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Product preserving gauge bundle functors on all principal bundle homomorphisms

by Włodzimierz M. Mikulski (Kraków)

Abstract. Let $\mathcal{PB}$ be the category of principal bundles and principal bundle homomorphisms. We describe completely the product preserving gauge bundle functors (ppgb-functors) on $\mathcal{PB}$ and their natural transformations in terms of the so-called admissible triples and their morphisms. Then we deduce that any ppgb-functor on $\mathcal{PB}$ admits a prolongation of principal connections to general ones. We also prove a "reduction" theorem for prolongations of principal connections into principal ones by means of Weil functors. We observe that there exist plenty of such prolongations. In Appendix, we classify the natural operators lifting vector-valued 1-forms (or vector-valued maps) to vector-valued 1-forms on Weil bundles.

1. Introduction. All manifolds and maps we consider in this paper are assumed to be smooth, i.e. of class $\mathcal{C}^\infty$. Manifolds are also assumed to be Hausdorff, finite-dimensional, second countable and without boundaries.

In this paper, $\mathcal{M}f$ denotes the category of manifolds, $\mathcal{M}f_m$ the category of m-dimensional manifolds and their local diffeomorphisms, $\mathcal{FM}$ the category of fibred manifolds (i.e. surjective submersions between manifolds) and fibred maps, $\mathcal{PB}$ the category of principal bundles and their principal bundle homomorphisms, and $\mathcal{PB}_m(G)$ the category of principal bundles with standard fibre being the Lie group G and m-dimensional bases and their local principal bundle isomorphisms with the identity map of G as the Lie group homomorphism. The tensor product $\otimes$ will always be over $\mathbb{R}$ (i.e. $\otimes := \otimes_{\mathbb{R}}$). Similarly, $\mathrm{Hom} := \mathrm{Hom}_{\mathbb{R}}$ and $\dim := \dim_{\mathbb{R}}$.

About 1953, A. Weil (see [W]) introduced the concept of a *near A-point* on a manifold M as an algebra homomorphism of the algebra $\mathcal{C}^\infty(M,\mathbb{R})$ of smooth functions on M into a local algebra A. Nowadays A is called a *Weil algebra* and the space $T^A(M)$ of all near A-points on M is called a

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Weil bundle. About 1985, D. Eck (see [E1]), O. O. Luciano (see [L]) and G. Kainz and P. W. Michor (see [KaMi]) proved independently that the product preserving bundle functors $F : \mathcal{M}f \to \mathcal{FM}$ (the ppb-functors F on $\mathcal{M}f$) are the Weil functors $T^A : \mathcal{M}f \to \mathcal{FM}$ for Weil algebras $A = F(\mathbb{R})$, and that the natural transformations $\eta : F \to F_1$ between ppb-functors on $\mathcal{M}f$ are in bijection with the algebra homomorphisms $\eta_{\mathbb{R}} : F(\mathbb{R}) \to F_1(\mathbb{R})$ between the corresponding Weil algebras. Moreover, in [KaMi], it was observed that if $F = T^A$ and $F_1 = T^{A_1}$, then $F \circ F_1 = T^{A \otimes A_1}$. Consequently, the exchange algebra isomorphism $A \otimes A_1 = A_1 \otimes A$ defines the natural isomorphism $F \circ F_1 = F_1 \circ F$. In particular, if $F_1 = T$ is the tangent functor, there is the flow isomorphism $F \circ T = T \circ F$. A detailed presentation of all the above results can also be found in the fundamental monograph [KMS].

Ppb-functors F on $\mathcal{M}f$ play an important role in differential geometry. For example, a ppb-functor $F = T^A : \mathcal{M}f \to \mathcal{FM}$ can be applied in prolongations of connections. Indeed, if $\Gamma : Y \times_M T(M) \to T(Y)$ is a general connection on a fibred manifold $p : Y \to M$, then $\mathcal{F}(\Gamma) := F(\Gamma) : F(Y) \times_{F(M)} T(F(M)) \to T(F(Y))$ (under the flow identifications $F(T(M)) = T(F(M))$ and $F(T(Y)) = T(F(Y))$) is a general connection on $F(p) : F(Y) \to F(M)$. The above connection $\mathcal{F}(\Gamma)$ was constructed by J. Slovák (see [S]).

For ppb-functors F on $\mathcal{M}f_m$ one can also deduce the following results. If G is a Lie group with the multiplication map $\mu_G : G \times G \to G$ and the unity $e_G : \mathrm{pt} \to G$, where $\mathrm{pt} = \{\emptyset\}$ is the trivial Lie group, then $F(G)$ is a Lie group with the multiplication map $\mu_{F(G)} := F(\mu_G) : F(G) \times F(G) = F(G \times G) \to F(G)$ and the unity $e_{F(G)} := F(e_G) : F(\mathrm{pt}) = \mathrm{pt} \to F(G)$, and if $\nu : G \to G_1$ is a Lie group homomorphism, then so is $F(\nu) : F(G) \to F(G_1)$. For the Lie algebra of G we have $F(\mathcal{L}(G)) = \mathcal{L}(F(G)) = \mathcal{L}(G) \otimes A$. For the exponential map we have $\mathrm{Exp}_{F(G)} = F(\mathrm{Exp}_G)$.

If $p : P \to M$ is a principal bundle with the Lie group G and the right action $r : P \times G \to P$ then $F(p) : F(P) \to F(M)$ is a principal bundle with the Lie group $F(G)$ and the right action $F(r) : F(P) \times F(G) = F(P \times G) \to F(P)$, and if $f : P \to P_1$ is a principal bundle homomorphism covering $\underline{f} : M \to M_1$ and with the Lie group homomorphism $\varphi_f : G \to G_1$ then $F(f) : F(P) \to F(P_1)$ is a principal bundle homomorphism covering $F(\underline{f}) : F(M) \to F(M_1)$ and with the Lie group homomorphism $F(\varphi_f) : F(G) \to F(G_1)$. If Γ is a principal (i.e. general right invariant) connection on a principal bundle $p : P \to M$, then $\mathcal{F}(\Gamma)$ (as above) is a principal connection on $F(p) : F(P) \to F(M)$. If $\eta : F \to F_1$ is a natural transformation, then $\eta_G : F(G) \to F_1(G)$ is a Lie group homomorphism and $\eta_P : F(P) \to F_1(P)$ is a principal bundle homomorphism covering $\eta_M : F(M) \to F_1(M)$ with the Lie group homomorphism $\eta_G : F(G) \to F_1(G)$. A more detailed presentation of the above results can be found in [K1]. (In the special case of $F = T^{p,r} =$

the bundle functor of p^r-velocities, some of the above facts can also be found in [GS].)

