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Permanence for a delayed Nicholson's blowflies model with a nonlinear density-dependent mortality term

by Bingwen Liu (Changde)

Abstract. We study a generalized Nicholson's blowflies model with a nonlinear density-dependent mortality term. Under appropriate conditions, we employ a novel proof to establish some criteria guaranteeing the permanence of this model. Moreover, we give an example to illustrate our main result.

1. Introduction. In a classic study of population dynamics, Nicholson [8] and W. S. Gurney et al. [2] proposed the following delay differential equation model:

$$x'(t) = -\delta x(t) + Px(t-\tau)e^{-ax(t-\tau)}, \tag{1.1}$$

where $x(t)$ is the size of the population at time t, P is the maximum per capita daily egg production, $1/a$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the generation time.

As a class of biological systems, Nicholson's blowflies model and the corresponding equation have attracted much attention. In particular, the existence of positive solutions, persistence, permanence, oscillation and stability of Nicholson's blowflies model have been extensively studied. We refer the reader to [4, 5, 6, 7, 10] and the references cited therein.

Recently, L. Berezansky et al. [1] pointed out that new studies in population dynamics indicate that a linear model of density-dependent mortality will be most accurate for populations at low densities, and marine ecologists are currently in the process of constructing new fishery models with nonlinear density-dependent mortality rates. However, few authors have considered dynamics of Nicholson's blowflies model with a nonlinear density-dependent mortality term. Therefore, L. Berezansky et al. [1] asked about

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the dynamic behavior of the following Nicholson's blowflies model with a nonlinear density-dependent mortality term:

$$x'(t) = -D(x(t)) + Px(t-\tau)e^{-x(t-\tau)}, \tag{1.2}$$

where the nonlinear density-dependent mortality term $D(x)$ might have one of the following forms: $D(x) = ax/(b+x)$ or $D(x) = a - be^{-x}$ with constants $a, b > 0$.

The main purpose of this paper is to give conditions ensuring the permanence of (1.2) with $D(x) = ax/(b+x)$. Since the coefficients and delays in differential equations of population and ecology problems are usually time-varying in the real world, we shall consider the following Nicholson's blowflies model with a nonlinear density-dependent mortality term:

$$x'(t) = -\frac{a(t)x(t)}{b(t)+x(t)} + \beta(t)x(t-\tau(t))e^{-\gamma(t)x(t-\tau(t))}, \tag{1.3}$$

where $a(t)$, $b(t)$, $\beta(t)$ and $\gamma(t)$ are continuous functions bounded above and below by positive constants, and $\tau(t) \geq 0$ is a bounded continuous function. Obviously, when $D(x) = ax/(b+x)$, (1.2) is a special case of (1.3).

Throughout this paper, let $r = \sup_{t\in\mathbb{R}} \tau(t)$, let $C = C([-r,0],\mathbb{R})$ be the space of continuous functions equipped with the usual supremum norm $\|\cdot\|$, and let $C_+ = C([-r,0],\mathbb{R}_+)$. If $x(t)$ is continuous and defined on $[-r+t_0,\sigma)$ with $t_0,\sigma \in \mathbb{R}$, then we define $x_t \in C$ by $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-r,0]$.

Due to the biological interpretation of model (1.3), only positive solutions are meaningful and therefore admissible. Thus we just consider admissible initial conditions

$$x_{t_0} = \varphi, \quad \varphi \in C_+ \quad \text{and} \quad \varphi(0) > 0. \tag{1.4}$$

Define a continuous map $f : \mathbb{R} \times C_+ \to \mathbb{R}$ by setting

$$f(t,\varphi) = -\frac{a(t)\varphi(0)}{b(t)+\varphi(0)} + \beta(t)\varphi(-\tau(t))e^{-\gamma(t)\varphi(-\tau(t))}.$$

Then f is a locally Lipschitz map with respect to $\varphi \in C_+$, which ensures the existence and uniqueness of the solution of (1.3) with admissible initial conditions (1.4).

We write $x_t(t_0,\varphi)$ or $x(t;t_0,\varphi)$ for a solution of the admissible initial value problem (1.3), (1.4). Also, let $[t_0,\eta(\varphi))$ be the maximal right-interval of existence of $x_t(t_0,\varphi)$.

The paper is organized as follows. In Section 2, we shall derive new sufficient conditions for the permanence of model (1.3). In Section 3, we shall give an example and a remark to illustrate our results obtained in the previous sections.

