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Enclosing solutions of second order equations

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Abstract. We apply Max Müller's Theorem to second order equations $u'' = f(t,u,u')$ to obtain solutions between given functions v, w.

1. Introduction. Let $I \subseteq \mathbb{R}$ be an interval, and let $v, w \in C^2(I,\mathbb{R})$ with $v(t) \le w(t)$ $(t \in I)$. Let

$$S := \{(t,x) : t \in I,\ v(t) \le x \le w(t)\},$$

and let $f : S \times \mathbb{R} \to \mathbb{R}$ be continuous. Consider the second order equation

$$u''(t) = f(t, u(t), u'(t)). \tag{1}$$

We are interested in the existence of a solution $u : I \to \mathbb{R}$ of (1). Then in particular graph $u \subseteq S$, that is, $v(t) \le u(t) \le w(t)$ on I.

Let $k, l : I \to \mathbb{R}$ be continuous and such that the equation

$$h''(t) + k(t)|h'(t)| + l(t)h(t) = 0 \tag{2}$$

has a positive solution $h : I \to (0,\infty)$. Under these assumptions we prove

Theorem 1. *If*

(i) $|f(t,x,p) - f(t,x,q)| \le k(t)|p-q|$ $((t,x) \in S,\ p,q \in \mathbb{R})$,
(ii) $v''(t) + l(t)v(t) \le f(t,x,v'(t)) + l(t)x$ $((t,x) \in S)$,
(iii) $w''(t) + l(t)w(t) \ge f(t,x,w'(t)) + l(t)x$ $((t,x) \in S)$,

then (1) *has a solution* $u : I \to \mathbb{R}$.

Remarks. If $f(t,x,p) = f(t,x)$ and $k(t) = 0$, conditions (i)–(iii) reduce to

$$v''(t) + l(t)v(t) \le f(t,x) + l(t)x \le w''(t) + l(t)w(t) \qquad ((t,x) \in S),$$

which are satisfied for example if $f(t,x) + l(t)x$ is increasing in $x \in [v(t), w(t)]$ for each $t \in I$ and if

$$v''(t) \le f(t,v(t)), \qquad w''(t) \ge f(t,w(t)) \qquad (t \in I).$$

This case is covered by the result in [2].

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Schrader [6] proved the existence of a solution u of (1) between v and w under the assumptions that

$$v''(t) \leq f(t, v(t), v'(t)), \quad w''(t) \geq f(t, w(t), w'(t)) \quad (t \in I),$$

that f is continuous on $I \times \mathbb{R}^2$, that all solutions of initial value problems for equation (1) exist on I, and that Dirichlet boundary value problems for (1) on compact subintervals of I have at most one solution.

Moreover, as described in [2] the differential inequalities above should not be mixed up with upper and lower solutions of boundary value problems in the sense of Nagumo [4], where the inequalities are in opposite direction. The following trivial example ($f = 0$) shows most clearly the difference from the method of upper and lower solutions for boundary value problems:

For $I = [a, b]$ there is an affine function between $v \leq w$ if $v'' \leq 0$ and $w'' \geq 0$, but in general it is not possible to prescribe boundary values between $v(a) \leq w(a)$ and $v(b) \leq w(b)$.

On the other hand, Rachůnková [5] proves the existence of solutions of (1) satisfying various boundary conditions, which satisfy $v(t_u) \leq u(t_u) \leq w(t_u)$ for some $t_u \in I$.

2. Max Müller's Theorem. Let $\mathbb{R}^2$ be ordered by the natural cone $K = \{(x, y) : x \geq 0, y \geq 0\}$. To prove Theorem 1 we make use of the following two-dimensional version of Max Müller's Theorem [3] (see also [7]):

Let $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in C^1([a, b], \mathbb{R}^2)$ with $\xi(t) \leq \eta(t)$ on $[a, b]$, and let

$$D := \{(t, x, y) \in [a, b] \times \mathbb{R}^2 : \xi(t) \leq (x, y) \leq \eta(t)\}.$$