For classical linear connections $\Gamma = \nabla$ on M (i.e. principal connections on the principal bundle $P^1(M)$ of linear frames of M), $\mathcal{F}(\nabla)$ coincides with (or more precisely, is the reduction to $T^A(P^1(M)) \subset P^1(T^A(M))$ of) the complete lift of ∇ to $T^A(M)$ in the sense of A. Morimoto [Mo]. $\mathcal{F}(\nabla)$ was also investigated in [GMP]. ([GRS] investigates a so-called horizontal lifting of classical linear connections on M to classical linear connections on $T^A(M)$ by means of an additional rth order linear connection on M, i.e. a principal connection on the rth order frame bundle $P^r(M)$ of M.) In [D2], all affine $\mathcal{M}f_m$-natural operators B lifting torsion-free classical linear connections ∇ on m-manifolds M to torsion-free classical linear connections $B(\nabla)$ on $T^A(M)$ were described.

The present paper is dedicated to the study of gauge bundle functors on principal bundles. In Section 2, we start from the definition and general properties of gauge bundle functors (gb-functors) $F : \mathcal{K} \to \mathcal{FM}$ on a subcategory $\mathcal{K} \subset \mathcal{PB}$. The definition of gb-functors covers all standard definitions of natural bundle functors, like natural bundles by A. Nijenhuis [Ni] (on the category $\mathcal{M}f_m$), prolongation functors by I. Kolář [K2] (on the category $\mathcal{M}f$) and gauge-natural bundles by D. Eck [E2] (on the category $\mathcal{PB}_m(G)$). We try to compare general properties of gb-functors on $\mathcal{K}$ with the ones of standard natural bundle functors mentioned above. Then we restrict our investigations to product preserving gauge bundle functors (ppgb-functors) $F : \mathcal{PB} \to \mathcal{FM}$ on the whole category $\mathcal{PB}$ only. A simple example of such a functor is the extended Weil functor $T^A : \mathcal{PB} \to \mathcal{FM}$ for a Weil algebra A sending any principal bundle $p : P \to M$ to $T^A(P) \to M$ (the composition of $T^A(p) : T^A(P) \to T^A(M)$ with the Weil bundle projection $T^A(M) \to M$) and any principal bundle homomorphism $f : P \to P_1$ covering $\underline{f} : M \to M_1$ to a fibred map $T^A(f) : T^A(P) \to T^A(P_1)$ covering $\underline{f} : M \to M_1$. We present many examples of ppgb-functors on $\mathcal{PB}$. We show essential differences between ppgb-functors on $\mathcal{PB}$ and ppb-functors on $\mathcal{M}f$. In particular, we exhibit a ppgb-functor $F : \mathcal{PB} \to \mathcal{FM}$ such that there is no exchanging isomorphism $F \circ T \cong T \circ F$.

In Sections 3–8, we describe all ppgb-functors on $\mathcal{PB}$ in terms of so called admissible triples and classify all natural transformations $F \to H$ between ppgb-functors $F, H : \mathcal{PB} \to \mathcal{FM}$ by means of morphisms between admissible triples. Moreover, for ppgb-functors $F, H : \mathcal{PB} \to \mathcal{FM}$, a Lie group G and a natural number $m \geq 2$, we classify all natural transformations $F_{|\mathcal{PB}_m(G)} \to H_{|\mathcal{PB}_m(G)}$ by means of so-called admissible pairs. In particular, we find explicitly all natural transformations $T^A \to T^B$ between the extended Weil functors $T^A, T^B : \mathcal{PB} \to \mathcal{FM}$ for Weil algebras A and B

and all natural transformations $T^A_{|\mathcal{PB}_m(G)} \to T^B_{|\mathcal{PB}_m(G)}$ for $m \in \mathbb{N}$, a Lie group G and Weil algebras A and B. As an application of the description of ppgb-functors on $\mathcal{PB}$, we present (in Section 9) a canonical construction of a general connection $\mathcal{F}(\Gamma)$ on $F(p) : F(P) \to F(M)$ from a principal connection Γ on $p : P \to M$ for an arbitrary ppgb-functor $F : \mathcal{PB} \to \mathcal{FM}$.

In Section 10, we prove a "reduction" theorem for $\mathcal{PB}_m(G)$-gauge-natural operators lifting principal connections to Weil bundles. In Section 11, using some results from Appendix (which consists of Sections 12 and 13), we prove that for commutative Lie groups G and sufficiently large m, there are plenty of affine $\mathcal{PB}_m(G)$-gauge-natural operators lifting principal connections to Weil bundles (we give a lower bound on the dimension of the affine space of such affine $\mathcal{PB}_m(G)$-gauge-natural operators). In Appendix, we present a full description of all $\mathcal{M}f_m$-natural operators lifting vector-valued 1-forms (or vector-valued maps) to vector-valued 1-forms on Weil bundles.

Full descriptions of product preserving (gauge) bundle functors on some other categories over manifolds can also be found in [KuMi], [Ku], [Mi3], [Mi4], [Mi5], [MT], [Sh] (the list is not complete); some other reduction theorems for gauge-natural operators on connections can also be found in [DM], [J1] (the list is not complete). In [JV], the reduction theorems were applied to obtain complete descriptions of gauge-natural operators lifting principal and classical connections to principal connections on higher order principal prolongations of principal bundles. In Section 11 of the present paper, the "reduction" theorem is used to obtain the estimate mentioned above.

2. Product preserving gauge bundle functors on principal bundles: definitions, simple properties, examples. Let $\mathcal{K} \subset \mathcal{PB}$ be a subcategory such that for any $\mathcal{K}$-object $P \to M$ and any open subset $U \subset M$ we have $P_{|U} \in \mathrm{Obj}(\mathcal{K})$ and the inclusion $\mathrm{i}_U : P_{|U} \to P$ is a $\mathcal{K}$-morphism.

DEFINITION 2.1. A *gauge bundle functor* (*gb-functor* for short) on $\mathcal{K}$ as above is a covariant functor $F : \mathcal{K} \to \mathcal{FM}$ satisfying the following conditions:

(i) *Base preservation.* For any $\mathcal{K}$-object $P = (p : P \to M)$ with the base M the induced $\mathcal{FM}$-object $F(P) = (\pi_P : F(P) \to M)$ is a fibred manifold over the same base M. For any $\mathcal{K}$-morphism $f : P_1 \to P_2$ covering $\underline{f} : M_1 \to M_2$ the induced $\mathcal{FM}$-map $F(f) : F(P_1) \to F(P_2)$ is also over $\underline{f}$.

(ii) *Locality property.* For any $\mathcal{K}$-object $p : P \to M$ and any open subset $U \subset M$ the $\mathcal{FM}$-map $F(\mathrm{i}_U) : F(P_{|U}) \to F(P)$ (induced by the inclusion $\mathrm{i}_U : P_{|U} \to P$) is a diffeomorphism onto $\pi_P^{-1}(U)$.