2. The permanence

LEMMA 2.1. *Suppose* $\inf_{t\in\mathbb{R}} a(t)\gamma(t)e/\beta(t)>1$. *Then the solution* $x_t(t_0,\varphi) \in C_+$ *for all* $t \in [t_0, \eta(\varphi))$, *the set of* $\{x_t(t_0,\varphi) : t \in [t_0, \eta(\varphi))\}$ *is bounded, and* $\eta(\varphi) = +\infty$. *Moreover,* $x(t; t_0, \varphi) > 0$ *for all* $t \geq t_0$.

Proof. Since $\varphi \in C_+$, using Theorem 5.2.1 in [9, p. 81], we have $x_t(t_0,\varphi) \in C_+$ for all $t \in [t_0, \eta(\varphi))$. Let $x(t) = x(t; t_0, \varphi)$. From (1.3) and the fact that $\frac{a(t)x}{b(t)+x} \leq \frac{a(t)x}{b(t)}$ for all $t \in \mathbb{R}$ and $x \geq 0$, we get

$$\begin{aligned}(2.1)\qquad x'(t) &= -\frac{a(t)x(t)}{b(t)+x(t)} + \beta(t)x(t-\tau(t))e^{-\gamma(t)x(t-\tau(t))}\\ &\geq -\frac{a(t)}{b(t)}x(t) + \beta(t)x(t-\tau(t))e^{-\gamma(t)x(t-\tau(t))}.\end{aligned}$$

In view of $x(t_0) = \varphi(0) > 0$, integrating (2.1) from t_0 to t, we have

$$\begin{aligned}(2.2)\qquad x(t) &\geq e^{-\int_{t_0}^{t}\frac{a(u)}{b(u)}\,du}x(t_0)\\ &\quad + e^{-\int_{t_0}^{t}\frac{a(u)}{b(u)}\,du}\int_{t_0}^{t} e^{\int_{t_0}^{s}\frac{a(v)}{b(v)}\,dv}\beta(s)x(s-\tau(s))e^{-\gamma(s)x(s-\tau(s))}\,ds\\ &> 0 \quad \text{for all } t \in [t_0, \eta(\varphi)).\end{aligned}$$

For each $t \in [t_0 - r, \eta(\varphi))$, we define

$$M(t) = \max\{\xi : \xi \leq t, x(\xi) = \max_{t_0-r\leq s\leq t} x(s)\}.$$

We now show that $x(t)$ is bounded on $[t_0, \eta(\varphi))$. In the contrary case, observing that $M(t) \to \eta(\varphi)$ as $t \to \eta(\varphi)$, we have

$$(2.3)\qquad \lim_{t\to\eta(\varphi)} x(M(t)) = +\infty.$$

But $x(M(t)) = \max_{t_0-r\leq s\leq t} x(s)$, and so $x'(M(t)) \geq 0$. Thus,

$$\begin{aligned}0 &\leq x'(M(t))\\ &= -\frac{a(M(t))x(M(t))}{b(M(t))+x(M(t))}\\ &\quad + \beta(M(t))x(M(t)-\tau(M(t)))e^{-\gamma(M(t))x(M(t)-\tau(M(t)))},\end{aligned}$$

and consequently

$$\begin{aligned}(2.4)\qquad &\frac{a(M(t))x(M(t))}{b(M(t))+x(M(t))}\\ &\qquad \leq \beta(M(t))x(M(t)-\tau(M(t)))e^{-\gamma(M(t))x(M(t)-\tau(M(t)))}.\end{aligned}$$

By the continuity and boundedness of the functions $a(t), b(t), \beta(t)$ and $\gamma(t)$, we can select a sequence $\{T_n\}_{n=1}^{+\infty}$ such that

$$
\begin{cases} \lim_{n\to+\infty} T_n = \eta(\varphi), & \lim_{n\to+\infty} x(M(T_n)) = +\infty, \\ \lim_{n\to+\infty} a(M(T_n)) = a^*, & \lim_{n\to+\infty} b(M(T_n)) = b^*, \\ \lim_{n\to+\infty} \beta(M(T_n)) = \beta^*, & \lim_{n\to+\infty} \gamma(M(T_n)) = \gamma^*. \end{cases} \tag{2.5}
$$

