ANNALES
POLONICI MATHEMATICI
101.2 (2011)

Linear systems over $\mathbb{P}^1 \times \mathbb{P}^1$ with base points of multiplicity bounded by three

by Tomasz Lenarcik (Kraków)

Abstract. We propose a combinatorial method of proving non-specialty of a linear system of curves with multiple points in general position. As an application, we obtain a classification of special linear systems on $\mathbb{P}^1 \times \mathbb{P}^1$ with multiplicities not exceeding 3.

1. Introduction. Let $p_1, \ldots, p_r \in \mathbb{P}^1 \times \mathbb{P}^1$ denote points in general position and let $m_1, \ldots, m_r$ be positive integers. Consider the blowing up $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$ at $p_1, \ldots, p_r$, and denote the exceptional divisors by $E_1, \ldots, E_r$ respectively. For given $d, e \geq 0$, we define $\mathcal{L}_{(d,e)}(p_1 m_1, \ldots, p_r m_r)$ to be a complete linear system of the divisor

$$dH_1 + eH_2 - m_1 E_1 - \cdots - m_r E_r,$$

where H_1 and H_2 are the pullbacks of the classes of $\mathbb{P}^1 \times \{a_1\}$ and $\{a_2\} \times \mathbb{P}^1$ respectively, and $a_1, a_2 \in \mathbb{P}^1$ are arbitrary. It can be understood as a linear space of curves of bidegree (d, e) that vanish at p_i with multiplicity at least m_i for any $i = 1, \ldots, r$. For a sufficiently general choice of affine coordinates, each curve from $\mathcal{L}_{(d,e)}(p_1 m_1, \ldots, p_r m_r)$ can be uniquely represented (up to a constant factor) by a polynomial in two variables, X and Y, which contains monomials of the form $X^\alpha Y^\beta$ with $0 \leq \alpha \leq d$ and $0 \leq \beta \leq e$. Therefore, it follows from linear algebra that the projective dimension of $\mathcal{L}_{(d,e)}(p_1 m_1, \ldots, p_r m_r)$ is not less than

$$\max\Big\{ -1, (d+1)(e+1) - \sum_i \binom{m_i + 1}{2} - 1 \Big\}. \tag{1.1}$$

The actual dimension, however, does not have to equal the *expected dimension* (1.1), as the equations may happen to be linearly dependent even for a general choice of $p_1, \ldots, p_r$. In such an instance, we say that the linear system is *special*. A similar definition can be formulated for special linear

2010 *Mathematics Subject Classification*: Primary 14H50; Secondary 14Q05.
Key words and phrases: linear systems, curves, surfaces, fat points, exact cover.

DOI: 10.4064/ap101-2-1

systems over $\mathbb{P}^2$ (see for example [Dum]). Further details about linear systems will be presented in Section 1.

In Section 2 we propose a combinatorial technique of proving non-specialty of linear systems. Several approaches of this kind have recently been developed, including degeneration techniques by Ciliberto, Dumitrescu and Miranda [CDM], an application of tropic geometry by Baur and Draisma [Dra, BD], and a reduction method by Dumnicki and Jarnicki [Dum, DJ].

Comprehensive research has been done on linear systems over $\mathbb{P}^2$ in connection with the Gimigliano–Harbourne–Hirschowitz Conjecture (see [H] for the original statement or [Dum] for further references). In [DJ], Dumnicki and Jarnicki gave a classification of all special systems over $\mathbb{P}^2$ with multiplicities at most 11, which made it possible to verify that Gimigliano–Harbourne–Hirschowitz Conjecture holds for all systems of this type. The case of $\mathbb{P}^1 \times \mathbb{P}^1$ seems to be less investigated in terms of such classification. As long as all multiplicities of the base points equal 2, the problem of specialty of linear systems has been widely studied by many authors for varieties of type $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. This is due to the fact that special linear systems of this kind are closely related to defective Segre–Veronese embeddings (see [CGG1, CGG2] by Catalisano, Geramita, Gimigliano and [BD, CDM]).

As an application of the method introduced in Section 2, we state Theorem 2.2, which gives a characterization of special linear systems over $\mathbb{P}^1 \times \mathbb{P}^1$ with base points of multiplicity at most 3. While writing this paper, we found that such a characterization had already been known for homogeneous systems, i.e. systems with base points of multiplicities all equal to 3 (Laface [L]). The proof of Theorem 2.2, which is the main result of this paper, will be presented in Section 3.

2. Linear systems. Let $\mathbb{K}$ be an arbitrary field, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \{1, 2, 3, \ldots\}$. For any $\delta \in \mathbb{N}^2$ we write $\delta = (\delta_1, \delta_2)$.

DEFINITION 2.1. Any finite and nonempty set $D \subset \mathbb{N}^2$ will be called a *diagram*. Let $r \in \mathbb{N}^*$ and $m_1, \ldots, m_r \in \mathbb{N}$ (1). Let L be a field of rational functions over $\mathbb{K}$ in variables $x_1, y_1, \ldots, x_r, y_r$. We define a *linear system* spanned over a diagram D with base points of multiplicities $m_1, \ldots, m_r$ to be an L-vector space $\mathcal{L}_D(m_1, \ldots, m_r) \subset L[X, Y]$ of polynomials $f = \sum_{\delta \in D} A_\delta X^{\delta_1} Y^{\delta_2}$ such that

$$(2.1) \qquad \frac{\partial^{\alpha+\beta} f}{\partial X^\alpha \partial Y^\beta}(x_i, y_i) = 0 \quad \text{for } i = 1, \ldots, r \text{ and } \alpha + \beta < m_i.$$

Denote by $M = M_D(m_1, \ldots, m_r)$ the matrix of the system of equations (2.1),

(1) There is a technical reason to consider zero; see for example the proof of Theorem 3.2.

which are linear with respect to the unknown coefficients $\{A_\delta\}_{\delta \in D}$. We say that the system $\mathcal{L}_D(m_1, \ldots, m_r)$ is *special* if M is not of maximal rank. Observe that the entries of M belong to the polynomial ring $\mathbb{K}[x_1, y_1, \ldots, x_r, y_r]$.