Let $F = (F_1, F_2) : D \to \mathbb{R}^2$ be continuous such that for $(t, x, y) \in D$,

$$\xi_1'(t) \leq F_1(t, \xi_1(t), y), \quad \xi_2'(t) \leq F_2(t, x, \xi_2(t)),$$
$$\eta_1'(t) \geq F_1(t, \eta_1(t), y), \quad \eta_2'(t) \geq F_2(t, x, \eta_2(t)),$$

and let $\xi(a) \leq (x_0, y_0) \leq \eta(a)$. Then the initial value problem

$$(x, y)'(t) = F(t, x(t), y(t)), \quad (x(a), y(a)) = (x_0, y_0)$$

has a solution $(x, y) : [a, b] \to \mathbb{R}^2$; in particular graph$(x, y) \subseteq D$, that is, $\xi(t) \leq (x(t), y(t)) \leq \eta(t)$ on $[a, b]$.

3. Proof of Theorem 1. First, we prove the assertion for any compact interval $[a, b] \subseteq I$. Let $h : [a, b] \to \mathbb{R}$ be a positive solution of (2), and let

$$\overline{v} := v/h, \quad \overline{w} := w/h.$$

Fix $t_0 \in [a, b]$ such that

$$\overline{w}(t_0) - \overline{v}(t_0) = \min\{\overline{w}(t) - \overline{v}(t) : t \in [a, b]\},$$

and note that

$$\begin{aligned} t_0 = a \ &\Rightarrow \ \overline{v}'(t_0) \le \overline{w}'(t_0), \\ t_0 \in (a,b) \ &\Rightarrow \ \overline{v}'(t_0) = \overline{w}'(t_0), \\ t_0 = b \ &\Rightarrow \ \overline{v}'(t_0) \ge \overline{w}'(t_0). \end{aligned}$$

We first consider the case $t_0 \in [a,b)$, and prove

$$\overline{v}'(t) \le \overline{w}'(t) \ (t \in [t_0, b]).$$

By using (2) we have

$$\overline{v}'' = \frac{v''}{h} + l\overline{v} + k\,\frac{|h'|}{h}\,\overline{v} - \frac{2h'}{h}\,\overline{v}',$$

and by (ii) with $x = v(t)$,

$$\begin{aligned} \overline{v}'' &\le \frac{1}{h}\,f(t,v,v') + l\overline{v} + k\,\frac{|h'|}{h}\,\overline{v} - \frac{2h'}{h}\,\overline{v}' \\ &= \frac{1}{h} f(t,v,h'\overline{v} + h\overline{v}') + l\overline{v} + k\,\frac{|h'|}{h}\,\overline{v} - \frac{2h'}{h}\,\overline{v}'. \end{aligned}$$

Analogously, from

$$\overline{w}'' = \frac{w''}{h} + l\overline{w} + k\,\frac{|h'|}{h}\,\overline{w} - \frac{2h'}{h}\,\overline{w}'$$

we get by (iii) and again for $x = v(t)$,

$$\begin{aligned} \overline{w}'' &\ge \frac{1}{h}\,f(t,v,w') + l\overline{v} + k\,\frac{|h'|}{h}\,\overline{w} - \frac{2h'}{h}\,\overline{w}' \\ &= \frac{1}{h} f(t,v,h'\overline{w} + h\overline{w}') + l\overline{v} + k\,\frac{|h'|}{h}\,\overline{w} - \frac{2h'}{h}\,\overline{w}'. \end{aligned}$$

Let $G : [a,b] \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$G(t,x,y) = \begin{pmatrix} y \\ \frac{1}{h(t)} f(t,v(t),h'(t)x + h(t)y) + l(t)\overline{v}(t) + k(t)\frac{|h'(t)|}{h(t)}x - \frac{2h'(t)}{h(t)}y \end{pmatrix}.$$