In view of (2.4) and the fact that $\sup_{u\geq 0} ue^{-\gamma(t)u} = 1/(\gamma(t)e)$, we get

$$
\begin{aligned} &\frac{a(M(T_n))x(M(T_n))}{b(M(T_n)) + x(M(T_n))} \\ &\leq \beta(M(T_n))x(M(T_n) - \tau(M(T_n)))e^{-\gamma(M(T_n))x(M(T_n)-\tau(M(T_n)))} \\ &\leq \beta(M(T_n))\frac{1}{\gamma(M(T_n))e}. \end{aligned} \tag{2.6}
$$

Letting $n \to +\infty$, (2.5) and (2.6) imply that

$$
\lim_{n\to+\infty} \frac{a(M(T_n)\gamma(M(T_n))e}{\beta(M(T_n))} = \frac{a^*\gamma^* e}{\beta^*} \leq 1,
$$

which contradicts the fact that $\inf_{t\in\mathbb{R}} a(t)\gamma(t)e/\beta(t) > 1$. This implies that $x(t)$ is bounded on $[t_0, \eta(\varphi))$. From Theorem 2.3.1 in [3], we easily obtain $\eta(\varphi) = +\infty$. This ends the proof of Lemma 2.1.

THEOREM 2.1. *Let*

$$
\inf_{t\in\mathbb{R}} \frac{a(t)\gamma(t)e}{\beta(t)} > 1 \quad \text{and} \quad \inf_{t\in\mathbb{R}} \frac{\beta(t)b(t)}{a(t)} > 1. \tag{2.7}
$$

Then system (1.3) *is permanent, i.e., there exist two positive constants k and K such that*

$$
k \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq K. \tag{2.8}
$$

Proof. From Lemma 2.1, the set $\{x_t(t_0, \varphi) : t \in [t_0, +\infty)\}$ is bounded, and there exists a positive constant K such that

$$
0 < x(t) \leq K \quad \text{for all } t > t_0. \tag{2.9}
$$

It follows that

$$
\limsup_{t\to+\infty} x(t) \leq K. \tag{2.10}
$$

We next prove that there exists a positive constant k such that

$$
\liminf_{t\to+\infty} x(t) \geq k. \tag{2.11}
$$

Suppose, for the sake of contradiction, $\liminf_{t\to+\infty} x(t) = 0$. For each $t \geq t_0$, we define

$$
m(t) = \max\{\xi : \xi \leq t,\, x(\xi) = \min_{t_0\leq s\leq t} x(s)\}.
$$

Observe that $m(t) \to +\infty$ as $t \to +\infty$ and that

$$\lim_{t\to+\infty} x(m(t)) = 0. \tag{2.12}$$

However, $x(m(t)) = \min_{t_0 \le s \le t} x(s)$, and so $x'(m(t)) \le 0$. According to (2.9), we have

$$\begin{aligned}
0 &\ge x'(m(t)) \\
&= -\frac{a(m(t))x(m(t))}{b(m(t)) + x(m(t))} + \beta(m(t))x(m(t) - \tau(m(t)))e^{-\gamma(m(t))x(m(t)-\tau(m(t)))} \\
&\ge -\frac{a(m(t))x(m(t))}{b(m(t)))} + \beta(m(t))x(m(t) - \tau(m(t)))e^{-\gamma(m(t))x(m(t)-\tau(m(t)))},
\end{aligned}$$

and consequently

$$\begin{aligned}
&\frac{a(m(t))x(m(t))}{b(m(t)))} \\
&\qquad \ge \beta(m(t))x(m(t) - \tau(m(t)))e^{-\gamma(m(t))x(m(t)-\tau(m(t)))}.
\end{aligned} \tag{2.13}$$

This together with (2.12) implies that

$$\lim_{t\to+\infty} x(m(t) - \tau(m(t))) = 0. \tag{2.14}$$

Now we select a sequence $\{t_n\}_{n=1}^{+\infty}$ such that

$$\begin{cases}
\lim_{n\to+\infty} t_n = +\infty, & \lim_{n\to+\infty} x(m(t_n)) = 0, \\
\lim_{n\to+\infty} a(m(t_n)) = a_*, & \lim_{n\to+\infty} b(m(t_n)) = b_*, \\
\lim_{n\to+\infty} \beta(m(t_n)) = \beta_*, & \lim_{n\to+\infty} \gamma(m(t_n)) = \gamma_*.
\end{cases} \tag{2.15}$$

