -\Delta v &= \lambda v \quad \text{in } G,\\ v &= 0 \quad \text{on } \partial G \end{aligned}$$

is positive and the corresponding eigenfunction $\Phi(x)$ may be chosen so that $\Phi(x) > 0$ in G (see Courant and Hilbert [1]).

We use the following notation:

$$F(t) = \Big(\int_G \Phi(x)\,dx\Big)^{-1}\int_G f(x,t)\Phi(x)\,dx, \qquad \widetilde{F}(t) = \frac{1}{|G|}\int_G f(x,t)\,dx,$$

$$\Psi(t) = \Big(\int_G \Phi(x)\,dx\Big)^{-1}\int_{\partial G} \psi\,\frac{\partial\Phi}{\partial\nu}(x)\,dS, \qquad \widetilde{\Psi}(t) = \frac{1}{|G|}\int_{\partial G}\widetilde{\psi}\,dS,$$

$$Q(t,\zeta) = \min_{x\in\overline{G}} q(x,t,\zeta),$$

where $|G| = \int_G dx$.

THEOREM 1. *Assume that hypotheses* (A_1)–(A_4) *hold*, *as well as*

(A_5) $\quad q_0(x,t) \geq \lambda_1 a(t)$, $q_i(x,t) \geq \lambda_1 b_i(t)$ $(i = 1,\ldots,k)$.

If the functional differential inequalities

$$(2)\qquad \frac{d}{dt}\left[p(t)\,\frac{d}{dt}\Big(y(t)+\int_{\alpha}^{\beta}h(t,\xi)y(\varrho(t,\xi))\,d\eta(\xi)\Big)\right] - \int_{\gamma}^{\delta}Q(t,\zeta)\varphi(y(\sigma(t,\zeta)))d\omega(\zeta) \geq \pm G(t)$$

have no eventually positive solutions, where

$$G(t) = F(t) - a(t)\Psi(t) - \sum_{i=1}^{k} b_i(t)\Psi(\tau_i(t)),$$

then every solution u of the boundary value problem (1), (B_1) *is oscillatory in Ω.*

Proof. Assume on the contrary that there is a nonoscillatory solution u of the problem (1), (B_1). First we assume that $u > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Then there exists a number $t_1 \geq t_0$ such that $u(x, \tau_i(t)) > 0$ in $G \times [t_1, \infty)$ $(i = 1, \ldots, k)$ and $u(x, \sigma(t, \zeta)) > 0$ in $G \times [t_1, \infty) \times [\gamma, \delta]$. Multiplying (1) by $(\int_G \Phi(x)\,dx)^{-1}\Phi(x)$ and then integrating over G yields

$$(3)\qquad \frac{d}{dt}\left[p(t)\frac{d}{dt}\Big(U(t)+\int_{\alpha}^{\beta}h(t,\xi)U(\varrho(t,\xi))\,d\eta(\xi)\Big)\right] - a(t)K_\Phi\int_G \Delta u(x,t)\Phi(x)\,dx - \sum_{i=1}^{k} b_i(t)K_\Phi\int_G \Delta u(x,\tau_i(t))\Phi(x)\,dx - K_\Phi\int_G q_0(x,t)u(x,t)\Phi(x)\,dx - \sum_{i=1}^{k} K_\Phi\int_G q_i(x,t)u(x,\tau_i(t))\Phi(x)\,dx - \int_{\gamma}^{\delta}Q(t,\zeta)K_\Phi\int_G \varphi(u(x,\sigma(t,\zeta)))\Phi(x)\,dx\,d\omega(\zeta) \geq F(t), \qquad t \geq t_1,$$

where $K_\Phi = (\int_G \Phi(x)\,dx)^{-1}$ and $U(t) = K_\Phi \int_G u(x,t)\Phi(x)\,dx$. We see from Green's formula that

$$(4)\qquad K_\Phi\int_G \Delta u(x,t)\Phi(x)\,dx = -\Psi(t) - \lambda_1 U(t), \qquad t \geq t_1,$$

$$(5)\qquad K_\Phi\int_G \Delta u(x,\tau_i(t))\Phi(x)\,dx = -\Psi(\tau_i(t)) - \lambda_1 U(\tau_i(t)), \qquad t \geq t_1$$

(see, e.g., [14, p. 79]). Applying Jensen's inequality, we obtain

$$(6)\qquad K_\Phi\int_G \varphi(u(x,\sigma(t,\zeta)))\Phi(x)\,dx \geq \varphi(U(\sigma(t,\zeta))), \qquad t \geq t_1.$$

Combining (3)–(6) yields

$$\frac{d}{dt}\Big[p(t)\,\frac{d}{dt}\Big(U(t)+\int_\alpha^\beta h(t,\xi)U(\varrho(t,\xi))\,d\eta(\xi)\Big)\Big]$$
$$- K_\Phi\int_G (q_0(x,t)-\lambda_1 a(t))u(x,t)\Phi(x)\,dx$$
$$-\sum_{i=1}^k K_\Phi\int_G (q_i(x,t)-\lambda_1 b_i(t))u(x,\tau_i(t))\Phi(x)\,dx$$
$$-\int_\gamma^\delta Q(t,\zeta)\varphi(U(\sigma(t,\zeta)))\,d\omega(\zeta)\ge G(t),\qquad t\ge t_1,$$

and hence

$$\frac{d}{dt}\Big[p(t)\,\frac{d}{dt}\Big(U(t)+\int_\alpha^\beta h(t,\xi)U(\varrho(t,\xi))\,d\eta(\xi)\Big)\Big]$$
$$-\int_\gamma^\delta Q(t,\zeta)\varphi(U(\sigma(t,\zeta)))\,d\omega(\zeta)\ge G(t),\qquad t\ge t_1.$$

It is obvious that $U(t)>0$ on $[t_1,\infty)$. Hence, $U(t)$ is an eventually positive solution of (2) with $+G(t)$. This contradicts the hypothesis. If $u<0$ in $G\times[t_0,\infty)$ for some $t_0>0$, we conclude that $V(t)=-U(t)$ is an eventually positive solution of (2) with $-G(t)$. This also contradicts the hypothesis. The proof is complete.