(iii) *Regularity property.* F transforms smoothly parametrized families of $\mathcal{K}$-morphisms into smoothly parametrized families of $\mathcal{FM}$-morphisms. More precisely, if $f : \mathbb{R} \times P \to Q$ is a smooth map such that for any $t \in \mathbb{R}$ the restricted map $f_t : P \to Q$, $f_t(p) = f(t,p)$, is a $\mathcal{K}$-map then the map $Ff : \mathbb{R} \times F(P) \to F(Q)$, $Ff(t,v) = F(f_t)(v)$, is smooth.

DEFINITION 2.2. Let F and H be gb-functors on $\mathcal{K}$. A *natural transformation* $\eta : F \to H$ is a family of maps $\eta_P : F(P) \to H(P)$ for all $\mathcal{K}$-objects P such that $H(f) \circ \eta_P = \eta_Q \circ F(f)$ for any $\mathcal{K}$-map $f : P \to Q$.

If $\mathcal{K} = \mathcal{M}f_m$, we obtain the classical concept of natural bundles in the sense of A. Nijenhuis (see [Ni]). If $\mathcal{K} = \mathcal{PB}_m(G)$ we obtain the classical concept of gauge-natural bundles on $\mathcal{PB}_m(G)$ in the sense of D. Eck (see [E2]). If $\mathcal{K} = \mathcal{M}f$ we obtain the classical concept of prolongation functors on $\mathcal{M}f$ in the sense of I. Kolář (see [K2]). Therefore the above concept of gb-functors is sufficiently general. Of course, one can consider the even more general concept of gauge bundle functors over local categories on manifolds (see [KMS, Remark 51.4]).

In the situation of $\mathcal{K} = \mathcal{M}f_m$ the regularity condition (iii) in Definition 2.1 is a consequence of conditions (i) and (ii) in that definition.This is a very deep result by D. B. A. Epstein and W. P. Thurston [ET]. Using it one can show that if $\mathcal{K} = \mathcal{PB}_m(G)$ or $\mathcal{K} = \mathcal{M}f$ then the regularity condition (iii) in Definition 2.1 also is a consequence of conditions (i) and (ii) there (see [KMS]). A similar regularity result for infinite-dimensional (or even topological Hausdorff) natural bundles $F(M)$ over m-manifolds M was proved in [Mi6]. In general, the regularity condition (iii) in Definition 2.1 cannot be omitted (even for product preserving functors $F : \mathcal{PB} \to \mathcal{FM}$): see Example 2.20.

The locality condition (ii) in Definition 2.1 cannot be omitted either (even for product preserving functors $F : \mathcal{PB} \to \mathcal{FM}$). For example, the product preserving functor $\mathcal{PB} \to \mathcal{FM}$ sending any principal bundle $P \to M$ with the Lie group G to its Lie groupoid $(P \times P)/G$ (considered as the fibre manifold over M with respect to the source projection) and any $\mathcal{PB}$-map $f : P \to Q$ covering $\underline{f} : M \to N$ with the Lie group homomorphism $\nu : G \to H$ to the induced $\mathcal{FM}$-map $\tilde{f} : (P \times P)/G \to (Q \times Q)/H$ covering $\underline{f} : M \to N$ ($\tilde{f}([p_1, p_2]_G) = [f(p_1), f(p_2)]_H$, $p_1, p_2 \in P$) satisfies conditions (i) and (iii) in Definition 2.1 and it does not satisfy condition (ii) in that definition. The theory of Lie groupoids and Lie algebroids can be found in [Ma]. From the locality property (ii) in Definition 2.1, we get the following lemma.

LEMMA 2.3. *Let $F, H : \mathcal{K} \to \mathcal{FM}$ be gb-functors. If $\eta : F \to H$ is a natural transformation, then η_P covers the identity map of M for any $\mathcal{K}$-object $P \to M$.*

Proof. Let $P \to M$ be a $\mathcal{K}$-object. Suppose $v \in F_x(P)$ and $\eta_P(v) \in H_y(P)$, $x \neq y$, $x, y \in M$. Let U be an open neighbourhood of x such that

$y \notin U$. Let $\mathrm{i}_U : P_{|U} \to P$ be the inclusion. Then (by the definition of natural transformations) $H(\mathrm{i}_U) \circ \eta_{P_{|U}} = \eta_P \circ F(\mathrm{i}_U)$. By the locality condition (ii) of Definition 2.1, there exists $\tilde{v} \in F(P_{|U})$ such that $v = F(\mathrm{i}_U)(\tilde{v})$. Then $\eta_P(v) = \eta_P \circ F(\mathrm{i}_U)(\tilde{v}) = H(\mathrm{i}_U) \circ \eta_{P_{|U}}(\tilde{v}) \in H(P)_{|U}$. Contradiction. ∎

We see that Lemma 2.3 essentially generalizes [KMS, Lemma 14.11]. Indeed, the proof of [KMS, Lemma 14.11] works if there are sufficiently many $\mathcal{K}$-morphisms (if $\mathcal{K}$ is so-called Whitney-extendable). Our proof of Lemma 2.3 works for all $\mathcal{K}$ as above (even if the inclusion maps are $\mathcal{K}$-morphisms only).

The locality property (ii) in Definition 2.1 ensures that if $f, g : P \to Q$ are $\mathcal{K}$-morphisms between $P = (p : P \to M)$ and Q such that $f_{|p^{-1}(U)} = g_{|p^{-1}(U)}$ for some open subset $U \subset M$ then $F(f)_{|\pi_P^{-1}(U)} = F(g)_{|\pi_P^{-1}(U)}$. Indeed, the assumption means that $f \circ \mathrm{i}_U = g \circ \mathrm{i}_U$. Then $F(f) \circ F(\mathrm{i}_U) = F(g) \circ F(\mathrm{i}_U)$.

Definition 2.4. Let $f_1, f_2 : P \to Q$ be principal bundle morphisms covering $\underline{f}_1, \underline{f}_2 : M \to N$ and with Lie group homomorphisms $\nu_1, \nu_2 : G \to H$ and let $x \in M$. We say that $\mathrm{j}^r_x(f_1) = \mathrm{j}^r_x(f_2)$ if the following equivalent conditions are satisfied:

(i) $\mathrm{j}^r_p(f_1) = \mathrm{j}^r_p(f_2)$ for all $p \in P_x =$ the fibre of P over x.
(ii) $\mathrm{j}^r_p(f_1) = \mathrm{j}^r_p(f_2)$ for some $p \in P_x$ and $\nu_1 = \nu_2$.

Just as the order of gauge-natural bundles (see [KMS]), one can define the order of gb-functors.

Definition 2.5. We say that a gb-functor $F : \mathcal{K} \to \mathcal{FM}$ is of *order* r if the following condition is satisfied:

- For any $\mathcal{K}$-morphisms $f_1, f_2 : P \to Q$ between $\mathcal{K}$-objects $P \to M$ and Q and any $x \in M$, from $\mathrm{j}^r_x(f_1) = \mathrm{j}^r_x(f_2)$ it follows $F(f_1)_{|F_x(P)} = F(f_2)_{|F_x(P)}$.