Given $d, e \in \mathbb{N}$, we denote by $\mathcal{L}_{(d,e)}(m_1, \ldots, m_r)$ a linear system spanned by the diagram $\{0, 1, \ldots, d\} \times \{0, 1, \ldots, e\}$. For any diagram D and for any numbers $m_1, \ldots, m_t \in \mathbb{N}$, $q_1, \ldots, q_t \in \mathbb{N}^*$ we shall use the following notation:

$$\mathcal{L}_D(m_1^{\times q_1}, \ldots, m_t^{\times q_t}) := \mathcal{L}_D(\overbrace{m_1, \ldots, m_1}^{q_1}, \ldots, \overbrace{m_t, \ldots, m_t}^{q_t}).$$

In Section 3 we will prove the following result:

THEOREM 2.2. *Assume that $d \geq e \geq 0$. A linear system of the form $\mathcal{L}_{(d,e)}(1^{\times p}, 2^{\times q}, 3^{\times r})$ is special if and only if one of the following conditions holds:*

(S0) *$e = 0$, $p + 2q + 3r \leq d$ and $\max\{q, r\} \geq 1$,*
(S1) *$e = 1$, $p + 3q + 5r \leq 2d + 1$ and $r \geq 1$,*
(S2) *$e = 2$, $p = 0$, $d = q + 2r - 1$ and $2 \nmid q + r$,*
(S3) *$e = 3$, and for some $n \geq 1$:*

(S3a) *$d = 3n$, $p = q = 0$ and $r = 2n + 1$,*
(S3b) *$d = 3n$, $p \leq 1$, $q = 1$ and $r = 2n$,*
(S3c) *$d = 3n + 1$, $p \leq 2$, $q = 0$ and $r = 2n + 1$,*
(S3d) *$d = 3n + 2$, $p = 0$, $q = 2$ and $r = 2n + 1$,*

(S4) *$d = 5$, $e = 4$, $p = q = 0$ and $r = 5$.*

Throughout the proof, we take advantage of a relation between geometrical properties of the diagram D and the rank of the matrix $M_D(m_1, \ldots, m_r)$ (see Theorem 3.2). For linear systems that contain only one base point this relation can be expressed as follows (see [Dum] for the proof):

PROPOSITION 2.3. *Let $D = \{\delta_1, \ldots, \delta_s\}$ be a diagram and suppose that $\#D = s = \binom{m+1}{2}$ for some $m \in \mathbb{N}^*$. Then $\det M_D(m) = 0$ (i.e. the system $\mathcal{L}_D(m)$ is special) if and only if there exists a curve of degree $m - 1$ that contains all points of D. Moreover, if $\det M_D(m) \neq 0$, then*

$$\det M_D(m) = A x^{\delta_{1,1} + \cdots + \delta_{s,1} - N(m)} y^{\delta_{1,2} + \cdots + \delta_{s,2} - N(m)}$$

for some $A \in \mathbb{K} \setminus 0$, with $N(m) \in \mathbb{Z}_{\geq 0}$ being some constant that depends only on m.

This motivates the following definition:

DEFINITION 2.4. Let D be a diagram and $\#D = \binom{m+1}{2}$. We say that D is *non-special* (resp. *special*) of degree m if $\det M_D(m) \neq 0$ (resp. $= 0$).

At this point, an attractive idea may come to mind that maybe non-special systems with multiple points can be somehow constructed from single-point non-special systems, in terms of their "supporting" diagrams. It turns out that this idea may be applied in some specific situations. We explain the details in the following section.

3. Unique tilings. We now introduce the notion of a *unique tiling*, and state Theorems 3.2 and 3.5. Thanks to these theorems, we will be able to prove non-specialty of linear systems in terms of finding a solution for some specific problem of exact covering.

DEFINITION 3.1. Given a diagram D, we define its *center of mass* $c(D)$ by

$$\mathbb{Q}^2 \ni (c_1(D), c_2(D)) = c(D) := \frac{1}{\#D} \sum_{d \in D} d. \tag{3.1}$$

We say that a finite and non-empty set of diagrams T is a *tiling* if any two elements of T are disjoint. Suppose that T and T' are tilings, and consider a mapping $f : T \to T'$. We say that T and T' are *congruent* through f, and write $f : T \simeq T'$, if the following conditions hold:

(i) $\#f(D) = \#D$, $c(f(D)) = c(D)$ for any $D \in T$,
(ii) f is one-to-one,
(iii) $\bigcup T = \bigcup T'$.

A tiling T that contains only non-special diagrams is said to be *unique* if for any tiling T' the condition $f : T \simeq T'$ implies that either $T = T'$ and $f = \mathrm{id}_T$, or T' contains a special diagram.

The following theorem states a relation between uniqueness of a particular tiling and non-specialty of a linear system.

THEOREM 3.2. *Let $T = \{D_1, \ldots, D_r\}$ be a unique tiling such that $\#D_i = \binom{m_i+1}{2}$. If D is a diagram for which one of the following conditions holds:*

(i) *$D_1 \cup \cdots \cup D_r \subset D$,*
(ii) *$D \subset D_1 \cup \cdots \cup D_r$,*

then $\mathcal{L}_D(m_1, \ldots, m_r)$ is non-special.

REMARK. The following notation will be used throughout the proof. Given a diagram D and numbers $m, r, i > 0$, where $i \le r$, we define

$$M_D^{(i)}(m) := M_D(0^{\times i-1}, m, 0^{\times r-i}).$$

Observe that the entries of the matrix $M_D^{(i)}(m)$ depend only on variables x_i, y_i (see Definition 2.1).

Proof. Let $M = M_D(m_1, \ldots, m_r)$. We shall group the rows of M into submatrices $M_1, \ldots, M_r$ such that M_i corresponds to $\binom{m_i+1}{2}$ equations that depend on variables x_i, y_i (see Definition 2.1). The columns of M are indexed by the elements of D in a natural way (note that each element of D corresponds to a monomial).

First, observe that it is sufficient to prove the theorem under the assumption of (i). Now define $D' = D_1 \cup \dots \cup D_r$. From (i) it follows that the minor $M(D')$ consisting of the columns indexed by the elements of D' is maximal. By the Laplace decomposition we get

$$\begin{aligned} \det M(D') &= \sum \varepsilon(D'_1, \dots, D'_r) \det M_1(D'_1) \cdots \det M_r(D'_r) \\ &= \sum \varepsilon(D'_1, \dots, D'_r) \det M^{(1)}_{D'_1}(m_1) \cdots \det M^{(r)}_{D'_r}(m_r) \end{aligned} \tag{3.2}$$

where $\varepsilon(D'_1, \dots, D'_r) \in \{-1, 1\}$, and the summation runs over all partitions $\{D'_1, \dots, D'_r\}$ of D' for which $\#D'_i = \binom{m_i+1}{2}$.