THEOREM 2. *Assume that hypotheses* (A_1)–(A_4) *hold. If the functional differential inequalities*

$$\text{(7)}\qquad \frac{d}{dt}\Big[p(t)\,\frac{d}{dt}\Big(y(t)+\int_\alpha^\beta h(t,\xi)y(\varrho(t,\xi))\,d\eta(\xi)\Big)\Big]$$
$$-\int_\gamma^\delta Q(t,\zeta)\varphi(y(\sigma(t,\zeta)))\,d\omega(\zeta)\ge \pm\widetilde{G}(t),$$

where

$$\widetilde{G}(t)=\widetilde{F}(t)+a(t)\widetilde{\Psi}(t)+\sum_{i=1}^k b_i(t)\widetilde{\Psi}(\tau_i(t)),$$

have no eventually positive solutions, then every solution u of the boundary value problem (1), (B_2) *is oscillatory in* Ω.

Proof. Suppose to the contrary that there exists a nonoscillatory solution u of the problem (1), (B_2). First we assume that $u>0$ in $G\times[t_0,\infty)$ for some $t_0>0$. Then there is a number $t_1\ge t_0$ such that $u(x,\tau_i(t))>0$ in $G\times(0,\infty)$

$(i = 1, \ldots, k)$ and $u(x, \sigma(t,\zeta)) > 0$ in $G \times (0, \infty) \times [\gamma, \delta]$. Dividing (1) by $|G|$ and then integrating over G yields

$$
\begin{aligned}
(8)\qquad & \frac{d}{dt}\left[p(t)\,\frac{d}{dt}\Big(\widetilde{U}(t) + \int_\alpha^\beta h(t,\xi)\widetilde{U}(\varrho(t,\xi))\,d\eta(\xi)\Big)\right] \\
&\quad - a(t)\,\frac{1}{|G|}\int_G \Delta u(x,t)\,dx - \sum_{i=1}^k b_i(t)\,\frac{1}{|G|}\int_G \Delta u(x,\tau_i(t))\,dx \\
&\quad - \frac{1}{|G|}\int_G q_0(x,t)u(x,t)\,dx - \sum_{i=1}^k \frac{1}{|G|}\int_G q_i(x,t)u(x,\tau_i(t))\,dx \\
&\quad - \int_\gamma^\delta Q(t,\zeta)\frac{1}{|G|}\int_G \varphi(u(x,\sigma(t,\zeta)))\,dx\,d\omega(\zeta) \ge \widetilde{F}(t), \qquad t \ge t_1,
\end{aligned}
$$

where $\widetilde{U}(t) = \frac{1}{|G|}\int_G u(x,t)\,dx$. It follows from the divergence theorem that

$$
\begin{aligned}
(9)\qquad \frac{1}{|G|}\int_G \Delta u(x,t)\,dx &= \frac{1}{|G|}\int_{\partial G} \frac{\partial u}{\partial \nu}(x,t)\,dS \\
&= \frac{1}{|G|}\int_{\partial G} (\mu \cdot u(x,t) + \widetilde{\psi})\,dS \ge \widetilde{\Psi}(t), \qquad t \ge t_1.
\end{aligned}
$$

Analogously we have

$$
(10)\qquad \frac{1}{|G|}\int_G \Delta u(x,\tau_i(t))\,dx \ge \widetilde{\Psi}(\tau_i(t)), \qquad t \ge t_1.
$$

An application of Jensen's inequality shows that

$$
(11)\qquad \frac{1}{|G|}\int_G \varphi(u(x,\sigma(t,\zeta)))\,dx \ge \varphi(\widetilde{U}(\sigma(t,\zeta))), \qquad t \ge t_1.
$$

Combining (8)–(11) and taking account of hypothesis (A_1), we obtain

$$
\begin{aligned}
(12)\qquad & \frac{d}{dt}\left[p(t)\,\frac{d}{dt}\Big(\widetilde{U}(t) + \int_\alpha^\beta h(t,\xi)\widetilde{U}(\varrho(t,\xi))\,d\eta(\xi)\Big)\right] \\
&\quad - \int_\gamma^\delta Q(t,\zeta)\varphi(\widetilde{U}(\sigma(t,\zeta)))\,d\omega(\zeta) \ge \widetilde{G}(t), \qquad t \ge t_1.
\end{aligned}
$$

Consequently, we find that $\widetilde{U}(t)$ is an eventually positive solution of (7) with $+\widetilde{G}(t)$. This contradicts the hypothesis. The case where $u < 0$ can be treated similarly, and we are also led to a contradiction. The proof is complete.

3. Functional differential inequalities. In this section we derive sufficient conditions for the functional differential inequality

$$\frac{d}{dt}\left[p(t)\,\frac{d}{dt}\Big(y(t)+\int_{\alpha}^{\beta} h(t,\xi)y(\varrho(t,\xi))\,d\eta(\xi)\Big)\right] - \int_{\gamma}^{\delta} Q(t,\zeta)\varphi(y(\sigma(t,\zeta)))\,d\omega(\zeta) \ge H(t) \tag{13}$$

to have no eventually positive solution, where $H(t)$ is a continuous function.