If $F : \mathcal{M}f_m \to \mathcal{FM}$ is a natural bundle, then F is of finite order $\mathrm{ord}(F) \le 2^f + 1$, where $f = \dim(S_F)$ is the dimension of the so-called standard fibre $S_F = F_0(\mathbb{R}^m)$ of F. This nice result was proved by R. Palais and C.-L. Terng in [PT]. In [ET], D. B. A Epstein and W. P. Thurston proved that $\mathrm{ord}(F) \le 2f + 1$ and that this estimate is sharp for $m = 1$. In [Z], A. Zajtz showed that if $m \ge 2$, then $\mathrm{ord}(F) \le \max(f/(m-1), f/m+1)$ and that this estimate is sharp. If $F : \mathcal{PB}_m(G) \to \mathcal{FM}$ is a gauge-natural bundle, then F is also of finite order. This fact was proved by D. Eck in [E2] (see also [KMS, Theorem 51.7]). On the other hand, there are gauge bundle functors $F : \mathcal{PB} \to \mathcal{FM}$ of strictly infinite order. For example, let $H : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor of strictly infinite order (see [Mi1]). Then the gb-functor $F : \mathcal{PB} \to \mathcal{FM}$ defined by $F(P) = H(M)$ for any $\mathcal{PB}$-object $P \to M$ and

$F(f) = H(\underline{f}) : H(M) \to H(N)$ for any $\mathcal{PB}$-morphism $f : P \to Q$ with the underlying map $\underline{f} : M \to N$ is of strictly infinite order.

In all the above-mentioned standard theories of natural bundle functors one of the most important properties of these functors is that they are in one-to-one correspondence with standard fibres. In the cases of fixed dimension of base manifolds (natural bundles on $\mathcal{M}f_m$ and gauge-natural bundles on $\mathcal{PB}_m(G)$) standard fibres are uniquely determined, they are manifolds with a left action of a certain Lie group, and natural bundles are fibred manifolds associated with a principal bundle (see the papers by D. Krupka [Kru] for the category $\mathcal{M}f_m$ and D. Eck [E2] for the category $\mathcal{PB}_m(G)$). In the case of prolongation functor of finite order r on the category $\mathcal{M}f$ of all manifolds the standard fibres form a sequence $S = (S_0, \ldots, S_n, \ldots)$ for any dimension with the action of the category L^r; see for instance [KMS] or the original paper by J. Janyška [J2]. What is the corresponding situation in the case of general definition? Even in the case of gb-functors on the whole $\mathcal{PB}$ an answer to this question is unknown.

Remark 2.6. The reviewer of the present paper supposes that in the case of gb-functors on the whole $\mathcal{PB}$ the sequence of standard fibres will be $S = (S_0^{\mathcal{G}r}, \ldots, S_n^{\mathcal{G}r}, \ldots)$ and he supposes there is an action of $L^r \times \mathcal{G}r$ on S. But (in the author's opinion) to realize this idea we need to introduce canonical smooth manifold structures on $\mathrm{Hom}(G_1, G_2)$ for any Lie groups G_1, G_2 and the author does not know if it is possible.

In the present paper we give (among other things) an answer to the general question in the case of product preserving gb-functors on the whole $\mathcal{PB}$.

To introduce product preserving gb-functors we assume additionally that $\mathcal{K}$ is closed with respect to taking products (i.e. if P_1 and P_2 are $\mathcal{K}$-objects then $P_1 \times P_2$ is a $\mathcal{K}$-object) and that any $\mathcal{PB}$-morphism between $\mathcal{K}$-objects is a $\mathcal{K}$-morphism and that any $\mathcal{PB}$-object $\mathcal{PB}$-isomorphic to a $\mathcal{K}$-object is a $\mathcal{K}$-object. Then it makes sense to introduce the following definition, quite similar to the one (see [KMS]) of ppb-functors on $\mathcal{M}f$.

Definition 2.7. A gb-functor $F : \mathcal{K} \to \mathcal{FM}$ is a *product preserving gauge bundle functor* (ppgb-functor) if it has the following property:

(i) *Product preserving property.* For any $\mathcal{K}$-objects P_1 and P_2, the map $(F(\mathrm{pr}_1), F(\mathrm{pr}_2)) : F(P_1 \times P_2) \to F(P_1) \times F(P_2)$ is an $\mathcal{FM}$-isomorphism, where $\mathrm{pr}_i : P_1 \times P_2 \to P_i$ $(i = 1, 2)$ are the usual projections.

For a ppgb-functor $F : \mathcal{K} \to \mathcal{FM}$ and $\mathcal{K}$-objects P_1 and P_2 we will always identify $F(P_1 \times P_2)$ with $F(P_1) \times F(P_2)$ by the $\mathcal{FM}$-isomorphism from the product preserving property (i) in Definition 2.7. So, if $F : \mathcal{K} \to \mathcal{FM}$ is a ppgb-functor then $F(P_1 \times P_2) = F(P_1) \times F(P_2)$ for any $\mathcal{K}$-objects P_1 and P_2

and (just as for ppb-functors on $\mathcal{M}f$)

$$F(f_1 \times f_2) = F(f_1) \times F(f_2) : F(P_1) \times F(P_2) \to F(Q_1) \times F(Q_2)$$

for any $\mathcal{K}$-morphisms $f_i : P_i \to Q_i$ for $i = 1, 2$. If $F, H : \mathcal{K} \to \mathcal{FM}$ are ppgb-functors and $\eta : F \to H$ is a natural transformation, then

$$\eta_{P_1 \times P_2} = \eta_{P_1} \times \eta_{P_2} : F(P_1) \times F(P_2) \to H(P_1) \times H(P_2)$$

for any $\mathcal{K}$-objects P_1 and P_2.

From now on we consider ppgb-functors which are defined on the whole category $\mathcal{PB}$. However some of the results obtained can be directly generalized to ppgb-functors $F : \mathcal{K} \to \mathcal{FM}$ for some "special" subcategories $\mathcal{K} \subset \mathcal{PB}$ instead of $\mathcal{PB}$ (see Remark 5.6).

We have the following examples of ppgb-functors $F : \mathcal{PB} \to \mathcal{FM}$ of finite order. (In Proposition 6.1 we observe that any ppgb-functor $F : \mathcal{PB} \to \mathcal{FM}$ is of finite order.)

Example 2.8. The forgetting functor $I : \mathcal{PB} \to \mathcal{FM}$ sending any $\mathcal{PB}$-object $P \to M$ to the $\mathcal{FM}$-object $P \to M$ and any $\mathcal{PB}$-morphism $f : P \to Q$ to the $\mathcal{FM}$-map $f : P \to Q$ is a ppgb-functor.

