On sectional curvature of a Riemannian manifold with semi-symmetric metric connection

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Abstract. We prove that if the sectional curvature of an \( n \)-dimensional pseudo-symmetric manifold with semi-symmetric metric connection is independent of the orientation chosen then the generator of such a manifold is gradient and also such a manifold is subprojective in the sense of Kagan.

1. Introduction. Let \((M_n, g)\) be an \( n \)-dimensional differentiable manifold of class \( C^\infty \) with the metric tensor \( g \), the Riemannian connection \( \nabla \) and a smooth linear connection \( \nabla^* \) on \( M_n \). A smooth linear connection \( \nabla^* \) on \( M_n \) is said to be semi-symmetric if its torsion tensor \( T \) satisfies the relation

\[
T(X, Y) = w(Y)X - w(X)Y
\]

where \( w \) is a smooth linear differential form and \( X \) and \( Y \) are any smooth vector fields on \( M_n \), [Y1]. The concept of a semi-symmetric connection has been studied on Kenmotsu manifolds [PD1], almost contact manifolds [DS], Sasakian manifolds [PD2] and Riemannian manifolds [D]. It is known [Y1] that if \( \nabla^* \) is a semi-symmetric metric connection then

\[
\nabla^*_X Y = \nabla_X Y + w(Y)X - g(X, Y)\rho, \tag{2}
\]

\[
g(X, \rho) = w(X), \tag{3}
\]

for any vector fields \( X \) and \( Y \). Further, it is also known [Y1] that if \( R^* \) and \( R \) denote the curvature tensors of the smooth linear connection \( \nabla^* \) and the Levi-Civita connection \( \nabla \), respectively, then

\[
R^*(X, Y)Z = R(X, Y)Z - \alpha(Y, Z)X + \alpha(X, Z)Y - g(Y, Z)AX + g(X, Z)AY \tag{4}
\]

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where $\alpha$ is a tensor field of type $(0,2)$ defined by

$$\alpha(X,Y) = (\nabla_X w)(Y) - w(X)w(Y) + \frac{1}{2} w(\rho)g(X,Y)$$

and $A$ is a tensor field of type $(1,1)$ defined by

$$g(AX,Y) = \alpha(X,Y)$$

for any vector fields $X$ and $Y$.

We shall use the following results in the next section:

In a local coordinate system, equations (4), (5) and (6) can be written as follows:

$$(7) \ \ R_{ijkh}^* = R_{ijkh} - \frac{1}{2} n(n-2) P_{ij} g_{kh} + \frac{1}{2} P_{ijkh} + \frac{1}{2} P_{jikh} + \frac{1}{2} P_{ijkh} - \alpha g_{ih}$$

where

$$(8) \ \ P_{jk} = \nabla_j w_k - w_j w_k + \frac{1}{2} g_{jk} w^h w^h, \quad P^h_k = P_{km} g^{mh}.$$  

From (7), we have (see [Y1])

$$(9) \ \ R_{ih}^* = R_{ih} - (n-2) P_{ih} - \alpha g_{ih},$$

$$(10) \ \ R^* = R - 2(n-1) \alpha,$$

where

$$(11) \ \ \alpha = g^{ih} P_{ih}.$$  

M. C. Chaki [CH] introduced a type of non-flat Riemannian manifold $(M_n, g)$ $(n \geq 2)$ whose curvature tensor $R_{hijk}$ satisfies the condition

$$(12) \ \ \nabla_l R_{hijk} = 2 \lambda_l R_{hijk} + \lambda_h R_{lijk} + \lambda_i R_{hilk} + \lambda_j R_{hijkl} + \lambda_k R_{hijl}$$

where $\lambda_l$ is a non-zero vector which is called the generator of the manifold. Such a manifold is called pseudo-symmetric and is denoted by $(PS)_n$.

A Riemannian manifold is called an Einstein manifold if its Ricci tensor is proportional to its metric.

Moreover, an $n$-dimensional manifold with a semi-symmetric metric connection is called an Einstein manifold with a semi-symmetric metric connection if the symmetric part of the Ricci tensor is proportional to the metric, i.e.,

$$(13) \ \ R^*_{(ij)} = \lambda g_{ij}$$

where $\lambda$ is a scalar function.