It is assumed that:

(A_6) $p(t)$ is bounded from above, that is, there exists a positive constant p_1 such that $0 < p(t) \le p_1$;

(A_7) there exists a positive constant h_0 satisfying

$$\int_{\alpha}^{\beta} h(t,\xi)\,d\eta(\xi) \le h_0 < 1;$$

(A_8) $\varrho(t,\xi) \le t$ for $(t,\xi) \in (0,\infty)\times[\alpha,\beta]$;

(A_9) $\widetilde{\sigma}(t) \equiv \min_{\zeta\in[\gamma,\delta]} \sigma(t,\zeta)$ is a nondecreasing C^1-function such that

$$\widetilde{\sigma}(t) \ge t, \qquad \widetilde{\sigma}'(t) \ge \frac{1}{\sigma_0} \quad \text{for some } \sigma_0 > 0;$$

(A_{10}) $\displaystyle\int_{T}^{\infty} \frac{1}{\varphi(v)}\,dv < \infty$ for some $T > 0$.

THEOREM 3. *Assume that hypotheses* (A_1)–(A_4) *and* (A_6)–(A_{10}) *hold, and also*

(A_{11}) *there is a C^2-function $\theta(t)$ such that $\theta(t)$ is bounded,*

$$\liminf_{t\to\infty} \theta(t) < 0 \quad \textit{and} \quad (p(t)\theta'(t))' = H(t).$$

If the following conditions are satisfied:

$$\int_{t_0}^{\infty}\left[\int_{t_0}^{t} \frac{1}{p(s)}\,ds \cdot \int_{\gamma}^{\delta} Q(t,\zeta)\varphi([c+\Theta(\sigma(t,\zeta))]_+)\,d\omega(\zeta)\right] dt = \infty, \tag{14}$$

$$\int_{t_0}^{\infty}\left[\int_{t}^{\widetilde{\sigma}(t)}\left[\int_{\gamma}^{\delta} Q(s,\zeta)\,d\omega(\zeta)\right] ds\right] dt = \infty \tag{15}$$

for some $t_0 > 0$ and any $c > 0$, where

$$\Theta(t) = \theta(t) - \int_\alpha^\beta h(t,\xi)\theta(\varrho(t,\xi))\,d\eta(\xi),$$
$$[c + \Theta(\sigma(t,\zeta))]_+ = \max\{c + \Theta(\sigma(t,\zeta)), 0\},$$

then (13) *has no eventually positive solution.*

Proof. Suppose that (13) has an eventually positive solution $y(t)$. Letting

$$z(t) = y(t) + \int_\alpha^\beta h(t,\xi)y(\varrho(t,\xi))\,d\eta(\xi) - \theta(t) \tag{16}$$

and taking into account (A_{11}), we find that

$$(p(t)z'(t))' \geq \int_\gamma^\delta Q(t,\zeta)\varphi(y(\sigma(t,\zeta)))\,d\omega(\zeta) \geq 0. \tag{17}$$

Therefore, $p(t)z'(t) \geq 0$ or $p(t)z'(t) < 0$ eventually. Since $p(t) > 0$, we see that $z'(t) \geq 0$ or $z'(t) < 0$. Hence, $z(t)$ is a monotone function, and $z(t) > 0$ or $z(t) \leq 0$ eventually. We claim that $z(t) > 0$ eventually. Suppose $z(t) \leq 0$ $(t \geq t_0)$ for some $t_0 > 0$. Then we have

$$0 < y(t) + \int_\alpha^\beta h(t,\xi)y(\varrho(t,\xi))\,d\eta(\xi) \leq \theta(t),$$

which contradicts the hypothesis $\liminf_{t\to\infty} \theta(t) < 0$. Hence, we conclude that $z(t) > 0$ eventually. Since $z(t)$ is a monotone function, the following three cases are possible:

(i) $\lim_{t\to\infty} z(t) = 0$,
(ii) $\lim_{t\to\infty} z(t) = z_0 > 0$,
(iii) $\lim_{t\to\infty} z(t) = \infty$.

First we consider case (i). It is clear from (16) that $\theta(t) \geq -z(t)$ and therefore

$$\liminf_{t\to\infty} \theta(t) \geq \liminf_{t\to\infty}(-z(t)) = 0,$$

which contradicts the hypothesis $\liminf_{t\to\infty} \theta(t) < 0$.

Next we consider case (ii). In this case we can show that $\lim_{t\to\infty} z'(t) = 0$. It follows from (16) that

$$y(t) = z(t) - \int_\alpha^\beta h(t,\xi)y(\varrho(t,\xi))\,d\eta(\xi) + \theta(t) \tag{18}$$

and hence

$$y(t) \leq z(t) + \theta(t). \tag{19}$$

We see from (18) and (19) that

$$(20)\qquad \begin{aligned} y(t) &\geq z(t) - \int_{\alpha}^{\beta} h(t,\xi)[z(\varrho(t,\xi)) + \theta(\varrho(t,\xi))]\, d\eta(\xi) + \theta(t) \\ &= z(t) - \int_{\alpha}^{\beta} h(t,\xi) z(\varrho(t,\xi))\, d\eta(\xi) + \Theta(t) \\ &\geq z(t) - h_0 \max_{\xi\in[\alpha,\beta]} z(\varrho(t,\xi)) + \Theta(t). \end{aligned}$$

Since

$$\lim_{t\to\infty} (z(t) - h_0 \max_{\xi\in[\alpha,\beta]} z(\varrho(t,\xi))) = z_0 - h_0 z_0 = (1-h_0)z_0 > 0,$$

it can be shown that

$$y(t) \geq C + \Theta(t),$$

where $C = (1-h_0)z_0/2$. In view of the positivity of $y(t)$, we observe that

$$(21)\qquad y(t) \geq [C + \Theta(t)]_+.$$

Combining (17) with (21) yields

$$(22)\qquad (p(t)z'(t))' \geq \widehat{Q}(t),$$

where

$$\widehat{Q}(t) = \int_{\gamma}^{\delta} Q(t,\zeta)\varphi([C + \Theta(\sigma(t,\zeta))]_+)\, d\omega(\zeta).$$