By (i), the functions

$$p \mapsto f(t,z,p) + k(t)p, \quad p \mapsto f(t,z,-p) + k(t)p$$

are increasing on $\mathbb{R}$, so the second coordinate of G is increasing in x, and the first coordinate is increasing in y. Hence G is quasimonotone increasing in (x,y) with respect to the cone $K = \{(x,y) : x \ge 0, y \ge 0\}$ (cf. [8]). Moreover G is continuous and Lipschitz continuous in (x,y). From the estimates for $\overline{v}''$, $\overline{w}''$ above we obtain

$$\begin{pmatrix} \overline{v} \\ \overline{v}' \end{pmatrix}' - G(t,\overline{v}(t),\overline{v}'(t)) \le \begin{pmatrix} 0 \\ 0 \end{pmatrix} \le \begin{pmatrix} \overline{w} \\ \overline{w}' \end{pmatrix}' - G(t,\overline{w}(t),\overline{w}'(t))$$

for $t \in [t_0, b]$. Together with $(\overline{v}(t_0), \overline{v}'(t_0)) \le (\overline{w}(t_0), \overline{w}'(t_0))$, a classical result on differential inequalities (see [8, Satz 2]) implies

$$(\overline{v}(t), \overline{v}'(t)) \le (\overline{w}(t), \overline{w}'(t)) \quad (t \in [t_0, b]).$$

Next, consider equation (1). The transformation $\overline{u} := u/h$ leads to

$$\begin{aligned} \overline{u}'' &= \frac{u''}{h} - \frac{h''}{h}\,\overline{u} - \frac{2h'}{h}\,\overline{u}' \\ &= \frac{1}{h}\,f(t, u, u') + \left(l + k\,\frac{|h'|}{h}\right)\overline{u} - \frac{2h'}{h}\,\overline{u}' \\ &= \frac{1}{h}\,f(t, h\overline{u}, h'\overline{u} + h\overline{u}') + \left(l + k\,\frac{|h'|}{h}\right)\overline{u} - \frac{2h'}{h}\,\overline{u}'. \end{aligned}$$

We fix $c_0 \in [\overline{v}(t_0), \overline{w}(t_0)]$ and $c_1 \in [\overline{v}'(t_0), \overline{w}'(t_0)]$, and consider the initial value problem

$$(3) \qquad (x'(t), y'(t)) = F(t, x(t), y(t)), \quad (x(t_0), y(t_0)) = (c_0, c_1),$$

with

$$D := \{(t, x, y) : t \in [t_0, b],\ (\overline{v}(t), \overline{v}'(t)) \le (x, y) \le (\overline{w}(t), \overline{w}'(t))\},$$

and $F = (F_1, F_2) : D \to \mathbb{R}^2$ defined by

$$F(t, x, y) = \begin{pmatrix} y \\ \frac{1}{h(t)} f(t, h(t)x, h'(t)x + h(t)y) + \big(l(t) + k(t)\frac{|h'(t)|}{h(t)}\big)x - \frac{2h'(t)}{h(t)}y \end{pmatrix}.$$

Note that if $(x, y) : [t_0, b] \to \mathbb{R}^2$ is a solution of (3), then $u(t) = h(t)x(t)$ is a solution of (1) on $[t_0, b]$.

For $(t, x, y) \in D$ we obviously have

$$\overline{v}'(t) \le F_1(t, \overline{v}(t), y) = y,$$

and

$$(\overline{v}')'(t) \le F_2(t, x, \overline{v}'(t))$$

follows from the following inequalities (note that $(t, h(t)x) \in S$): From (i) we obtain

$$f(t, hx, h'\overline{v} + h\overline{v}') - f(t, hx, h'x + h\overline{v}') \le k|h'|(x - \overline{v}).$$