Now, we can state the following lemma which will be used in our subsequent work:

**Lemma.** Suppose that $S$ is a $(0,2)$ covariant tensor. If for all linearly independent vectors $X$ and $Y$,

$$(14) \ \ S_{\alpha\beta\lambda\mu} X^\alpha Y^\beta X^\lambda Y^\mu = 0,$$
then
\[ S_{\alpha\beta\lambda\mu} + S_{\lambda\mu\alpha\beta} + S_{\alpha\mu\lambda\beta} + S_{\lambda\beta\alpha\mu} = 0. \]
Here \( X^\alpha \) and \( Y^\beta \) are the contravariant components of \( X \) and \( Y \), respectively, [LR].

2. Sectional curvatures of a Riemannian manifold having a semi-symmetric metric connection. Let \( P(x^k) \) be any point of \( M_n(\nabla^*, g) \) and denote by \( X^\alpha, Y^\alpha \) the components of two linearly independent vectors \( X, Y \in T_P(M_n) \). These vectors determine a two-dimensional subspace (plane) \( \pi \) in \( T_P(M_n) \).

The scalar
\[ K^*(\pi) = \frac{R^*_{\alpha\beta\lambda\mu}X^\alpha Y^\beta X^\lambda Y^\mu}{(g^\beta\lambda g^\alpha\mu - g^\alpha\lambda g^\beta\mu)X^\alpha Y^\beta X^\lambda Y^\mu} \]
is called the sectional curvature of \( M_n(\nabla^*, g) \) at \( P \) with respect to the plane \( \pi \).

From (16), it follows that
\[ S_{\alpha\beta\lambda\mu}X^\alpha Y^\beta X^\lambda Y^\mu = 0 \]
where we have put
\[ S_{\alpha\beta\lambda\mu} = R^*_{\alpha\beta\lambda\mu} - K^*(\pi)(g^\beta\lambda g^\alpha\mu - g^\alpha\lambda g^\beta\mu). \]
Assume that at any point \( P \in M_n(\nabla^*, g) \), the sectional curvatures for all planes in \( T_P(M_n) \) are the same. A two-dimensional Riemannian manifold having semi-symmetric metric connection need not be considered, since it has only one plane at each point. Then, according to the Lemma, the condition (15) gives
\[ R^*_{\alpha\beta\lambda\mu} + R^*_{\lambda\mu\alpha\beta} + R^*_{\alpha\mu\lambda\beta} + R^*_{\lambda\beta\alpha\mu} = 2K^*(\pi)(g_{\mu\alpha}g_{\lambda\beta} + g_{\alpha\beta}g_{\mu\lambda}) \]
\[ R^*_{\lambda\beta} = \frac{2}{n-1}K^*(\pi)g_{\lambda\beta}. \]
Multiply the equation (19) by \( g^\alpha\mu \) to find
\[ \frac{R^*_{\lambda\beta} + R^*_{\beta\lambda}}{2} = (n-1)K^*(\pi)g_{\lambda\beta}. \]
This can be rewritten in the form
\[ R^*_{(\lambda\beta)} = (n-1)K^*(\pi)g_{\lambda\beta} \]
where
\[ R^*_{(\lambda\beta)} = \frac{R^*_{\lambda\beta} + R^*_{\beta\lambda}}{2}. \]
Transvecting (21) by \( g^\lambda\beta \), we get
\[ R^* = n(n-1)K^*(\pi). \]
From (9), we have
\begin{equation}
R_{[\lambda\beta]}^* = (2 - n)P_{[\lambda\beta]}.
\end{equation}

Since the sectional curvatures at \( P \in M_n(\nabla^*, g) \) are the same for all planes in \( T_P(M_n) \), by using (16), we have
\begin{equation}
R_{\alpha\beta\lambda\mu}^* = K^* (\pi) (g_{\beta\lambda}g_{\alpha\mu} - g_{\alpha\lambda}g_{\beta\mu}).
\end{equation}

Multiplying (25) by \( g^{\alpha\mu} \) and summing over \( \alpha \) and \( \mu \), we get
\begin{equation}
R_{\lambda\beta}^* = K^* (\pi) (n - 1)g_{\lambda\beta}.
\end{equation}

From (8), (21), (25) and (26), it follows that
\begin{align}
& (27) \quad R_{[\lambda\beta]}^* = 0, \\
& (28) \quad \nabla_{[\lambda} w_{\beta]} = 0.
\end{align}

(21) means that \( M_n(\nabla^*, g) \) is an Einstein manifold with a semi-symmetric metric connection. (28) implies that the 1-form \( w \) is closed.