Integrating (22) over $[t, \widetilde{t}\,]$ yields

$$p(\widetilde{t})z'(\widetilde{t}) - p(t)z'(t) \geq \int_{t}^{\widetilde{t}} \widehat{Q}(s)\, ds.$$

Letting $\widetilde{t} \to \infty$ and taking account of (A_6), we see that $\widehat{Q}(t)$ is integrable on $[t_0, \infty)$ and that

$$-p(t)z'(t) \geq \int_{t}^{\infty} \widehat{Q}(s)\, ds$$

and therefore

$$(23)\qquad -z'(t) \geq \frac{1}{p(t)} \int_{t}^{\infty} \widehat{Q}(s)\, ds.$$

Integrating (23) over $[T, t]$ yields

$$\begin{aligned}
-z(t)+z(T) &\geq \int_T^t \left[\frac{1}{p(s)}\int_s^\infty \widehat{Q}(r)\,dr\right] ds \\
&= \int_T^t \left[\int_T^r \frac{1}{p(s)}\,ds\cdot\widehat{Q}(r)\right] dr + \int_t^\infty \left[\int_T^t \frac{1}{p(s)}\,ds\cdot\widehat{Q}(r)\right] dr \\
&= \int_T^t (\widehat{p}(r)-\widehat{p}(T))\widehat{Q}(r)\,dr + \int_t^\infty (\widehat{p}(t)-\widehat{p}(T))\widehat{Q}(r)\,dr \\
&= \int_T^t \widehat{p}(r)\widehat{Q}(r)\,dr - \widehat{p}(T)\int_T^\infty \widehat{Q}(r)\,dr + \widehat{p}(t)\int_t^\infty \widehat{Q}(r)\,dr \\
&\geq \int_T^t \widehat{p}(r)\widehat{Q}(r)\,dr - \widehat{p}(T)\int_T^\infty \widehat{Q}(r)\,dr,
\end{aligned}$$

where

$$\widehat{p}(t) = \int_{t_0}^t \frac{1}{p(s)}\,ds.$$

Hence, we obtain

$$\int_T^t \widehat{p}(r)\widehat{Q}(r)\,dr \leq -z(t)+z(T)+\widehat{p}(T)\int_T^\infty \widehat{Q}(r)\,dr \leq z(T)+\widehat{p}(T)\int_T^\infty \widehat{Q}(r)\,dr.$$

Since $\widehat{Q}(t)$ is integrable on $[t_0,\infty)$, we see that $\int_T^t \widehat{p}(r)\widehat{Q}(r)\,dr$ is bounded from above. This contradicts hypothesis (14).

Finally, we treat case (iii). In this case it is easily seen that $z'(t) \geq 0$. Hypothesis (A_8) implies

$$z(\varrho(t,\xi)) \leq z(t).$$

From (20) we find that

$$y(t) \geq z(t) - z(t)\int_\alpha^\beta h(t,\xi)\,d\eta(\xi) + \Theta(t) \geq (1-h_0)z(t) + \Theta(t).$$

Since $\theta(t)$ is bounded, we observe that so is $\Theta(t)$. Since $\Theta(t)$ is bounded and $\lim_{t\to\infty} z(t) = \infty$, for any sufficiently small $\varepsilon > 0$ there is a sufficiently large number T such that $\Theta(t) \geq -\varepsilon z(t)$ $(t \geq T)$. Hence

$$y(t) \geq (1-h_0-\varepsilon)z(t)$$

and therefore

$$y(\sigma(t,\zeta)) \geq (1-h_0-\varepsilon)z(\sigma(t,\zeta)).$$

Inequality (17) implies that

$$(p(t)z'(t))' \geq \int_\gamma^\delta Q(t,\zeta)\varphi((1-h_0-\varepsilon)z(\sigma(t,\zeta)))\,d\omega(\zeta) \tag{24}$$

$$\geq \varphi((1-h_0-\varepsilon)z(\widetilde{\sigma}(t)))\int_\gamma^\delta Q(t,\zeta)\,d\omega(\zeta).$$

Integrating (24) over $[t,\widetilde{\sigma}(t)]$, we obtain

$$p(\widetilde{\sigma}(t))z'(\widetilde{\sigma}(t)) - p(t)z'(t)$$
$$\geq \int_t^{\widetilde{\sigma}(t)} \Big[\varphi((1-h_0-\varepsilon)z(\widetilde{\sigma}(s)))\int_\gamma^\delta Q(s,\zeta)\,d\omega(\zeta)\Big]\,ds$$
$$\geq \varphi((1-h_0-\varepsilon)z(\widetilde{\sigma}(t)))\int_t^{\widetilde{\sigma}(t)} \Big[\int_\gamma^\delta Q(s,\zeta)\,d\omega(\zeta)\Big]\,ds,$$

which yields

$$\int_t^{\widetilde{\sigma}(t)} \Big[\int_\gamma^\delta Q(s,\zeta)\,d\omega(\zeta)\Big]\,ds \leq \frac{p_1 z'(\widetilde{\sigma}(t))}{\varphi((1-h_0-\varepsilon)z(\widetilde{\sigma}(t)))}.$$

Taking into account (A_9), we observe that

$$\int_t^{\widetilde{\sigma}(t)} \Big[\int_\gamma^\delta Q(s,\zeta)\,d\omega(\zeta)\Big]\,ds \leq \frac{p_1\sigma_0}{1-h_0-\varepsilon}\,\frac{(1-h_0-\varepsilon)z'(\widetilde{\sigma}(t))\widetilde{\sigma}'(t)}{\varphi((1-h_0-\varepsilon)z(\widetilde{\sigma}(t)))}.$$