Let $s_i = \#D'_i$. From Proposition 2.3 it follows that each component of the sum (3.2) is non-zero if and only if $D'_1, \dots, D'_r$ are non-special. Moreover, such a component is a monomial of the form

$$\begin{aligned} &\det M^{(1)}_{D'_1}(m_1) \cdots \det M^{(r)}_{D'_r}(m_r) \\ &\quad = A x_1^{c_1(D'_1)s_1 - N(m_1)} y_1^{c_2(D'_1)s_1 - N(m_1)} \cdots x_r^{c_1(D'_r)s_r - N(m_r)} y_r^{c_2(D'_r)s_r - N(m_r)} \end{aligned} \tag{3.3}$$

with some $A \in \mathbb{K} \setminus 0$. Thanks to the uniqueness of T, the non-zero monomial

$$\det M^{(1)}_{D_1}(m_1) \cdots \det M^{(r)}_{D_r}(m_r)$$

turns up as a component of (3.2) only once. Since it cannot be reduced, we get $\det M(D') \neq 0$. ∎

Remark. A weak point of Theorem 3.2 is the assumption about uniqueness of the tiling. Verifying whether a given tiling is unique or not may seem even more challenging than evaluating the rank of the matrix related to a linear system "by hand". This problem is partially addressed by Theorem 3.5, which gives a sufficient condition for uniqueness of a tiling. Before stating this result, we introduce some definitions.

Definition 3.3. Given a diagram D, we define its *boundary distributions* $\varphi_1(D)$, $\varphi_2(D)$ and *inertial momentum* $i(D)$ as follows:

$$\varphi_1(D) : \mathbb{N} \ni \alpha \mapsto \#(D \cap \{\alpha\} \times \mathbb{N}) \in \mathbb{N},$$
$$\varphi_2(D) : \mathbb{N} \ni \beta \mapsto \#(D \cap \mathbb{N} \times \{\beta\}) \in \mathbb{N},$$
$$i(D) = \sum_{\delta \in D} \|\delta - c(D)\|^2$$

where $\|\cdot\|$ denotes the Euclidean norm. We say that a diagram D is *stable* if it is non-special, its vertical and horizontal sections are segments, and for any diagram D' the equalities $\#D = \#D'$ and $c(D) = c(D')$ imply that at least one of the following conditions holds:

(i) D' is special,
(ii) $i(D') > i(D)$,
(iii) $i(D') = i(D)$, $\varphi_1(D') = \varphi_1(D)$ and $\varphi_2(D') = \varphi_2(D)$.

We will also need the following relation, defined on the set of all diagrams:

$$D \preceq D' \;\Leftrightarrow\; \text{there exist } \delta = (\delta_1, \delta_2) \in D \text{ and } \delta' = (\delta'_1, \delta'_2) \in D', \text{ such that } \delta_1 = \delta'_1 \text{ and } \delta_2 \leq \delta'_2.$$

When restricted to a particular tiling, this relation may be extended to a partial ordering. The following observation contains further details. We omit the simple proof.

Observation 3.4. *Let T be a tiling. Suppose that the projection on the first coordinate of any diagram in T is a segment. Then the following conditions are equivalent:*

(i) *$\preceq$ can be extended to a partial ordering on T,*
(ii) *for any $D, D' \in T$ the relations $D \preceq D'$ and $D' \preceq D$ imply $D = D'$.*

The following fact will be our main tool throughout the proof of Theorem 2.2 in the next section.

Theorem 3.5. *Suppose that a tiling T consists of stable diagrams. If the relation $\preceq$ can be extended to a partial ordering on T, then T is unique.*

Example 3.6. Any 1-diagram, i.e. a diagram which is singleton, is stable. A simple consequence of this trivial observation, thanks to Theorems 3.2 and 3.5, is the well known fact that any linear system with base points of multiplicities all equal to 1 is non-special. More examples of stable diagrams can be found in Figure 1.

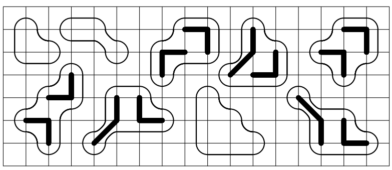


Fig. 1. All 3-diagrams and 6-diagrams up to isometry. Note that any 6-diagram, apart from the "triangular" one, can be covered with two 3-diagrams. We will make use of this property in the proof of Theorem 2.2.

Remark. Whether a given diagram is special or not can be usually verified with the help of the Bézout Theorem and Proposition 2.3. Thanks to Observation 3.4(ii), it is very easy to check whether $\preceq$ can be extended to a partial ordering on a given tiling. Meanwhile, determining if a diagram is stable or not seems to be a more complex task. As the condition of being a stable diagram is an invariant of isometry, the problem of finding all stable diagrams of a bounded degree leads to a finite number of cases, and so it can be solved through effective, but tedious, computation.

Figure 1 represents all stable diagrams consisting of three or six elements, up to isometry. Every diagram that is isometric to one of them will be called either a *3-diagram* or a *6-diagram*.

Proof of Theorem 3.5. We proceed by induction on the number of elements of T. It is clear that a tiling consisting of one diagram is always unique.

Let $T = \{D_1, \ldots, D_s\}$ and $D = \bigcup T$. Suppose that $T' = \{D'_1, \ldots, D'_s\}$ consists of non-special diagrams and $f : T \simeq T'$ where $f : D_j \mapsto D'_j$ for $j = 1, \ldots, s$. Our goal is to prove that $D_j = D'_j$ for any j. From the elementary properties of inertial momentum one has

$$\sum_{j=1}^{s} \#D_j \|c(D_j)\|^2 + i(D_j) = \sum_{d\in D} \|d\|^2 = \sum_{j=1}^{s} \#D'_j \|c(D'_j)\|^2 + i(D'_j). \tag{3.4}$$

As the D_j are stable diagrams and the D'_j are non-special, it follows that $i(D_j) \le i(D'_j)$. From (3.4) we get $i(D_j) = i(D'_j)$ for any $j = 1, \ldots, s$. Since D_j is a stable diagram, it follows that $\varphi_1(D_j) = \varphi_1(D'_j)$ for any $j = 1, \ldots, s$.

By assumption, we can extend $\preceq$ to a partial ordering on T. Without loss of generality we may assume that D_1 is minimal. For any $\alpha \in \mathbb{N}$ we define

$$m(\alpha) := \begin{cases} \min\{\beta \in \mathbb{N} : (\alpha, \beta) \in D_1\} & \text{if } \varphi_1(D_1)(\alpha) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since the vertical sections of D_1 are segments and $\varphi_1(D_1)(\alpha) = \varphi_1(D'_1)(\alpha)$, and since D_1 is minimal, we get

$$\begin{aligned} &m(\alpha) + (m(\alpha)+1) + \ldots + (m(\alpha) + \varphi_1(D_1)(\alpha) - 1) \\ &\quad = \frac{(\varphi_1(D_1)(\alpha) - 1)\varphi_1(D_1)(\alpha)}{2} + \varphi_1(D_1)(\alpha) \cdot m(\alpha) \le \sum_{(\alpha,\beta)\in D'_1} \beta. \end{aligned} \tag{3.5}$$

Summing up (3.5) for all possible α we get

$$\begin{aligned} \#D_1 \cdot c_2(D_1) &= \sum_{\alpha\in\mathbb{N}} \frac{(\varphi_1(D_1)(\alpha) - 1)\varphi_1(D_1)(\alpha)}{2} + \varphi_1(D_1)(\alpha) \cdot m(\alpha) \\ &\le \sum_{\alpha\in\mathbb{N}} \sum_{(\alpha,\beta)\in D'_1} \beta = \#D'_1 \cdot c_2(D'_1). \end{aligned} \tag{3.6}$$

From $\#D_1 \cdot c_2(D_1) = \#D'_1 \cdot c_2(D'_1)$ one actually has equalities in both (3.6) and (3.5). This implies $D_1 = D'_1$. According to the induction hypothesis, the tiling $T \setminus \{D_1\}$ is unique. Hence, from $f : T \setminus \{D_1\} \simeq T' \setminus \{D_1\}$ it follows that $D_2 = D'_2, \ldots, D_r = D'_r$. ∎

4. The proof of Theorem 2.2. We will divide the proof into several lemmas. The first of these gives an explanation why linear systems which fulfill one of the conditions from Theorem 2.2 are special.