In view of (2.13), we get

$$\begin{aligned}
&\frac{a(m(t_n))}{b(m(t_n))} \\
&\qquad \ge \beta(m(t_n))\frac{x(m(t_n) - \tau(m(t_n)))e^{-\gamma(m(t_n))x(m(t_n)-\tau(m(t_n)))}}{x(m(t_n))} \\
&\qquad \ge \beta(m(t_n))\frac{x(m(t_n) - \tau(m(t_n)))e^{-\gamma(m(t_n))x(m(t_n)-\tau(m(t_n)))}}{x(m(t_n) - \tau(m(t_n)))} \\
&\qquad = \beta(m(t_n))e^{-\gamma(m(t_n))x(m(t_n)-\tau(m(t_n)))}.
\end{aligned} \tag{2.16}$$

Letting $n \to +\infty$, (2.14)–(2.16) imply that

$$\lim_{n\to+\infty} \frac{\beta(m(t_n))b(m(t_n))}{a(m(t_n))} = \frac{\beta_* b_*}{a_*} \le 1,$$

which contradicts (2.7). Hence, (2.11) holds. This completes the proof of Theorem 2.1.

3. An example. In this section, we present an example illustrating our results of the previous sections.

EXAMPLE 3.1. Consider the following Nicholson's blowflies model with a nonlinear density-dependent mortality term:

$$(3.1)\qquad x'(t) = -\frac{(3+|\sin\sqrt{2}t|)x(t)}{(5+\frac{t}{t^2+1})+x(t)} + (1+\cos^2 t)x(t-2e^{|\arctan t|})e^{-(1+|\arctan t|)x(t-2e^{|\arctan t|})}.$$

Then $r = 2e^{\pi/2}$,

$$\inf_{t\in\mathbb{R}} \frac{a(t)\gamma(t)e}{\beta(t)} = \inf_{t\in\mathbb{R}} \frac{(3+|\sin\sqrt{2}t|)(1+|\arctan t|)e}{1+\cos^2 t} > 1,$$

and

$$\inf_{t\in\mathbb{R}} \frac{\beta(t)b(t)}{a(t)} = \inf_{t\in\mathbb{R}} \frac{(1+\cos^2 t)(5+\frac{t}{t^2+1})}{3+|\sin\sqrt{2}t|} > 1.$$

It follows that the model (3.1) satisfies all the conditions in Theorem 2.1. Hence, it is permanent.

REMARK 3.1. It is clear that the results in [1, 4, 5, 6, 7, 10] and the references therein cannot be applied to prove the permanence of (3.1). This implies that the results of this paper are new and they complement previously known results.

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References

[1] L. Berezansky, E. Braverman and L. Idels, *Nicholson's blowflies differential equations revisited: Main results and open problems*, Appl. Math. Modelling 34 (2010), 1405–1417.

[2] W. S. Gurney, S. P. Blythe and R. M. Nisbet, *Nicholson's blowflies (revisited)*, Nature 287 (1980), 17–21.

[3] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer, New York, 1993.

[4] V. L. Kocić and G. Ladas, *Oscillation and global attractivity in the discrete model of Nicholson's blowflies*, Appl. Anal. 38 (1990), 21–31.

[5] Y. Lenbury and D. V. Giang, *Nonlinear delay differential equations involving population growth*, Math. Comput. Modelling 40 (2004), 583–590.
[6] B. W. Liu, *Global stability of a class of Nicholson's blowflies model with patch structure and multiple time-varying delays*, Nonlinear Anal. Real World Appl. 11 (2010), 2557–2562.
[7] E. Liz and G. Röst, *Dichotomy results for delay differential equations with negative Schwarzian derivative*, ibid. 11 (2010), 1422–1430.
[8] A. J. Nicholson, *An outline of the dynamics of animal populations*, Austral. J. Zoology 2 (1954), 9–65.
[9] H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, Math. Surveys Monogr. 41, Amer. Math. Soc., Providence, RI, 1995.
[10] H. Zhou, W. T. Wang and H. Zhang, *Convergence for a class of non-autonomous Nicholson's blowflies model with time-varying coefficients and delays*, Nonlinear Anal. Real World Appl. 11 (2010), 3431–3436.

Bingwen Liu
Department of Mathematics
Hunan University of Arts and Science
Changde, Hunan 415000, P.R. China
E-mail: liubw007@yahoo.com.cn

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