Integrating the above inequality over $[T,t]$, we obtain

$$\int_T^t \Big[\int_s^{\widetilde{\sigma}(s)} \Big[\int_\gamma^\delta Q(r,\zeta)\,d\omega(\zeta)\Big]\,dr\Big]\,ds \leq \frac{p_1\sigma_0}{1-h_0-\varepsilon}\int_T^t \frac{(1-h_0-\varepsilon)z'(\widetilde{\sigma}(s))\widetilde{\sigma}'(s)}{\varphi((1-h_0-\varepsilon)z(\widetilde{\sigma}(s)))}\,ds$$
$$= \frac{p_1\sigma_0}{1-h_0-\varepsilon}\int_{(1-h_0-\varepsilon)z(\widetilde{\sigma}(T))}^{(1-h_0-\varepsilon)z(\widetilde{\sigma}(t))} \frac{1}{\varphi(v)}\,dv$$
$$\leq \frac{p_1\sigma_0}{1-h_0-\varepsilon}\int_{(1-h_0-\varepsilon)z(\widetilde{\sigma}(T))}^{\infty} \frac{1}{\varphi(v)}\,dv < \infty,$$

which contradicts hypothesis (15). The proof of Theorem 3 is complete.

An important special case of (13) is the second order neutral differential equation

$$\Big(y(t) + \sum_{i=1}^{l} h_i(t)y(t-\varrho_i)\Big)'' - q(t)\varphi(y(t+\sigma)) = H(t), \tag{25}$$

where $q(t) \in C([0,\infty);[0,\infty))$ and $H(t) \in C([0,\infty);\mathbb{R})$, $0 \le \sum_{i=1}^{l} h_i(t) \le h_0 < 1$ for some positive constant h_0, and ϱ_i and σ are positive constants. As a corollary of Theorem 3 we derive the following theorem.

THEOREM 4. *Assume that hypotheses* (A_4) *and* (A_{10}) *hold, and that there is a* C^2*-function* $\theta(t)$ *such that* $\theta(t)$ *is bounded*, $\liminf_{t\to\infty}\theta(t) < 0$, $\limsup_{t\to\infty}\theta(t) > 0$ *and* $\theta''(t) = H(t)$. *If*

$$\int_{t_0}^{\infty} tq(t)\varphi([c \pm \Theta(t+\sigma)]_+)\,dt = \infty, \qquad \int_{t_0}^{\infty}\Big[\int_{t}^{t+\sigma} q(s)\,ds\Big]\,dt = \infty$$

for some $t_0 > 0$ *and any* $c > 0$, *where*

$$\Theta(t) = \theta(t) - \sum_{i=1}^{l} h_i(t)\theta(t-\varrho_i),$$

then every solution of (25) *is oscillatory at* $t = \infty$.

We note that Theorem 3 does not apply to the linear equation because of hypothesis (A_{10}). Instead of (A_{10}) we assume that

(A_{12}) there is a constant K_0 such that $\varphi(v)/v \ge K_0 > 0$ for $v \ne 0$.

THEOREM 5. *Assume that hypotheses* (A_1)–(A_3), (A_6)–(A_9), (A_{11}), *and* (A_{12}) *hold. If*

$$\int_{t_0}^{\infty}\Big[\int_{t_0}^{t}\frac{1}{p(s)}\,ds \cdot \int_{\gamma}^{\delta} Q(t,\zeta)[c+\Theta(\sigma(t,\zeta))]_+\,d\omega(\zeta)\Big]\,dt = \infty$$

and

$$\limsup_{t\to\infty}\int_{t}^{\widetilde{\sigma}(t)}\Big[\int_{s}^{\widetilde{\sigma}(t)}\frac{1}{p(r)}\,dr \cdot \int_{\gamma}^{\delta} Q(s,\zeta)\,d\omega(\zeta)\Big]\,ds > \frac{1}{K_0(1-h_0)} \tag{26}$$

for some $t_0 > 0$ *and any* $c > 0$, *then* (13) *has no eventually positive solution.*

Proof. Let $y(t)$ be an eventually positive solution of (13). Proceeding as in the proof of Theorem 3, we see that (17) holds, and therefore

$$(p(t)z'(t))' \ge K_0 \int_{\gamma}^{\delta} Q(t,\zeta)y(\sigma(t,\zeta))\,d\omega(\zeta).$$

Hence, in cases (i) and (ii) of the proof of Theorem 3 we are led to a contradiction by the same arguments as in Theorem 3. We consider case (iii), i.e. $\lim_{t\to\infty} z(t) = \infty$. Then (24) can be reduced to

$$(p(t)z'(t))' \ge K_0(1-h_0-\varepsilon)z(\widetilde{\sigma}(t))\int_{\gamma}^{\delta} Q(t,\zeta)\,d\omega(\zeta).$$

Integrating the above inequality over $[t,s]$ yields

$$p(s)z'(s) - p(t)z'(t) \ge K_0(1-h_0-\varepsilon)\int_t^s z(\widetilde{\sigma}(r))\int_\gamma^\delta Q(r,\zeta)\,d\omega(\zeta)\,dr$$

and hence

$$z'(s) \ge K_0(1-h_0-\varepsilon)\,\frac{1}{p(s)}\int_t^s z(\widetilde{\sigma}(r))\int_\gamma^\delta Q(r,\zeta)\,d\omega(\zeta)\,dr.$$