REMARK. One can consider using the Cremona transformation as a method of verifying the specialty of these linear systems (see for example [DJ]). This is due to the fact that every complete linear system of the form $\mathcal{L}_{(d,e)}(m_1,\dots,m_r)$ (over $\mathbb{P}^1\times\mathbb{P}^1$) is isomorphic to some linear system over $\mathbb{P}^2$ (see [CGG1] for more details).

LEMMA 4.1. *The linear systems listed in the hypothesis of Theorem* 2.2 *are special.*

Proof. Later, we will refer to the following two observations. We omit their proofs as they are very simple.

OBSERVATION 4.2. *Consider a diagram D and numbers $m_1,\dots,m_r\ge 1$. Then the following properties hold:*

(i) *if $\#D>\sum_i\binom{m_i+1}{2}$, then the system $\mathcal{L}_D(m_1,\dots,m_r)$ is non-empty (i.e. it contains a non-zero polynomial);*
(ii) *if we have "$\le$" in* (i), *then the system $\mathcal{L}_D(m_1,\dots,m_r)$ is non-empty if and only if it is special.*

OBSERVATION 4.3. *Let D, D' be diagrams, and $m_1,m_1',\dots,m_r,m_r'\in\mathbb{N}$* (*some of them may be zero*). *Then*

$$\mathcal{L}_D(m_1,\dots,m_r)\cdot\mathcal{L}_{D'}(m_1',\dots,m_r')\subset\mathcal{L}_{D+D'}(m_1+m_1',\dots,m_r+m_r')$$

where $D+D'=\{d+d'\mid d\in D,\ d'\in D'\}$. Furthermore, if the systems on the left-hand side of the inclusion are non-empty, then

$$\begin{aligned}\dim_L&\mathcal{L}_{D+D'}(m_1+m_1',\dots,m_r+m_r')\\&\ge\max\{\dim_L\mathcal{L}_D(m_1,\dots,m_r),\dim_L\mathcal{L}_D'(m_1',\dots,m_r')\}.\end{aligned}$$

Let us begin with a system of the form (S2), i.e. we assume that $q+r=2n+1$ for some $n\ge 0$ and claim that the following system is special:

$$\mathcal{L}_{(q+2r-1,2)}(2^{\times q},3^{\times r}).$$

Thanks to Observation 4.2(ii) it is sufficient to show that the system is non-empty. From Observation 4.3 it follows that

$$(\mathcal{L}_{(n,1)}(1^{\times q},1^{\times r}))^2\cdot\mathcal{L}_{(r,0)}(0^{\times q},1^{\times r})\subset\mathcal{L}_{(q+2r-1,2)}(2^{\times q},3^{\times r}).$$

The factors are non-empty thanks to Observation 4.2(i) as the corresponding diagrams have respectively $2(n+1)$ and $r+1$ elements. The assertion is now a consequence of Observation 4.3.

The reader may verify that a similar argument can be applied to (S3a), (S3d) and (S4) as long as one considers the following inclusions:

$$\mathcal{L}_{(2n,2)}(2^{\times 2n+1}) \cdot \mathcal{L}_{(n,1)}(1^{\times 2n+1}) \subset \mathcal{L}_{(3n,3)}(3^{\times 2n+1}),$$
$$\mathcal{L}_{(2n+2,2)}(2^{\times 2}, 2^{\times 2n+1}) \cdot \mathcal{L}_{(n,1)}(0^{\times 2}, 1^{\times 2n+1}) \subset \mathcal{L}_{(3n+2,3)}(2^{\times 2}, 3^{\times 2n+1}),$$
$$\mathcal{L}_{(4,2)}(2^{\times 5}) \cdot \mathcal{L}_{(1,2)}(1^{\times 5}) \subset \mathcal{L}_{(5,4)}(3^{\times 5}).$$

From the case (S2) we already know that the factors containing base points of multiplicity 2 are non-empty.

We will need a more detailed estimation for (S3b) and (S3c). Let us consider the following inclusions:

$$\mathcal{L}_{(2n,2)}(0,2,2^{\times 2n}) \cdot \mathcal{L}_{(n,1)}(0,0,1^{\times 2n}) \subset \mathcal{L}_{(3n,3)}(0,2,3^{\times 2n}),$$
$$\mathcal{L}_{(2n,2)}(0,2,2^{\times 2n}) \cdot \mathcal{L}_{(n,1)}(1,0,1^{\times 2n}) \subset \mathcal{L}_{(3n,3)}(1,2,3^{\times 2n}),$$
$$\mathcal{L}_{(2n,2)}(0,0,2^{\times 2n+1}) \cdot \mathcal{L}_{(n+1,1)}(0,0,1^{\times 2n+1}) \subset \mathcal{L}_{(3n+1,3)}(0,0,3^{\times 2n+1}),$$
$$\mathcal{L}_{(2n,2)}(0,0,2^{\times 2n+1}) \cdot \mathcal{L}_{(n+1,1)}(0,1,1^{\times 2n+1}) \subset \mathcal{L}_{(3n+1,3)}(0,1,3^{\times 2n+1}),$$
$$\mathcal{L}_{(2n,2)}(0,0,2^{\times 2n+1}) \cdot \mathcal{L}_{(n+1,1)}(1,1,1^{\times 2n+1}) \subset \mathcal{L}_{(3n+1,3)}(1,1,3^{\times 2n+1}).$$

In each case we can easily compute the dimension of the second factor, which equals $2, 1, 3, 2, 1$ respectively ([2]). Since the first factor is non-empty, by Observation 4.3 the dimension of the system on the right-hand side of the inclusion is at least $2, 1, 3, 2, 1$, and this is more than the expected dimension. Therefore, all systems on the right are special.