Example 2.9. The group functor $F^{\mathrm{gr}} : \mathcal{PB} \to \mathcal{FM}$ sending any $\mathcal{PB}$-object $P \to M$ with the group G to the trivial $\mathcal{FM}$-object $M \times G$ over M and any $\mathcal{PB}$-morphism $f : P \to Q$ covering $\underline{f} : M \to N$ with the Lie group homomorphism $\nu : G \to H$ to the $\mathcal{FM}$-map $\underline{f} \times \nu : M \times G \to N \times H$ is a ppgb-functor. We see that $F^{\mathrm{gr}} : \mathcal{PB} \to \mathcal{PB}$.

Example 2.10. The extended tangent functor $T : \mathcal{PB} \to \mathcal{FM}$ sending any $\mathcal{PB}$-object $p : P \to M$ to the $\mathcal{FM}$-object $T(P) \to M$ (the composition of $T(p) : T(P) \to T(M)$ with the tangent bundle projection $T(M) \to M$) and any $\mathcal{PB}$-morphism $f : P \to P_1$ covering $\underline{f} : M \to M_1$ to the induced $\mathcal{FM}$-map $T(f) : T(P) \to T(P_1)$ (the tangent map of f) covering $\underline{f} : M \to M_1$ is a ppgb-functor.

Example 2.11. The Lie algebroid functor $L : \mathcal{PB} \to \mathcal{FM}$ sending any $\mathcal{PB}$-object $P \to M$ with the Lie group G to its Lie algebroid $L(P) = T(P)/G$ (considered as a fibre manifold over M) and any $\mathcal{PB}$-map $f : P \to P_1$ with the Lie group homomorphism $\nu : G \to G_1$ to $L(f) : L(P) \to L(P_1)$ given by $L(f)([v]_G) = [T(f)(v)]_{G_1}$ is a ppgb-functor.

Example 2.12. Let A be a Weil algebra. Applying the Weil functor $T^A : \mathcal{M}f \to \mathcal{FM}$ we get the extended Weil functor $T^A : \mathcal{PB} \to \mathcal{FM}$ sending any principal bundle $p : P \to M$ with the Lie group G to the fibred manifold $T^A(P) \to M$ (the composition of $T^A(p) : T^A(P) \to T^A(M)$ with the Weil bundle projection $T^A(M) \to M$) and sending any principal bundle homomorphism $f : P \to P_1$ covering $\underline{f} : M \to M_1$ with the Lie group homomorphism $\nu : G \to G_1$ to the induced $\mathcal{FM}$-map $T^A(f) : T^A(P) \to$

$T^A(P_1)$ (the usual A-prolongation of f) covering $\underline{f}$. Clearly, $T^A : \mathcal{PB} \to \mathcal{FM}$ is a ppgb-functor. In particular, if $A = \mathbb{D}$ is the Weil algebra of dual numbers we obtain the extended tangent functor $T : \mathcal{PB} \to \mathcal{FM}$. If $A = \mathbb{R}$, we obtain the forgetting functor $I : \mathcal{PB} \to \mathcal{FM}$.

EXAMPLE 2.13. Let A be a Weil algebra. We have a functor $L^A : \mathcal{PB} \to \mathcal{FM}$ sending any $\mathcal{PB}$-object $P \to M$ with the Lie group G to the factor bundle $L^A(P) = T^A(P)/G$ over M and any principal bundle homomorphism $f : P \to P_1$ with the Lie group homomorphism $\nu : G \to G_1$ to $L^A(f) : T^A(P)/G \to T^A(P_1)/G_1$ given by $L^A(f)([v]_G) = [T^A(f)(v)]_{G_1}$ for any $v \in T^A(P)$. Here (by definition) $[v]_G = [w]_G$ iff $v = T^A(r^g)(w) \in T^AP$ for some $g \in G$, where $r^g : P \to P$ is the right translation by $g \in G$. Clearly, $L^A : \mathcal{PB} \to \mathcal{FM}$ is a ppgb-functor. In particular, if $A = \mathbb{D}$ we obtain the Lie algebroid functor $L : \mathcal{PB} \to \mathcal{FM}$.

EXAMPLE 2.14. Let $\mu : A \to B$ be an algebra homomorphism between Weil algebras (considered as the corresponding natural transformation $\tilde{\mu} : T^A \to T^B$) and let $T^\mu : \mathcal{FM} \to \mathcal{FM}$ be the corresponding product preserving bundle functor (see [Mi3]). Then by "restriction" we get a functor $T^\mu : \mathcal{PB} \to \mathcal{FM}$ sending any principal bundle (and then fibred manifold) $p : P \to M$ to the fibred manifold $T^\mu(P) \to M$ and any principal bundle homomorphism (and then fibred map) $f : P \to P_1$ to the fibred map $T^\mu(f) : T^\mu(P) \to T^\mu(P_1)$. More explicitly, $T^\mu : \mathcal{PB} \to \mathcal{FM}$ transforms any $\mathcal{PB}$-object $p : P \to M$ into

$$T^\mu(P) = \{(v, w) \in T^A(M) \times T^B(P) \mid \tilde{\mu}_M(v) = T^B(p)(w)\}$$

over M and any $\mathcal{PB}$-map $f : P \to P_1$ covering $\underline{f} : M \to M_1$ into the restriction of $T^A(\underline{f}) \times T^B(f)$. Clearly, $T^\mu : \mathcal{PB} \to \mathcal{FM}$ is a ppgb-functor. In particular, if $\mu = \mathrm{id}_A : A \to A$, we obtain the extended Weil functor $T^A : \mathcal{PB} \to \mathcal{FM}$. If $\mu = \kappa : \mathbb{R} \to B$, we obtain the B-vertical functor $V^B : \mathcal{PB} \to \mathcal{FM}$ such that

$$V^B(P) = \bigcup_{x \in M} T^B(P_x) \quad \text{and} \quad V^B(f) = \bigcup_{x \in M} T^B(f_x) : V^B(P) \to V^B(P_1)$$

for any $\mathcal{PB}$ object $P \to M$ and any $\mathcal{PB}$-map $f : P \to P_1$. If $\mu : \mathbb{R} \to \mathbb{D}$ we obtain the (classical) vertical functor $V : \mathcal{PB} \to \mathcal{FM}$.