Integrating the above inequality over $[t,\widetilde{\sigma}(t)]$, we have

$$\begin{aligned} z(\widetilde{\sigma}(t)) - z(t) &\ge K_0(1-h_0-\varepsilon)\int_t^{\widetilde{\sigma}(t)}\left[\frac{1}{p(s)}\int_t^s z(\widetilde{\sigma}(r))\int_\gamma^\delta Q(r,\zeta)\,d\omega(\zeta)\,dr\right]ds \\ &= K_0(1-h_0-\varepsilon)\int_t^{\widetilde{\sigma}(t)}\left[\int_r^{\widetilde{\sigma}(t)}\frac{1}{p(s)}\,ds\cdot z(\widetilde{\sigma}(r))\int_\gamma^\delta Q(r,\zeta)\,d\omega(\zeta)\right]dr \end{aligned}$$

and hence

$$\begin{aligned} z(\widetilde{\sigma}(t)) &\ge z(t) + K_0(1-h_0-\varepsilon)z(\widetilde{\sigma}(t)) \\ &\quad\times\int_t^{\widetilde{\sigma}(t)}\left[\int_r^{\widetilde{\sigma}(t)}\frac{1}{p(s)}\,ds\cdot\int_\gamma^\delta Q(r,\zeta)\,d\omega(\zeta)\right]dr. \end{aligned}$$

Consequently, we obtain

$$1 - K_0(1-h_0-\varepsilon)\int_t^{\widetilde{\sigma}(t)}\left[\int_r^{\widetilde{\sigma}(t)}\frac{1}{p(s)}\,ds\cdot\int_\gamma^\delta Q(r,\zeta)\,d\omega(\zeta)\right]dr \ge \frac{z(t)}{z(\widetilde{\sigma}(t))} > 0$$

and therefore

$$\limsup_{t\to\infty}\int_t^{\widetilde{\sigma}(t)}\left[\int_r^{\widetilde{\sigma}(t)}\frac{1}{p(s)}\,ds\cdot\int_\gamma^\delta Q(r,\zeta)\,d\omega(\zeta)\right]dr \le \frac{1}{K_0(1-h_0-\varepsilon)}.$$

Letting $\varepsilon \to 0$, we find that

$$\limsup_{t\to\infty}\int_t^{\widetilde{\sigma}(t)}\left[\int_r^{\widetilde{\sigma}(t)}\frac{1}{p(s)}\,ds\cdot\int_\gamma^\delta Q(r,\zeta)\,d\omega(\zeta)\right]dr \le \frac{1}{K_0(1-h_0)},$$

which contradicts the hypothesis (26). The proof is complete.

We consider the linear differential equation of neutral type

$$\text{(27)}\qquad \Big(y(t) + \sum_{i=1}^{l} h_i(t)y(t-\varrho_i)\Big)'' - q(t)y(t+\sigma) = H(t),$$

which is a special case of (25).

THEOREM 6. *Assume that there is a C^2-function $\theta(t)$ such that $\theta(t)$ is bounded*, $\liminf_{t\to\infty}\theta(t)<0$, $\limsup_{t\to\infty}\theta(t)>0$ *and* $\theta''(t)=H(t)$. *If*

$$\int_{t_0}^{\infty} tq(t)[c\pm\Theta(t+\sigma)]_+\,dt=\infty,\qquad \limsup_{t\to\infty}\int_{t}^{t+\sigma}(t+\sigma-s)q(s)\,ds>\frac{1}{1-h_0}$$

for some $t_0>0$ *and any* $c>0$, *then every solution of* (27) *is oscillatory at* $t=\infty$.

4. Oscillation results for boundary value problems. In this section we present oscillation results for the boundary value problems for (1), (B_i) ($i=1,2$) by combining the results of Sections 2 and 3.

THEOREM 7. *Assume that hypotheses* (A_1)–(A_{10}) *hold, and that there exists a C^2-function $\theta(t)$ such that $\theta(t)$ is bounded*, $\liminf_{t\to\infty}\theta(t)<0$, $\limsup_{t\to\infty}\theta(t)>0$ *and*

$$(p(t)\theta'(t))'=G(t).$$

If

$$\int_{t_0}^{\infty}\left[\int_{t_0}^{t}\frac{1}{p(s)}\,ds\cdot\int_{\gamma}^{\delta}Q(t,\zeta)\varphi([c\pm\Theta(\sigma(t,\zeta))]_+)\,d\omega(\zeta)\right]dt=\infty, \tag{28}$$

$$\int_{t_0}^{\infty}\left[\int_{t}^{\widetilde{\sigma}(t)}\left[\int_{\gamma}^{\delta}Q(s,\zeta)\,d\omega(\zeta)\right]ds\right]dt=\infty \tag{29}$$

for some $t_0>0$ *and any* $c>0$, *then every solution u of the boundary value problem* (1), (B_1) *is oscillatory in* Ω.

Proof. The conclusion follows by combining Theorems 1 and 3.

THEOREM 8. *Assume that hypotheses* (A_1)–(A_4), (A_6)–(A_{10}) *hold, and that there is a C^2-function $\theta(t)$ such that $\theta(t)$ is bounded*, $\liminf_{t\to\infty}\theta(t)<0$, $\limsup_{t\to\infty}\theta(t)>0$ *and*

$$(p(t)\theta'(t))'=\widetilde{G}(t).$$

If (28) *and* (29) *are satisfied, then every solution u of the boundary value problem* (1), (B_2) *is oscillatory in* Ω.

Proof. A combination of Theorems 2 and 3 yields the conclusion.