We now move to the cases (S0) and (S1). First, consider the system $\mathcal{L}_{(d,1)}(1^{\times p}, 2^{\times q}, 3^{\times r})$ and suppose that

$$p + 3q + 5r \leq 2d + 1 \quad \text{and} \quad r \geq 1.$$

Let $M = M_{(d,1)}(1^{\times p}, 2^{\times q}, 3^{\times r})$. Our goal is to show that each maximal minor of M is zero. Since any six points of the diagram $\{0, \ldots, d\} \times \{0, 1\}$ are contained in two lines, from Proposition 2.3 it follows that the rows of M corresponding to any point of multiplicity 3 (we assumed that there is at least one such point) are linearly dependent. Hence, if only $p+3q+6r \leq 2d+2$ (i.e. the number of rows does not exceed the number of columns), then M cannot have the maximal rank. If $p+3q+6r > 2d+2$ (i.e. there are more rows than columns), then from $p + 3q + 5r \leq 2d + 1$ it follows that among any $2d + 2$ rows there are at least six corresponding to the same point of multiplicity 3. Moreover, these rows are linearly dependent, as observed before.

Finally, consider an (S0) system $\mathcal{L}_{(d,0)}(1^{\times p}, 2^{\times q}, 3^{\times r})$; i.e. assume that $p + 2q + 3r \leq d$ and $\max\{q, r\} \geq 1$. Let $M = M_{(d,0)}(1^{\times p}, 2^{\times q}, 3^{\times r})$. First, observe that as long as the number of rows does not exceed the number of

(2) Let us recall that systems containing only points of multiplicity 1 are always non-special (see Example 3.6).

columns, we can proceed as before. Otherwise, from $p+2q+3r \le d$ it follows that among any $d+1$ rows there are at least three corresponding to a point of multiplicity 2, or at least four corresponding to a point of multiplicity 3. Clearly, we can apply one of the previous arguments to the former case. To deal with the latter, observe that given any four rows corresponding to a point of multiplicity 3, at least one of them is zero. Namely, the zero row corresponds to one of the three linear equations that arise from calculating partial derivatives $\partial/\partial Y$, $\partial^2/\partial Y^2$ and $\partial^2/\partial X \partial Y$. They are all zero, because polynomials in any system of the form $\mathcal{L}_{(d,0)}(\ldots)$ do not depend on the variable Y. ∎

Remark 4.4. In the following lemmas we prove non-specialty of some class of linear systems. In fact, we only need to verify the following property:

Given a system of the form $\mathcal{L}_{(d,e)}(1^{\times p}, 2^{\times q}, 3^{\times r})$, for which none of the conditions (S0–4) holds, there exists a tiling that fulfills the hypotheses of Theorems 3.2 and 3.5, i.e. it either fits into the diagram $\{0,\ldots,d\}\times\{0,\ldots,e\}$ or covers all of its nodes.

The assumption in Theorem 3.5 concerning the possibility of extending $\preceq$ to a partial ordering will always be fulfilled. This will follow immediately from Observation 3.4(ii).

Lemma 4.5. *Any system of the form* $\mathcal{L}_{(d,0)}(1^{\times p}, 2^{\times q}, 3^{\times r})$ *where* $p+2q+3r > d$ *or* $q=r=0$ (*i.e.* (S0) *does not hold*) *is non-special.*

Proof. The case of $q=r=0$ follows from the fact that any system with all multiplicities equal to 1 is non-special (see Example 3.6).

Now suppose that $p+2q+3r > d$. According to Theorem 3.2 we only need to check whether it is possible to cover the diagram $\{0,\ldots,d\}\times\{0\}$ with 1-diagrams, 3-diagrams and 6-diagrams, numbering p, q and r respectively. Thanks to the inequality $p+2q+3r > d$, the greedy algorithm presented in Figure 2 will do. ∎

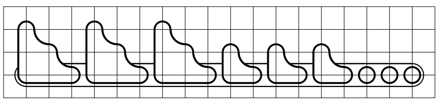


Fig. 2. The greedy algorithm for covering diagram $\{0,\ldots,d\}\times\{0\}$ under the assumption $p+2q+3r > d$

Lemma 4.6. *Any system of the form* $\mathcal{L}_{(d,1)}(1^{\times p}, 2^{\times q}, 3^{\times r})$ *where* $p+3q+5r > 2d+1$ *or* $r=0$ (*i.e.* (S1) *does not hold*) *is non-special.*

Proof. If $p+3q+5r > 2d+1$, then one can easily cover all nodes of the diagram $D=\{0,\ldots,d\}\times\{0,1\}$ using the greedy algorithm suggested in Figure 3.

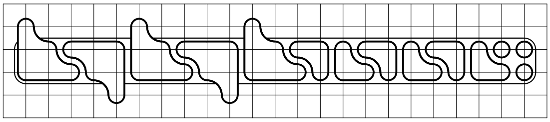


Fig. 3. The greedy algorithm for covering the diagram $\{0,\dots,d\} \times \{0,1\}$ under the assumption $p + 3q + 5r > 2d + 1$

Now suppose that $r = 0$ and $p + 3q \le 2d + 1$. Consider a tiling consisting of L-shaped 3-diagrams arranged from left to right along D (as shown in Figure 4). Thanks to the assumption $p + 3q \le 2d + 1$, all q diagrams will fit into D. Moreover, this tiling may be extended by any number of 1-diagrams (p in this case) so that condition (i) from Theorem 3.2 remains true. ∎

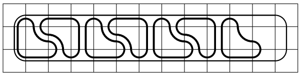


Fig. 4. A tiling that fits into $\{0,\dots,d\} \times \{0,1\}$ under the assumption $r = 0$, $p+3q \le 2d+1$

Remark 4.7. We will make use of the last argument from the proof of Lemma 4.6 a few more times. It is generally true that given a tiling which satisfies the hypotheses of Theorems 3.2 and 3.5, one can always add a single 1-diagram such that these hypotheses remain satisfied. It seems reasonable to call this relative ([3]) property of a tiling “extendability by a 1-diagram”. It is clear that one can introduce the same notion for diagrams of higher order (like 3 or 6). Note that an arbitrary tiling does not have to be extendable by a 6- or even 3-diagram.

From the fact that every tiling is extendable by a 1-diagram it follows that every tiling is extendable by an arbitrary number of 1-diagrams. This will allow us to simplify the proofs by assuming $p = 0$.

Lemma 4.8. *Any system of the form* $\mathcal{L}_{(d,2)}(1^{\times p}, 2^{\times q}, 3^{\times r})$ *where* $p > 0$ *or* $d \ne q + 2r - 1$ *or* $2 \,|\, q + r$ (*i.e.* (S2) *does not hold*) *is non-special.*

Proof. First, assume that $2 \,|\, r + q$. Then one can combine all 3-diagrams and 6-diagrams into 2×3, 3×3 and 4×3 blocks (see Figure 5). It is clear that any tiling created by arranging these blocks from left to right along the diagram $D = \{0,\dots,d\} \times \{0,1,2\}$ will satisfy the hypothesis of Theorem 3.2, i.e. the tiling will either fit into D or cover all of its nodes.