With the help of (7), (8) and (28), we find that
\begin{equation}
R_{\alpha\beta\lambda\mu}^* + R_{\beta\lambda\alpha\mu}^* + R_{\lambda\alpha\beta\mu}^* = 0,
\end{equation}
i.e., the first Bianchi identity holds for the linear connection.

From (9) and (10) we have
\begin{equation}
P_{ij} = -\lambda_{ij} - \frac{R_{ih}^*}{n - 2} - \frac{R^* g_{ih}}{2(n - 1)(n - 2)}
\end{equation}
where
\begin{equation}
\lambda_{ij} = -\frac{1}{n - 2} R_{ij} + \frac{1}{2(n - 1)(n - 2)} R g_{ij}.
\end{equation}

From (21), (23) and (27), we have \( R_{ih}^* = R^* g_{ih}/n \). Then, by using (30), we find
\begin{equation}
P_{ij} = -\lambda_{ij} - \frac{R^* g_{ij}}{2n(n - 1)}.
\end{equation}

By the aid of the equations (7), (23) and (32), we get
\begin{equation}
R_{ijkl}^* = C_{ijkl} + K^* (\pi) (g_{ih}g_{jk} - g_{ik}g_{jh}).
\end{equation}

By using (25) and (33), we can easily see that this space is conformally flat.

In [1], by using a different method, it has been shown that if a Riemannian manifold admits a semi-symmetric metric connection with closed \( \pi \) constant curvature, then the manifold is conformally flat.
Since this manifold is conformally flat, we have

\[ R_{ijkh} = \frac{1}{(n-2)} (g_{jk}R_{ih} - g_{ik}R_{jh} + g_{ih}R_{jk} - g_{jh}R_{ik}) \]

\[ - \frac{1}{(n-1)(n-2)} R (g_{jk}g_{ih} - g_{jh}g_{ik}). \]

By using (31), the equation (34) can be rewritten as

\[ R_{ijkh} = -g_{jk}\lambda_{ih} - g_{ih}\lambda_{jk} + g_{ik}\lambda_{jh} + g_{jh}\lambda_{ik}. \]

If we multiply the equation (12) by \( g^{hk} \), we obtain

\[ 2\lambda_l R_{jk} + \lambda_j R_{lk} + \lambda_k R_{lj} = \nabla_l R_{jk}. \]

Multiplying (36) by \( g^{jk} \), we find

\[ 2\lambda_l R + 4\lambda_i g^{ih} R_{lh} = \nabla_l R. \]

By cyclic permutation of the indices \( l, j \) and \( k \) and by using the last two equations and (36), we have the relation

\[ \lambda_l R_{jk} + \lambda_j R_{lk} + \lambda_k R_{lj} = \frac{1}{4} (\nabla_l R_{jk} + \nabla_j R_{kl} + \nabla_k R_{lj}). \]

It is known [CH] that a conformally flat \((PS)_n\) \((n \geq 3)\) cannot be of zero scalar curvature and in a conformally flat \((PS)_n\), it is also known [T] that

\[ R_{ij} = \frac{R - t}{n - 1} g_{ij} + \frac{nt - R}{(n-1)\lambda_p\lambda^p} \lambda_i \lambda_j \]

where \( R \) denotes the scalar curvature and \( t \) is a scalar.

The expression (39) can be written as

\[ R_{ij} = \theta g_{ij} + \beta v_i v_j \]

where

\[ \theta = \frac{R - t}{n - 1}, \quad \beta = \frac{nt - R}{n - 1}, \quad \lambda^h R_{hk} = t \lambda_k, \quad v_i = \frac{\lambda_i}{\sqrt{\lambda_m\lambda^n}} \]

and \( v_i \) is a unit vector.

Thus, from (34) and (40), we have

\[ R_{ijkl} = b(-g_{jl}v_i v_k + g_{jk}v_i v_l - g_{ik}v_j v_l + g_{il}v_j v_k) + a (g_{dl}g_{jk} - g_{jl}g_{ik}) \]

where \( a = \frac{R - 2t}{(n-1)(n-2)} \) and \( b = \frac{nt - R}{(n-1)(n-2)}. \)

D. Smaranda [S] calls a Riemannian manifold whose curvature tensor satisfies (42) a manifold of almost constant curvature. Hence, we have the following theorem:

**Theorem 2.1.** If a \((PS)_n\) admits a semi-symmetric metric connection with constant sectional curvature then this manifold is of almost constant curvature.
For a conformally flat \((PS)_n\), the following condition holds \([T]\):

\[
\lambda^j \nabla_l R_{jk} = \lambda^j \lambda_j R_{lk} + \frac{3n-2}{n-1} t \lambda_l \lambda_k - \frac{t}{n-1} g_{lk} \lambda^j \lambda_j.
\]