THEOREM 9. *Assume that hypotheses* (A_1)–(A_9) *and* (A_{12}) *hold, and that there exists a C^2-function $\theta(t)$ such that $\theta(t)$ is bounded*, $\liminf_{t\to\infty}\theta(t)<0$, $\limsup_{t\to\infty}\theta(t)>0$ *and*

$$(p(t)\theta'(t))'=G(t).$$

If

$$\int_{t_0}^{\infty}\left[\int_{t_0}^{t}\frac{1}{p(s)}\,ds\cdot\int_{\gamma}^{\delta}Q(t,\zeta)[c\pm\Theta(\sigma(t,\zeta))]_+\,d\omega(\zeta)\right]dt=\infty, \tag{30}$$

$$\limsup_{t\to\infty}\int_{t}^{\widetilde{\sigma}(t)}\left[\int_{s}^{\widetilde{\sigma}(t)}\frac{1}{p(r)}\,dr\cdot\int_{\gamma}^{\delta}Q(s,\zeta)\,d\omega(\zeta)\right]ds>\frac{1}{K_0(1-h_0)} \tag{31}$$

for some $t_0>0$ *and any* $c>0$, *then every solution* u *of the boundary value problem* (1), (B_1) *is oscillatory in* Ω.

Proof. The conclusion follows from Theorems 1 and 5.

THEOREM 10. *Assume that hypotheses* (A_1)–(A_4), (A_6)–(A_9) *and* (A_{12}) *hold, and that there exists a* C^2*-function* $\theta(t)$ *such that* $\theta(t)$ *is bounded*, $\liminf_{t\to\infty}\theta(t)<0$, $\limsup_{t\to\infty}\theta(t)>0$ *and*

$$(p(t)\theta'(t))'=\widetilde{G}(t).$$

If (30) *and* (31) *are satisfied, then every solution* u *of the boundary value problem* (1), (B_2) *is oscillatory in* Ω.

Proof. The conclusion follows by combining Theorems 2 and 5.

A special case of (1) is

$$\frac{\partial^2}{\partial t^2}\Big(u(x,t)-\sum_{i=1}^{l}h_i(t)u(x,\varrho_i(t))\Big)-\Delta u(x,t) \\ -q_0(t)u(x,t)-q(t)\varphi(u(x,\sigma(t)))=f(x,t), \tag{32}$$

where $h_i(t)\in C([0,\infty);[0,\infty))$ $(i=1,\dots,l)$, $q_0(t),q(t)\in C([0,\infty);[0,\infty))$, $\varrho_i(t)\in C([0,\infty);\mathbb{R})$ $(i=1,\dots,l)$, $\sigma(t)\in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty}\varrho_i(t)=\infty$, $\lim_{t\to\infty}\sigma(t)=\infty$, $\varrho_i(t)\le t$, $\sigma(t)\ge t$, $\sigma'(t)\ge 1/\sigma_0$ for some $\sigma_0>0$, $0\le\sum_{i=1}^{l}h_i(t)\le h_0<1$ for some $h_0>0$, $q_0(t)\ge\lambda_1$ and $f(x,t)\in C(\overline{\Omega};\mathbb{R})$.

The following two corollaries are direct consequences of Theorems 7 and 9, and the proofs will be omitted.

COROLLARY 1. *Assume that hypotheses* (A_4) *and* (A_{10}) *hold, and that there exists a* C^2*-function* $\theta(t)$ *such that* $\theta(t)$ *is bounded*, $\liminf_{t\to\infty}\theta(t)<0$, $\limsup_{t\to\infty}\theta(t)>0$ *and*

$$\theta''(t)=G(t).$$

If

$$\int_{t_0}^{\infty}tq(t)\varphi([c\pm\Theta(\sigma(t))]_+)\,dt=\infty,\qquad\int_{t_0}^{\infty}\left[\int_{t}^{\sigma(t)}q(s)\,ds\right]dt=\infty$$

for some $t_0 > 0$ *and any* $c > 0$, *where*

$$\Theta(t) = \theta(t) - \sum_{i=1}^{l} h_i(t)\theta(\varrho_i(t)),$$

then every solution u *of the boundary value problem* (32), (B_1) *is oscillatory in* Ω.

COROLLARY 2. *Assume that hypotheses* (A_4) *and* (A_{12}) *hold, and that there exists a* C^2*-function* $\theta(t)$ *such that* $\theta(t)$ *is bounded*, $\liminf_{t\to\infty}\theta(t) < 0$, $\limsup_{t\to\infty}\theta(t) > 0$ *and*

$$\theta''(t) = G(t).$$

If

$$\int_{t_0}^{\infty} tq(t)[c \pm \Theta(\sigma(t))]_+ \, dt = \infty,$$

$$\limsup_{t\to\infty} \int_{t}^{\sigma(t)} (\sigma(t) - s)q(s)\, ds > \frac{1}{K_0(1-h_0)}$$

for some $t_0 > 0$ *and any* $c > 0$, *then every solution* u *of the boundary value problem* (32), (B_1) *is oscillatory in* Ω.

REMARK. We can establish the analogous oscillation results for the problem (32), (B_2) if we delete the hypothesis $q_0(t) \geq \lambda_1$ and we only replace $G(t)$ by $\widetilde{G}(t)$ in Corollaries 1 and 2.