Now suppose that $2 \nmid q + r$ and $d \ne q + 2r - 1$. The difference between the current and the previous case is that after creating 2×3, 3×3 and 4×3 blocks, one 3-diagram or one 6-diagram remains unpaired. Using the same

([3]) It depends on the “support” diagram of the linear system.

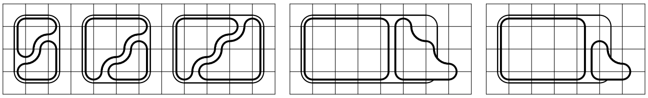


Fig. 5. Left: 3- and 6-diagrams combined into 2×3, 3×3 and 4×3 blocks. Right: if $2 \nmid q+r$ and $\#D = 3q + 6r$, then one node remains unoccupied (see the proof below).

algorithm as before, we may end up with a tiling which neither fits into D, nor covers all of its nodes (see Figure 5). However, this is possible only when $\#D = 3q + 6r \Leftrightarrow 3d + 3 = 3q + 6r$, which we assumed to be false.

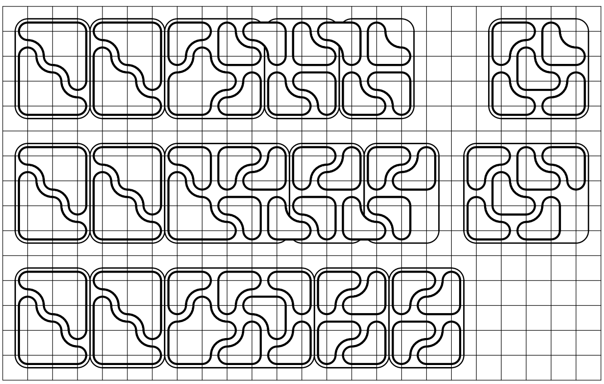
Finally, observe that if $2 \nmid q + r$ and $d = q + 2r - 1$, then the problem identified in the previous case can be easily addressed by covering the unoccupied node (see Figure 5) with a single 1-diagram. This is possible, since according to the hypothesis, one has $p > 0$ as long as $2 \nmid q + r$ and $d = q + 2r - 1$. ■

Lemma 4.9. *Assume that* $3 \le d$ *and* $p, q, r \ge 0$. *Furthermore,* n *is chosen to satisfy* $0 \le d - 3n < 3$. *Suppose that none of the following conditions holds* (*see Theorem* 2.2)*:*

(S3a) $d = 3n$, $p = q = 0$ *and* $r = 2n + 1$,
(S3b) $d = 3n$, $p \le 1$, $q = 1$ *and* $r = 2n$,
(S3c) $d = 3n + 1$, $p \le 2$, $q = 0$ *and* $r = 2n + 1$,
(S3d) $d = 3n + 2$, $p = 0$, $q = 2$ *and* $r = 2n + 1$.

Then the system $\mathcal{L}_{(d,3)}(1^{\times p}, 2^{\times q}, 3^{\times r})$ *is non-special.*

Proof. Observe that if $r \ge 2n + 2$, hence $6r \ge 12n + 12 \ge 4(d + 1)$, then the diagram $D = \{0, \ldots, d\} \times \{0, 1, 2, 3\}$ can be covered with $n + 1$ blocks of size 3×4, each tiled with a pair of 6-diagrams.



$d + 1 \equiv 1 \bmod 3$

$d + 1 \equiv 2 \bmod 3$

$d + 1 \equiv 0 \bmod 3$

Fig. 6. Three algorithms for covering the diagram $D = \{0, \ldots, d\} \times \{0, 1, 2, 3\}$ that can be applied if $6r \le \#D - 10$.

If $r \leq 2n-1$, so $6r \leq 12n-6 \leq \#D-10$ (i.e. after using all possible 6-diagrams there are at least 10 unoccupied nodes), then one of the algorithms presented in Figure 6 (the choice of the algorithm depends on $d+1$ modulo 3) can be used to construct a tiling that either fits into D, or covers all of its nodes. The blocks on the right in Figure 6 need to be used if the number of 6-diagrams is even. Even though the algorithm is given only for $p=0$, one can simply add the desired number of 1-diagrams after constructing the tiling from 3- and 6-diagrams (see Remark 4.7).

We still need to construct some tiling for $r=2n$ and $r=2n+1$. Observe that using $n-1$ blocks of size 3×4 one reduces the problem to $r=2$ or $r=3$, and $d+1 \in \{4,5,6\}$. Since we assumed that none of the conditions (S3a–d) hold, it can be easily verified that to get the proper tiling one can always choose a subset of one of the tilings presented in Figure 7 and extend it by a desired number of 1-diagrams, if needed. ∎

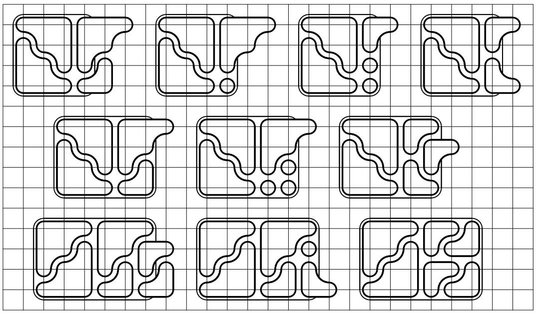


Fig. 7. If $r \in \{2,3\}$ and $d+1 \in \{4,5,6\}$, then the desired tiling is a subset of one of these tilings (possibly extended by a number of 1-diagrams).

Lemma 4.10. *If $p>0$ or $q>0$ or $r \neq 5$ (i.e.* (S4) *does not hold), then the system $\mathcal{L}_{(5,4)}(1^{\times p}, 2^{\times q}, 3^{\times r})$ is non-special.*

Proof. There exists a 6-diagram covering of $D=\{0,\dots,5\}\times\{0,\dots,4\}$ that uses precisely six diagrams. Hence, if only $r \geq 6$, the problem is solved. Supposing $r=0$, one can construct the desired tiling by using 3-diagrams combined into 3×2 and 2×3 blocks, so that as along as $3q \geq \#D$ it would cover all nodes of D, and it would fit into D if the opposite inequality holds.

When $r=5$ and $p>0$, the first tiling on the right in Figure 8 can be used. If p happens to be zero, then according to our assumptions $q>0$, and so the 1-diagram covering of the single node in D can be replaced by a 3-diagram.