Hence

$$\begin{aligned} F_2(t, x, \overline{v}') &= \frac{1}{h}\,f(t, hx, h'x + h\overline{v}') + \left(l + k\,\frac{|h'|}{h}\right)x - \frac{2h'}{h}\,\overline{v}' \\ &\ge \frac{1}{h}\,f(t, hx, h'\overline{v} + h\overline{v}') - k\,\frac{|h'|}{h}\,(x - \overline{v}) + \left(l + k\,\frac{|h'|}{h}\right)x - \frac{2h'}{h}\,\overline{v}' \end{aligned}$$

$$= \frac{1}{h} f(t, hx, v') + lx + k \frac{|h'|}{h} \overline{v} - \frac{2h'}{h} \overline{v}'$$
$$= \frac{1}{h} f(t, hx, v') + \frac{l}{h} (hx) + k \frac{|h'|}{h} \overline{v} - \frac{2h'}{h} \overline{v}',$$

which by (ii) is

$$\geq \frac{v''}{h} + l\overline{v} + k \frac{|h'|}{h} \overline{v} - \frac{2h'}{h} \overline{v}' = \overline{v}''.$$

Analogously

$$\overline{w}'(t) \geq F_1(t, \overline{w}(t), y) = y,$$

and

$$(\overline{w}')'(t) \geq F_2(t, x, \overline{w}'(t)).$$

According to Max Müller's Theorem we have a solution of (3), hence a solution $u : [t_0, b] \to \mathbb{R}$ of the initial value problem

$$u''(t) = f(t, u(t), u'(t)), \quad u(t_0) = h(t_0)c_0,\ u'(t_0) = h'(t_0)c_0 + h(t_0)c_1, \tag{4}$$

on $[t_0, b]$.

In case $t_0 \in (a, b]$ we consider the initial value problem (4) to the left, i.e., for any $\varphi : [a, b] \to \mathbb{R}$ we set

$$\varphi_-(t) = \varphi(a + b - t) \qquad (t \in [a, b]),$$

and define S_- and $f_- : S_- \times \mathbb{R} \to \mathbb{R}$ by

$$S_- = \{(t, x) : t \in [a, b],\ v_-(t) \leq x \leq w_-(t)\}$$

and

$$f_-(t, x, p) = f(a + b - t, x, -p).$$

Now, (2) and (i)–(iii) hold for h, k, l, v, w, and S, f replaced by h_-, k_-, l_-, v_-, w_-, and S_-, f_-, respectively. Since also

$$\overline{v}'(t_0) \geq \overline{w}'(t_0) \ \Rightarrow\ (\overline{v}_-)'(a + b - t_0) \leq (\overline{w}_-)'(a + b - t_0),$$

the first part of our proof, where t_0 is replaced by $a + b - t_0$, gives a solution $u_- : [a + b - t_0, b] \to \mathbb{R}$ of

$$(u_-)''(t) = f_-(t, u_-(t), (u_-)'(t)),$$
$$u_-(a + b - t_0) = h(t_0)c_0, \qquad (u_-)'(a + b - t_0) = -h'(t_0)c_0 - h(t_0)c_1,$$

and $u = (u_-)_-$ solves (4) on $[a, t_0]$.

If, in case $t_0 \in (a, b)$, we choose $c_0 \in [\overline{v}(t_0), \overline{w}(t_0)]$, $c_1 = \overline{v}'(t_0) = \overline{w}'(t_0)$, we may put together the solutions obtained by the above procedure to get a solution of (4) on $[a, b]$, which a fortiori satisfies $v \leq u \leq w$ on $[a, b]$.

To prove the theorem on the given interval I, which we may assume to be noncompact, we choose an increasing sequence $(I_n)_{n=1}^{\infty}$ of compact intervals such that

$$I = \bigcup_{n=1}^{\infty} I_n.$$

If I contains one of its boundary points, it belongs to some I_{n_0}, and we assume $n_0 = 1$ without loss of generality. Next, for each n we choose a solution $u_n : I_n \to \mathbb{R}$ of (1) such that

$$v(t) \le u_n(t) \le w(t) \quad (t \in I_n).$$

We fix $n \in \mathbb{N}$ and consider u_m, $m \ge n$. Then

$$|u_m''(t)| \le \max\{|f(\tau, x, 0)| : \tau \in I_n,\ v(\tau) \le x \le w(\tau)\} + k(t)|u_m'(t)| \ (t \in I_n),$$

from which we get (by [1, Chapter XII, Lemma 5.1]) a constant $L_n \ge 0$ such that

$$|u_m'(t)| \le L_n \quad (m \ge n,\ t \in I_n).$$

By a standard diagonal procedure and Ascoli–Arzelà's Theorem we get a subsequence (u_{n_k}) which (together with the first and second derivatives) is locally uniformly convergent on I. Its limit is then a solution $u : I \to \mathbb{R}$ of (1) such that $v(t) \le u(t) \le w(t)$ on I. ∎