EXAMPLE. We consider the problem

$$(33)\quad \frac{\partial}{\partial t}\left[\frac{1}{t+1}\frac{\partial}{\partial t}\left(u(x,t) + \int_0^{\pi}\frac{1}{4}\cdot u\left(x, t-\frac{5}{2}\pi+\xi\right)d\xi\right)\right] - \frac{\partial^2 u}{\partial x^2}(x,t) - 3u(x,t) - \int_0^{\pi} u\left(x, t+\frac{\pi}{2}+\zeta\right)d\zeta = f(x,t), \quad (x,t)\in(0,\pi)\times(0,\infty),$$

$$(34)\quad u(0,t) = u(\pi,t) = 0, \qquad t > 0,$$

where

$$f(x,t) = -\frac{3}{2}\sin x\left[\frac{\cos t}{(t+1)^2} + \frac{\sin t}{t+1}\right].$$

Here $n = 1$, $G = (0,\pi)$, $\Omega = (0,\pi)\times(0,\infty)$, $p(t) = 1/(t+1)$, $[\alpha,\beta] = [0,\pi]$, $h(t,\xi) = 1/4$, $\varrho(t,\xi) = t - (5/2)\pi + \xi$, $\eta(\xi) = \xi$, $a(t) = 1$, $b_i(t) \equiv 0$, $q_0(x,t) = 3$, $q_i(x,t) \equiv 0$, $[\gamma,\delta] = [0,\pi]$, $q(x,t,\zeta) = Q(t,\zeta) = 1$, $\varphi(s) = s$, $\sigma(t,\zeta) = t + \pi/2 + \zeta$ and $\omega(\zeta) = \zeta$. It is easily seen that $\lambda_1 = 1$ and $\Phi(x) = \sin x$. We easily see that (A_5), (A_6) and (A_{12}) hold for $p_1 = 1$ and

$K_0 = 1$. Since

$$\int_0^\pi h(t,\xi)\,d\eta(\xi) = \int_0^\pi \frac{1}{4}\,d\xi = \frac{\pi}{4} < 1,$$

we can choose $h_0 = \pi/4$, and hence (A_7) is satisfied. It is easy to check that

$$\varrho(t,\xi) = t - \frac{5}{2}\pi + \xi \le t - \frac{5}{2}\pi + \pi = t - \frac{3}{2}\pi \le t, \qquad \xi \in [0,\pi],$$

and hence (A_8) is satisfied. Since

$$\widetilde{\sigma}(t) = \min_{\zeta\in[0,\pi]} \left(t + \frac{\pi}{2} + \zeta\right) = t + \frac{\pi}{2} \ge t$$

and $\widetilde{\sigma}'(t) = 1$, we find that (A_9) holds for $\sigma_0 = 1$. An easy computation shows that

$$G(t) = F(t) = -\frac{3\pi}{8}\left[\frac{\cos t}{(t+1)^2} + \frac{\sin t}{t+1}\right].$$

Choosing $\theta(t) = (3/8)\pi \sin t$, we find that $\theta(t)$ is bounded, $\liminf_{t\to\infty} \theta(t) < 0$, $\limsup_{t\to\infty} \theta(t) > 0$, and $(p(t)\theta'(t))' = G(t)$. Then we have

$$\Theta(t) = \frac{3\pi}{16}\sin t.$$

A simple calculation implies that (30) and (31) hold. Consequently, from Theorem 9 it follows that every solution of the problem (33), (34) is oscillatory in $(0,\pi)\times(0,\infty)$. In fact

$$u(x,t) = \sin x \cdot \sin t$$

is such a solution.

References

[1] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, *Vol. I*, Interscience, New York, 1966.

[2] L. H. Deng, *Oscillation criteria for certain hyperbolic functional differential equations with Robin boundary condition*, Indian J. Pure Appl. Math. 33 (2002), 1137–1146.

[3] L. H. Deng and W. G. Ge, *Oscillation for certain delay hyperbolic equations satisfying the Robin boundary condition*, ibid. 32 (2001), 1269–1274.

[4] L. H. Deng, W. G. Ge and P. G. Wang, *Oscillation of hyperbolic equations with continuous deviating argument under the Robin boundary condition*, Soochow J. Math. 29 (2003), 1–6.

[5] K. Kreith, T. Kusano and N. Yoshida, *Oscillation properties of nonlinear hyperbolic equations*, SIAM J. Math. Anal. 15 (1984), 570–578.

[6] X. Z. Liu and X. L. Fu, *Oscillation criteria for nonlinear inhomogeneous hyperbolic equations with distributed deviating arguments*, J. Appl. Math. Stochastic Anal. 9 (1996), 21–31.

[7] D. P. Mishev and D. D. Bainov, *Oscillation properties of the solutions of hyperbolic equations of neutral type*, in: Differential Equations: Qualitative Theory (Szeged, 1984), Colloq. Math. Soc. János Bolyai 47, North-Holland, Amsterdam, 1987, Vol. II, 771–780.

[8] Y. Shoukaku, *On the oscillatory properties of parabolic equations with continuous distributed arguments*, Math. J. Toyama Univ. 26 (2003), 93–107.

[9] P. G. Wang, *Forced oscillation of a class of delay hyperbolic equation boundary value problem*, Appl. Math. Comput. 103 (1999), 15–25.

[10] —, *Oscillation of certain neutral hyperbolic equations*, Indian J. Pure Appl. Math. 31 (2000), 949–956.

[11] P. G. Wang and Y. H. Yu, *Oscillation criteria for a nonlinear hyperbolic equation boundary value problem*, Appl. Math. Lett. 12 (1999), 91–98.

[12] N. Yoshida, *An oscillation theorem for characteristic initial value problems for nonlinear hyperbolic equations*, Proc. Amer. Math. Soc. 76 (1979), 95–100.

[13] —, *Oscillation of nonlinear parabolic equations with functional arguments*, Hiroshima Math. J. 16 (1986), 305–314.

[14] —, *Oscillation criteria for a class of hyperbolic equations with functional arguments*, Kyungpook Math. J. 41 (2001), 75–85.

Department of Electrical Engineering
Hachinohe National College of Technology
Hachinohe 039-1192, Japan
E-mail: tanaka@xmath.ous.ac.jp

Department of Mathematics
Faculty of Science
Toyama University
Toyama 930-8555, Japan
E-mail: nori@sci.toyama-u.ac.jp

Reçu par la Rédaction le 1.12.2003 (1491)