We still need to consider $r \in \{1,2,3,4\}$, in which case a subset of one of the four remaining tilings presented in Figure 8 can be used, depending on r and the number of available 3-diagrams. ∎

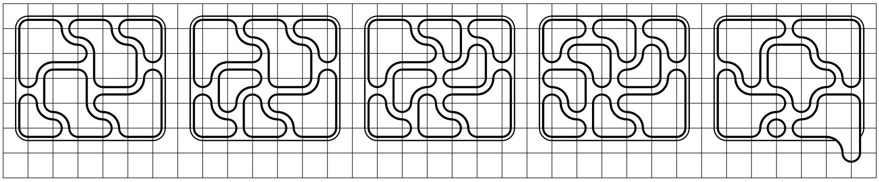


Fig. 8. Tilings that can be used if $r \in \{1, 2, 3, 4, 5\}$

Lemma 4.11. *For any $d \geq 6$ and $p, q, r \geq 0$ a linear system of the form $\mathcal{L}_{(d,4)}(1^{\times p}, 2^{\times q}, 3^{\times r})$ is non-special.*

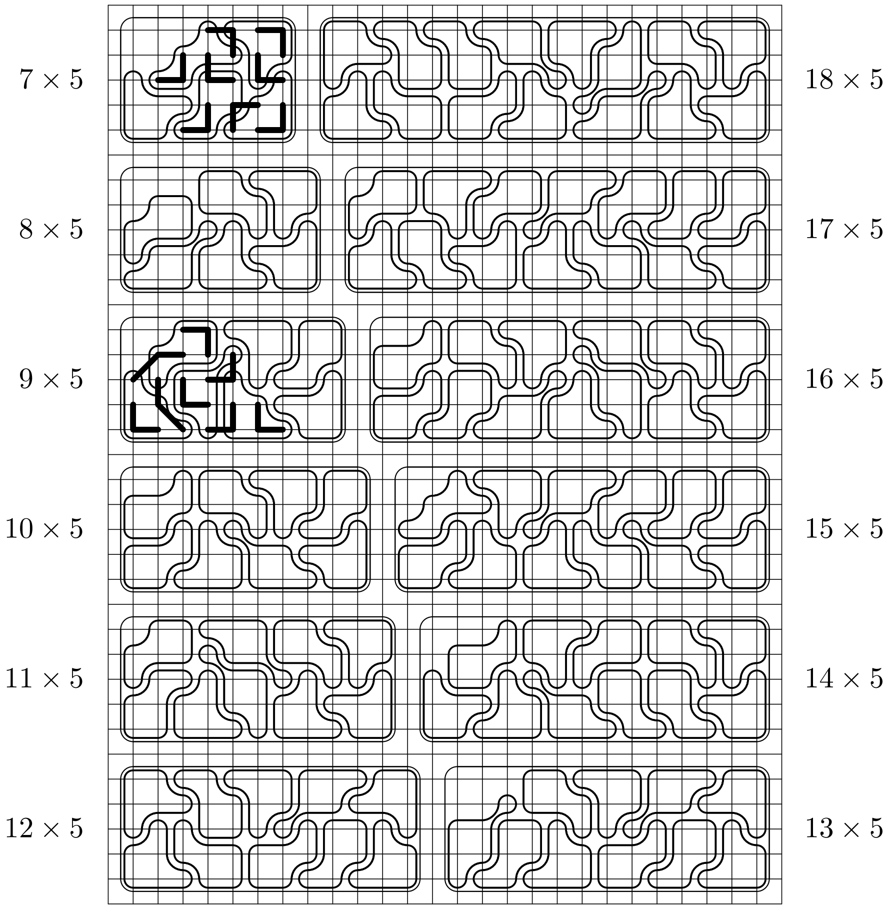


Fig. 9. Solutions for $6 \leq d < 18$, and $q = 0$. As long as $q \neq 0$ one should consider replacing an appropriate number of 6-diagrams with the same number of pairs of 3-diagrams. This can be done (see for example Figure 1 where it is shown explicitly) for every "non-triangular" 6-diagram.

Proof. Without loss of generality we may assume that $p = 0$. First, let $q = 0$ and observe that tilings presented in Figure 9 (or their subsets) can be used to prove non-specialty as long as $d < 18$. Also observe that those tilings are always extendable by a single "non-triangular" 6-diagram (see Remark 4.7).

Now assume that $d \geq 18$. If $r < 10$, then one can easily fit all 6-diagrams into D. For example, one can choose any subset of a tiling presented in Figure 9. If $r \geq 10$, then one reduces the problem of finding an appropriate tiling to a smaller d, namely $d-12$, by using the 12×5 tiling in the bottom-left corner of Figure 9.

Finally, suppose that q is arbitrary and define $n = \lfloor q/2 \rfloor$. Let us first construct a tiling with $n+r$ 6-diagrams according to the algorithm described above. When it is done, remove n (not arbitrary) 6-diagrams, and replace them with $2n$ 3-diagrams. Note that this can be always done, as any 6-diagram, apart from the "triangular" one, can be covered with two 3-diagrams (see Remark 4.7). However, some of the tilings from Figure 9 consist of the "triangular" diagram. In such an instance, the replacing strategy can be worked out explicitly.

For example, the top-left 7×5 tiling consists of one triangular 6-diagram. As long as $n \leq 4$, we keep replacing the 4 "non-triangular" diagrams. As soon as $n \geq 4$, we can start by using the strategy presented in Figure 9, which solves the problem of the "triangular" diagram, and then keep on replacing the remaining 6-diagrams. The same type of reasoning can be applied to the 9×5 tiling.

If $q = 2n$, then we are already done. If not, then we have to show that, relative to D (see Remark 4.7), the resulting tiling is extendable by a 3-diagram.

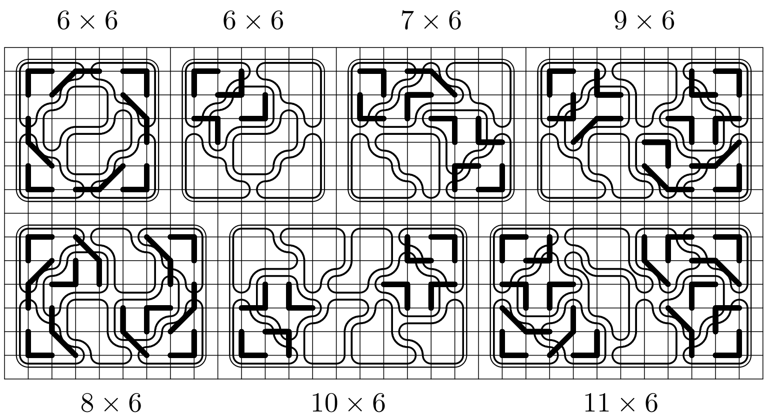


Fig. 10. Solutions for $5 = e \leq d < 11$ together with a suggested strategy of replacing the "triangular" 6-diagram