Taking the covariant derivative of \((41)_3\) with respect to \(x^m\) and using equation \((43)\), we find

\[
\lambda^h \lambda_h R_{km} + \frac{3n-2}{n-1} t \lambda_m \lambda_k - \frac{t}{n-1} g_{km} \lambda^h \lambda_h = \lambda_k \nabla_m t + t \nabla_m \lambda_k - R_{hk} \nabla_m \lambda^h.
\]

From \((40)\), \((41)\) and \((44)\), we get

\[
R - t n^{-1} g_{km} \lambda^h \lambda_h + \frac{nt - R}{n-1} \lambda_k \lambda_m + \frac{(3n-2)t}{n-1} \lambda_k \lambda_m - \frac{t}{n-1} g_{km} \lambda^h \lambda_h
\]

\[
= \lambda_k \nabla_m t + t \nabla_m \lambda_k - R_{hk} \nabla_m \lambda^h.
\]

If we multiply \((45)\) by \(\lambda^k\) then we find

\[
\nabla_m t = 4t \lambda_m.
\]

With the help of \((37)\) and \((40)\), we get

\[
\nabla_l R = 2((n+2) \theta + 3 \beta) \lambda_l.
\]

From equation \((47)\), it is clear that the covariant vector \(\lambda_l\) is a gradient. Thus, we have the following theorem:

**Theorem 2.2.** If a \((PS)_n\) admits a semi-symmetric metric connection with constant sectional curvature then the covariant vector \(\lambda_l\) of this manifold is a gradient.

Now, for a conformally flat manifold \((PS)_n\), we have (see \([DG]\))

\[
v_l \nabla_k \beta - v_k \nabla_l \beta + \beta (\nabla_k v_l - \nabla_l v_k) = 0.
\]

By using \((41)_2\) and \((46)\), we obtain

\[
v_l \nabla_k \beta - v_k \nabla_l \beta = 0.
\]

By using \((48)\) and \((49)\), we get

\[
\beta = 0 \quad \text{or} \quad \nabla_k v_l - \nabla_l v_k = 0.
\]

If \(\beta = 0\) then the manifold is flat. This contradicts the hypotheses. Thus, from \((50)\),

\[
\nabla_k v_l - \nabla_l v_k = 0.
\]

It is known \([DG]\) that the covariant vector \(v_i\) of a conformally flat \((PS)_n\) is a proper concircular vector field. Hence, we have the following theorem:

**Theorem 2.3.** A \((PS)_n\) admitting a semi-symmetric metric connection with a constant sectional curvature has a proper concircular vector field.
It is known [A] that if a conformally flat manifold admits a proper concircular vector field then the manifold is a subprojective manifold in the sense of Kagan. Thus, we can state the following theorem:

**Theorem 2.4.** If a $(PS)_n$ admits a semi-symmetric metric connection with a constant sectional curvature then this manifold is subprojective.

In [Y3], K. Yano proved that for a Riemannian manifold to admit a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

$$ds^2 = (dx^1)^2 + c^q g^{*}_{\alpha \beta} dx^\alpha dx^\beta$$

where

$$g^{*}_{\alpha \beta} = g^{*}_{\alpha \beta}(x')$$

are functions of $x'$ ($\alpha, \beta, \nu = 2, 3, \ldots, n$) and $q = q(x^1) \neq \text{const}$ is a function of $x^1$ only. Since a conformally flat $(PS)_n$ admits a proper concircular vector field $v_i$, the manifold under consideration is the warped product $1 \times_{e^q} M^*$ where $(M^*, g^*)$ is an $(n-1)$-dimensional Riemannian manifold.

Since this manifold is conformally flat, from (34), the following equation is satisfied:

$$\nabla_k R_{jl} - \nabla_l R_{jk} = \frac{1}{2(n-1)} (g_{jl} \nabla_k R - g_{jk} \nabla_l R).$$

Gębarowski [G] proved that the warped product $1 \times_{e^q} M^*$ satisfies (52) if and only if $M^*$ is an Einstein manifold.

Thus, we can state the following theorem:

**Theorem 2.5.** If a $(PS)_n$ admits a semi-symmetric metric connection with a constant sectional curvature then this manifold is the warped product $1 \times_{e^q} M^*$ where $M^*$ is an Einstein manifold.

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**References**


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