4. Examples. Let $g : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and bounded ($\alpha \le g \le \beta$), and Lipschitz continuous in its third variable. Let $\|\cdot\|$ denote Euclid's norm on $\mathbb{R}^n$, $n \ge 2$. The classical Ansatz for rotationally symmetric solutions of

$$\Delta z(\xi) = g(\|\xi\|, z(\xi), \|(\operatorname{grad} z)(\xi)\|) \quad (\xi \in \mathbb{R}^n) \tag{5}$$

is the transformation $u(\|\xi\|) = z(\xi)$, leading to the singular problem

$$u''(t) = g(t, u(t), |u'(t)|) - \frac{n-1}{t}\, u'(t) \quad (t \in (0, \infty)). \tag{6}$$

We may choose $l(t) = 0$ and $k(t) = k_0 + (n-1)/t$ with k_0 any Lipschitz constant of $p \mapsto g(t, x, p)$. Then $h(t) = 1$ solves (2). Fix $c \in \mathbb{R}$ and consider

$$v(t) = \frac{\alpha}{2n}\, t^2 + c, \quad w(t) = \frac{\beta}{2n}\, t^2 + c.$$

Then

$$\begin{aligned} v''(t) &= \frac{\alpha}{n} = \alpha - \frac{\alpha}{n}\,(n-1) \\ &\le g(t, x, v'(t)) - \frac{n-1}{t}\, v'(t) \quad (x \in \mathbb{R},\ t \in (0, \infty)), \end{aligned}$$

and

$$\begin{aligned} w''(t) &= \frac{\beta}{n} = \beta - \frac{\beta}{n}\,(n-1) \\ &\ge g(t, x, w'(t)) - \frac{n-1}{t}\, w'(t) \quad (x \in \mathbb{R},\ t \in (0, \infty)). \end{aligned}$$

By Theorem 1 there is a solution $u:(0,\infty)\to\mathbb{R}$ of (6) with $v(t)\le u(t)\le w(t)$ $(t\in(0,\infty))$. In particular the extension $u(0)=c$ leads to $u'(0)=0$. By elementary calculus, $u\in C^2([0,\infty),\mathbb{R})$. Therefore $z(\xi):=u(\|\xi\|)$ is in $C^2(\mathbb{R}^n,\mathbb{R})$, and is a symmetric solution of equation (5) such that

$$\frac{\alpha}{n}\,\|\xi\|^2+c\le z(\xi)\le\frac{\beta}{n}\,\|\xi\|^2+c\quad(\xi\in\mathbb{R}^n).$$

REMARK. In general there is no harmonic function between $v\le w$ if v is superharmonic and w is subharmonic [2].

In our second example we consider the case $f(t,x,p)=f(t,x)$, $k(t)=0$, and constant functions $v(t)=m$, $w(t)=M$ $(t\in I)$. Then conditions (i)–(iii) reduce to

$$l(t)m\le f(t,x)+l(t)x\le l(t)M\qquad(t\in I,\ m\le x\le M).$$

If f is of the form $f(t,x)=l(t)g(t,x)$ and $l(t)\ge 0$ these inequalities hold if

$$m\le g(t,x)+x\le M\quad(t\in I,\ m\le x\le M).$$

Consider for example $I=(0,1)$,

$$h(t)=t(1-t),\quad l(t)=\frac{2}{t(1-t)},$$

for which (2) holds, and $g(t,x)=\cos(tx)-x$. By Theorem 1 there is a solution $u:(0,1)\to\mathbb{R}$ of

$$u''(t)=\frac{2(\cos(tu(t))-u(t))}{t(1-t)}$$

with $-1\le u(t)\le 1$ $(t\in(0,1))$.

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