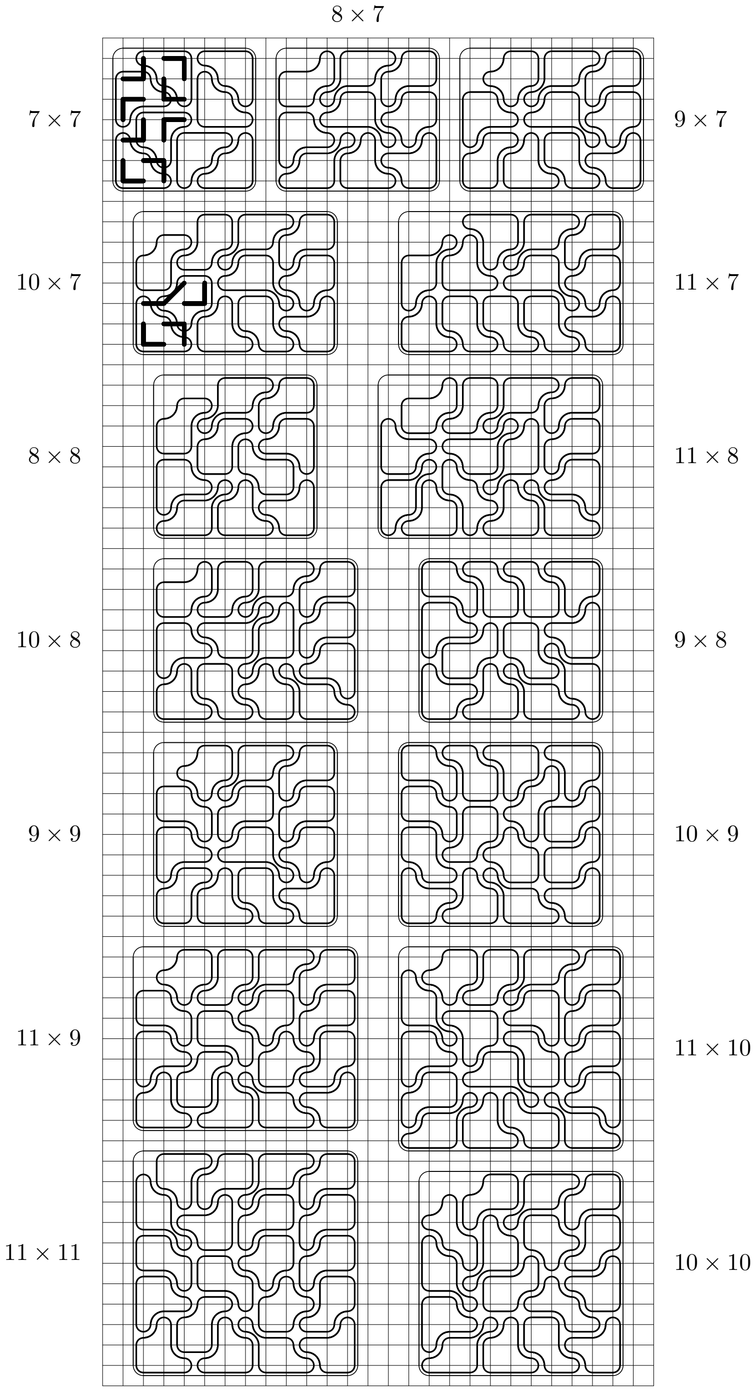


Fig. 11. Solutions for $5 < e \leq d < 12$ (see Lemma 4.12)

A problem can only occur when not all nodes of D have been covered so far. However, in such an instance the tiling must have been entirely created with blocks that contain tilings (or their subsets) presented in Figure 9 ([4]). It can be easily verified that these tilings are all extendable by a single 3-diagram (relative to the corresponding blocks). ∎

LEMMA 4.12. *For any* $5 \leq e \leq d$ *and* $p, q, r \geq 0$ *a linear system of the form* $\mathcal{L}_{(d,e)}(1^{\times p}, 2^{\times q}, 3^{\times r})$ *is non-special.*

Proof. Observe that any rectangular diagram of height 6 and width ≥ 6 can be tiled with 6-diagrams combined into blocks presented in Figure 10. We can proceed as in the proof of Lemma 4.11. First, assume that $p = q = 0$. If $d, e < 11$, then the tilings presented in Figure 11 will do.

Now suppose that $d \geq 11$. If $r \leq d + 1$, then the existence of a tilling that fits into D follows from the previous observation. As soon as $r > d+1$, one can use a $6 \times (e + 1)$ tiling to reduce the problem to smaller $d' = d - 6$ and smaller $r' = r - (e + 1)$, while still having $d' > 5$. If $d' < e$, then one may swap their roles, and eventually proceed by induction.

For arbitrary q the replacing strategy can be used as described in the proof of Lemma 4.11. The analysis of the "triangular" cases can be easily performed with the help of the strategies suggested in Figures 10 and 11. ∎

Acknowledgements. The author would like to thank Marcin Dumnicki for inspiration and help while writing this paper, and the anonymous referee for a great deal of constructive remarks that helped to improve it.

References

[BD] K. Baur and J. Draisma, *Secant dimensions of low-dimensional homogeneous varieties*, Adv. Geom. 10 (2010), 1–29.

[CGG1] M. V. Catalisano, A. V. Geramita and A. Gimigliano, *Higher secant varieties of Segre–Veronese varieties*, in: Projective Varieties with Unexpected Properties, de Gruyter, 2005, 81–107.

[CGG2] —, —, —, *Segre–Veronese embeddings of* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *and their secant varieties*, Collect. Math. 58 (2007), 1–24.

[CDM] C. Ciliberto and R. Miranda, *Degeneration of planar linear systems*, J. Reine Angew. Math. 501 (1998), 191–220.

[Dra] J. Draisma, *A tropical approach to secant dimensions*, J. Pure Appl. Algebra 212 (2008), 349–363.

[Dum] M. Dumnicki, *Reduction method for linear systems of plane curves with base points*, Ann. Polon. Math. 90 (2007), 131–143.

[DJ] M. Dumnicki and W. Jarnicki, *New effective bounds on the dimension of a linear system in* $\mathbb{P}^2$, J. Symbolic Comput. 42 (2007), 621–635.

([4]) No additional 6-diagrams needed to be used.

[H] A. Hirschowitz, *Une conjecture pour la cohomologie des diviseurs sur les surfaces rationnelles génériques*, J. Reine Angew. Math. 397 (1989), 208–213.
[L] A. Laface, *On linear systems of curves on rational scrolls*, Geom. Dedicata 90 (2002), 127–144.

Tomasz Lenarcik
Institute of Mathematics
Jagiellonian University
30-348 Kraków, Poland
E-mail: Tomasz.Lenarcik@im.uj.edu.pl

Received 30.10.2008
and in final form 26.8.2010 (2084)