EXAMPLE 2.15. Let (K, α) be a pair consisting of a regular (i.e. transforming smoothly parametrized families of Lie group homomorphisms into smoothly parametrized families of maps) product preserving functor $K : \mathcal{G}r \to \mathcal{M}f$ and a $\mathcal{G}r$-invariant family α of actions $\alpha_G : G \times K(G) \to K(G)$ for any Lie group G (the invariance of α means that for any Lie group homomorphism $\nu : G \to G_1$ the map $K(\nu) : K(G) \to K(G_1)$ is (α_G, α_{G_1})-invariant over ν). For example, let (K, α) be one of the following pairs:

(i) The pair $(\mathcal{L}, \mathrm{Ad})$ consisting of the Lie algebra functor $\mathcal{L} : \mathcal{G}r \to \mathcal{M}f$ sending any Lie group G to the Lie algebra $\mathcal{L}(G)$ of G and any Lie group homomorphism $\nu : G \to G_1$ to the induced Lie algebra map $\mathcal{L}(\nu) : \mathcal{L}(G) \to \mathcal{L}(G_1)$ and the family $\mathrm{Ad} = \{\mathrm{Ad}_G\}_{G \in \mathrm{Obj}(\mathcal{G}r)}$, where Ad_G is the adjoint action of G on $\mathcal{L}(G)$ for any Lie group G.

(ii) The pair $(\mathrm{Id}, \mathrm{Ad})$ consisting of the forgetting functor $\mathrm{Id} : \mathcal{G}r \to \mathcal{M}f$ sending any Lie group G to G and any Lie group homomorphism $\nu : G \to G_1$ to $\nu : G \to G_1$ and the family $\mathrm{Ad} = \{\mathrm{Ad}_G\}_{G \in \mathrm{Obj}(\mathcal{G}r)}$, where Ad_G is the adjoint action of G on G for any Lie group G.

(iii) The pair $(\mathrm{Id}^o, \mathrm{Ad}^o)$ consisting of the connected component functor $\mathrm{Id}^o : \mathcal{G}r \to \mathcal{M}f$ sending any Lie group G to the connected component $G^o \subset G$ of the unity of G and any Lie group homomorphism $\nu : G \to G_1$ to the restriction $\nu_{|G^o} : G^o \to G_1^o$ of ν, and the family $\mathrm{Ad}^o = \{\mathrm{Ad}^o_G\}_{G \in \mathrm{Obj}(\mathcal{G}r)}$, where Ad^o_G is the action of G on G^o given by the restriction of the adjoint action Ad_G for any Lie group G.

(iv) The pair $(\mathrm{Ab}, \mathrm{triv})$ consisting of the abelianization functor $\mathrm{Ab} : \mathcal{G}r \to \mathcal{M}f$ sending any Lie group G to the commutative Lie group $G/\overline{[G,G]}$ (where $\overline{[G,G]}$ is the closure of the algebraic commutant of G) and any Lie group homomorphism $\nu : G \to G_1$ to the quotient $[\nu] : G/\overline{[G,G]} \to G_1/\overline{[G_1,G_1]}$ of ν, and the family $\mathrm{triv} = \{\mathrm{triv}_G\}_{G \in \mathrm{Obj}(\mathcal{G}r)}$, where triv_G is the trivial action of G on $G/\overline{[G,G]}$ for any Lie group G.

Then we can define a functor $F^{K,\alpha} : \mathcal{PB} \to \mathcal{FM}$ by

$$F^{K,\alpha}(P) = P[K(G), \alpha_G] \quad \text{and} \quad F^{K,\alpha}(f) = f[K(\nu)]$$

for any principal bundle P with the Lie group G and any principal bundle homomorphism $f : P \to P_1$ with the Lie group homomorphism $\nu : G \to G_1$, where $P[K(G), \alpha_G]$ is the associated (with P) bundle with the standard fibre $K(G)$ and the action α_G and where $f([K(\nu)])$ is well-defined by $f([K(\nu)])([p,v]) = [f(p), K(\nu)(v)]$ for any $[p,v] \in P[K(G), \alpha_G]$. Clearly, $F^{K,\alpha} : \mathcal{PB} \to \mathcal{FM}$ is a ppgb-functor.

Composing the ppgb-functors on $\mathcal{PB}$ from previous examples with the Weil functors on manifolds we can obtain new ppgb-functors on $\mathcal{PB}$. Indeed, we have the following example.

EXAMPLE 2.16. Let $T^A : \mathcal{M}f \to \mathcal{FM}$ be the Weil functor corresponding to a Weil algebra A and $F : \mathcal{PB} \to \mathcal{FM}$ be a ppgb-functor.

(i) The composition $F \circ T^A : \mathcal{PB} \to \mathcal{FM}$ is defined as follows. Given a principal bundle $p : P \to M$, we have the principal bundle $T^A(P) := (T^A(p) : T^A(P) \to T^A(M))$. Applying $F : \mathcal{PB} \to \mathcal{FM}$ to $T^A(P)$, we obtain the fibred manifold $F(T^A(P)) \to T^A(M)$. Composing this projection with the Weil bundle projection $T^A(M) \to M$ we obtain the fibred manifold $(F \circ T^A)(P)$ over M. Given a principal bundle map $f : P \to P_1$ covering $\underline{f} : M \to M_1$, $T^A(f) : T^A(P) \to T^A(P_1)$ is a principal bundle map covering $T^A(\underline{f}) : T^A(M) \to T^A(M_1)$. Applying F, we obtain the fibred map $F(T^A(f)) : F(T^A(P)) \to F(T^A(P_1))$ covering $T^A(\underline{f}) : T^A(M) \to T^A(M_1)$, which can be considered as the fibred map $(F \circ T^A)(f) : (F \circ T^A)(P) \to (F \circ T^A)(P_1)$ covering $\underline{f}$.

(ii) The composition $T^A \circ F : \mathcal{PB} \to \mathcal{FM}$ is defined as follows. Applying F to $p : P \to M$ we obtain the fibred manifold $F(P) \to M$. Then applying T^A, we obtain the fibred manifold $T^A(F(P)) \to T^A(M)$. Then composing it with the Weil bundle projection $T^A(M) \to M$, we obtain the fibred manifold $(T^A \circ F)(P)$ over M. Applying F to $f : P \to P_1$, we obtain the fibred map $F(f) : F(P) \to F(P_1)$ covering $\underline{f}$. Applying T^A, we produce the fibred map $T^A(F(f)) : T^A(F(P)) \to T^A(F(P_1))$ covering $T^A(\underline{f}) : T^A(M) \to T^A(M_1)$, which can be considered as the fibred map $(T^A \circ F)(f) : (T^A \circ F)(P) \to (T^A \circ F)(P_1)$ covering $\underline{f}$.

Clearly, both functors $F \circ T^A : \mathcal{PB} \to \mathcal{FM}$ and $T^A \circ F : \mathcal{PB} \to \mathcal{FM}$ are ppgb-functors.

For any ppb-functor $F : \mathcal{M}f \to \mathcal{FM}$ there exists the "exchanging" isomorphism $F \circ T \cong T \circ F$. In contrast, we have the following example of a ppgb-functor $F : \mathcal{PB} \to \mathcal{FM}$ such that there is no natural isomorphism $F \circ T \cong T \circ F$.

EXAMPLE 2.17. Any connected abelian Lie group G is isomorphic to $(\mathbf{S}^1)^n \times \mathbb{R}^m$ for some n and m. Since $H_1((\mathbf{S}^1)^n \times \mathbb{R}^m) \cong \mathbb{R}^n$, where $H_1 : \mathcal{T}op \to \mathcal{V}ect$ (the category of all vector spaces over $\mathbb{R}$) is the first singular homology group functor with real coefficients, we have (by restriction) the functor $H_1 : \mathcal{G}r_{\text{ab-con}} \to \mathcal{V}ect_{\text{fin}}$ (the category of finite-dimensional vector spaces over $\mathbb{R}$), where $\mathcal{G}r_{\text{ab-con}}$ is the category of all connected abelian Lie groups and their homomorphisms. We have the abelianization functor $\mathrm{Ab} : \mathcal{G}r \to \mathcal{G}r_{\text{ab}}$ (see Example 2.15(iv)). We also have the connected component functor $\mathrm{Id}^o : \mathcal{G}r \to \mathcal{G}r_{\text{con}}$ (see Example 2.15(iii)). Define $E = H_1 \circ \mathrm{Id}^o \circ \mathrm{Ab} : \mathcal{G}r \to \mathcal{V}ect_{\text{fin}}$ (composition of functors). Define a functor $F : \mathcal{PB} \to \mathcal{FM}$ by

$$F(P) = M \times E(G) \quad \text{and} \quad F(f) = \underline{f} \times E(\nu)$$

for any $\mathcal{PB}$-object $P \to M$ with the Lie group G and any $\mathcal{PB}$-map $f : P \to P_1$ covering $\underline{f} : M \to M_1$ and with the Lie group homomorphism $\nu : G \to G_1$. From the two lemmas below it follows that F is a ppgb-functor such that there is no natural isomorphism $T \circ F \cong F \circ T$.

LEMMA 2.18. *The functor $E : \mathcal{G}r \to \mathcal{V}ect_{\text{fin}}$ from the last example is regular and product preserving.*

Proof. We see that $E(\mathrm{pt}) = \{0\}$. Moreover, if $G_1 \cong (\mathbf{S}^1)^{n_1} \times \mathbb{R}^{m_1}$ and $G_2 = (\mathbf{S}^1)^{n_2} \times \mathbb{R}^{m_2}$, then $H_1(G_1 \times G_2) \cong \mathbb{R}^{n_1+n_2} \cong H_1(G_1) \times H_1(G_2)$, and then

$$\dim(E(G_1 \times G_2)) = \dim(E(G_1)) + \dim(E(G_2))$$

for any Lie groups G_1 and G_2. Taking into account the above properties of E, we can show that E is product preserving as follows. Let $\mathrm{pr}_i : G_1 \times G_2 \to G_i$ (for $i = 1, 2$) be the projections. We have to show that

$$\Psi := (E(\mathrm{pr}_1), E(\mathrm{pr}_2)) : E(G_1 \times G_2) \to E(G_1) \times E(G_2)$$

is a diffeomorphism. Clearly, Ψ is linear. Then (by a dimension argument) it is sufficient to show that Ψ is surjective. Let $j_1 : G_1 \to G_1 \times G_2$ and $j_2 : G_2 \to G_1 \times G_2$ be the homomorphisms given by $j_1(g_1) = (g_1, e_{G_2})$, $j_2(g_2) = (e_{G_1}, g_2)$. Define $\Phi : E(G_1) \times E(G_2) \to E(G_1 \times G_2)$ by $\Phi(u, v) = E(j_1)(u) + E(j_2)(v)$. Then $\Psi \circ \Phi(u, v) = (u, v)$ as

$$\begin{aligned} E(\mathrm{pr}_1)(E(j_1)(u) + E(j_2)(v)) &= E(\mathrm{pr}_1)(E(j_1)(u)) + E(\mathrm{pr}_1)(E(j_2)(v)) \\ &= E(\mathrm{id}_{G_1})(u) + E(e_{G_1})(v) = u + 0 = u \end{aligned}$$

and (similarly) $E(\mathrm{pr}_2)(E(j_1)(u) + E(j_2)(v)) = v$. Hence Ψ is surjective. Therefore, E is product preserving. The regularity of E is trivial because Ab and Id^o are regular and H_1 has the same values on homotopic maps. ∎

LEMMA 2.19. *There is no natural isomorphism $T \circ F \cong F \circ T$, where F is the ppgb-functor from the last example.*

Proof. We see that $E(\mathbf{S}^1) = H_1(\mathbf{S}^1)$ and $E(T(\mathbf{S}^1)) = H_1(T(\mathbf{S}^1))$. Moreover, since $H_1(\mathbf{S}^1) \cong H_1(T(\mathbf{S}^1))$ (as $\mathbf{S}^1$ can be deformed onto $\mathbf{S}^1 \subset T(\mathbf{S}^1)$ by fibre homotheties), we see that $\dim(E(T(\mathbf{S}^1))) = \dim(E(\mathbf{S}^1)) = 1$. Now, let $P = \mathbb{R} \times \mathbf{S}^1 \to \mathbb{R}$ be the trivial principal bundle over $\mathbb{R}$ with the Lie group $\mathbf{S}^1$. Then $T(P)$ is the trivial principal bundle $T(\mathbb{R}) \times T(\mathbf{S}^1)$ over $T(\mathbb{R})$ with the Lie group $T(\mathbf{S}^1)$. Hence $F(T(P)) = T(\mathbb{R}) \times E(T(\mathbf{S}^1))$. On the other hand, $F(P) = \mathbb{R} \times E(\mathbf{S}^1)$, and so $T(F(P)) = T(\mathbb{R}) \times T(E(\mathbf{S}^1))$. Consequently, $\dim(F(T(P))) = 2 + 1 = 3 \neq 4 = 2 + 2 = \dim(T(F(P)))$. Thus there is no natural isomorphism $F \circ T \cong T \circ F$. ∎

The next example shows that the regularity condition (iii) in Definition 2.1 is not a consequence of conditions (i), (ii) in that definition and condition (i) of Definition 2.7, even for functors $F : \mathcal{PB} \to \mathcal{FM}$.

EXAMPLE 2.20. Let $c : \mathbb{C} \to \mathbb{C}$ be a discontinuous field morphism (it may be standardly obtained by means of the Kuratowski–Zorn lemma). Using c, we can construct a functor $H : \mathcal{V}ect_{\mathrm{fin}} \to \mathcal{M}f$ as follows. By means of the usual bases $(e_j)_{j=1}^k$ in $\mathbb{R}^k$ (or $\mathbb{C}^k$) for any k (where $e_j = (0, \ldots, 1, \ldots, 0)$, 1 at jth position), we identify $m \times n$-matrices with real (or complex) coefficients with $\mathbb{R}$-linear (or $\mathbb{C}$-linear) maps $\mathbb{R}^m \to \mathbb{R}^n$ (or $\mathbb{C}^m \to \mathbb{C}^n$) and vice versa.

Let $A = [a_{ij}]$ be an $m \times n$-matrix with real coefficients. We consider A as the $m \times n$-matrix with complex coefficients in the obvious way (because of $\mathbb{R} \subset \mathbb{C}$). Then $c(A) := [c(a_{ij})]$ is an $m \times n$-matrix with complex coefficients. Since c is a field morphism, we easily see that $c(\mathrm{I}_m) = \mathrm{I}_m$ for the $m \times m$ identity matrix I_m and $c(A_1 \circ A_2) = c(A_1) \circ c(A_2)$ for any $m \times n$-matrix A_1 and $n \times q$-matrix A_2 of real coefficients. For any $\mathcal{V}ect_{\mathrm{fin}}$-object V, $\dim(V) = m$, we define $H(V) = \mathrm{Iso}(\mathbb{R}^m, V) \times \mathbb{C}^m)/\cong$, where $\mathrm{Iso}(\mathbb{R}^m, V)$ is the space of all $\mathbb{R}$-linear isomorphisms (i.e. $\mathcal{V}ect_{\mathrm{fin}}$-isomorphisms) $\mathbb{R}^m \to V$ and $\cong$ is the equivalence relation given by $(\varphi, u) \cong (\varphi_1, u_1)$ iff there is an invertible $m \times m$-matrix B with real coefficients such that $\varphi_1 = \varphi \circ B$ and $u = c(B)(u_1)$. Then $H(V)$ is a manifold. Every $\varphi \in \mathrm{Iso}(\mathbb{R}^m, V)$ induces a chart $\tilde{\varphi} : H(V) \to \mathbb{C}^m$ by $\tilde{\varphi}(W) = u$, where $W = [\varphi, u] \in H(V)$ (u is uniquely determined by φ and W). If $\varphi_1 \in \mathrm{Iso}(\mathbb{R}^m, V)$ is another isomorphism, then $\tilde{\varphi} \circ \tilde{\varphi}_1^{-1} = c(\varphi^{-1} \circ \varphi_1) : \mathbb{C}^m \to \mathbb{C}^m$ is a $\mathbb{C}$-linear map (and therefore smooth).

For any $\mathbb{R}$-linear map (i.e. $\mathcal{V}ect_{\mathrm{fin}}$-map) $f : V \to V_1$, $\dim(V) = m$, $\dim(V_1) = m_1$, we define $H(f) : H(V) \to H(V_1)$ by $H(f)([\varphi, u]) := [\psi, c(\psi^{-1} \circ f \circ \varphi)(u)]$, where $\psi \in \mathrm{Iso}(\mathbb{R}^{m_1}, V_1)$. If $\psi_1 = \psi \circ A \in \mathrm{Iso}(\mathbb{R}^{m_1}, V_1)$ and $\varphi_1 = \varphi \circ B \in \mathrm{Iso}(\mathbb{R}^m, V)$ are some other isomorphisms and $[\varphi, u] = [\varphi_1, u_1]$, then $[\psi_1, c(\psi_1^{-1} \circ f \circ \varphi_1)(u_1)] = [\psi, c(\psi^{-1} \circ f \circ \varphi)(u)]$ because of $c(A) \circ c(\psi_1^{-1} \circ f \circ \varphi_1) \circ c(B^{-1}) = c(\psi^{-1} \circ f \circ \varphi)$ (see above). Using the induced charts we see that $H(f) : H(V) \to H(V_1)$ is smooth as $\tilde{\psi} \circ H(f) \circ \tilde{\varphi}^{-1} = c(\psi^{-1} \circ f \circ \varphi) : \mathbb{C}^m \to \mathbb{C}^{m_1}$ is $\mathbb{C}$-linear (and then smooth).

It is easily seen that $H : \mathcal{V}ect_{\mathrm{fin}} \to \mathcal{M}f$ is a product preserving functor (not necessarily regular). Given a $\mathcal{PB}$-object $P \to M$ with the Lie group G let $F(P) = M \times H(\mathcal{L}(G))$ be the trivial fibred manifold over M, and given a $\mathcal{PB}$-map $f : P \to P_1$ covering $\underline{f} : M \to M_1$ and with the Lie group homomorphism $\nu : G \to G_1$ let $F(f) := \underline{f} \times H(\mathcal{L}(\nu)) : F(P) \to F(P_1)$, where $\mathcal{L} : \mathcal{G}r \to \mathcal{V}ect_{\mathrm{fin}}$ is the Lie algebra functor.

It is clear that $F : \mathcal{PB} \to \mathcal{FM}$ satisfies conditions (i) and (ii) in Definition 2.1 and condition (i) in Definition 2.7. The functor F does not satisfy condition (iii) of Definition 2.1. Indeed, let $P = \mathbb{R} \times \mathbb{R}$ be the trivial $\mathcal{PB}$-object over $\mathbb{R}$ with the Lie group $G = (\mathbb{R}, +)$ acting (on the right) on P by $(x, y) \cdot z = (x, y + z)$. We have $\mathcal{L}(G) = \mathbb{R}$ and the exponent Exp_G is the identity map. The fibre homotheties $a_t : P \to P$, $a_t(x, y) = (x, ty)$, are $\mathcal{PB}$-maps with the Lie group homomorphism $\nu_t = t\,\mathrm{id}_{\mathbb{R}} : G \to G$. Then $\mathcal{L}(\nu_t) = t\,\mathrm{id}_{\mathbb{R}}$, and so $H(\nu_t) = c(t)\,\mathrm{id}_{\mathbb{C}}$, where we identify $H(\mathbb{R})$ with $\mathbb{C}$ via the map induced by the identity $\mathrm{id}_{\mathbb{R}} \in \mathrm{Iso}(\mathbb{R}, \mathbb{R})$. Then $F(a_t)(0, 1) = (0, c(t))$ for any $t \in \mathbb{R}$. But $c_{|\mathbb{R}}$ is discontinuous. (If $c_{|\mathbb{R}}$ is continuous, then so is c, as $c(x + iy) = c(x) + c(i)c(y)$ for any $x, y \in \mathbb{R}$.) Hence a_t is a smoothly parametrized family of $\mathcal{PB}$-maps and $F(a_t)$ is not a smoothly parametrized family. In other words $F : \mathcal{PB} \to \mathcal{FM}$ does not satisfy condition (iii) of Definition 2.1.

We end Section 2 observing that $\mathcal{M}f$ and $\mathcal{G}r$ are "subcategories" in $\mathcal{PB}$.

Example 2.21. Given a manifold M we have the principal bundle $i_1(M) = (\mathrm{id}_M : M \to M)$ with the Lie group pt (the trivial Lie group). Given a map $f : M_1 \to M_2$ we have $i_1(f) = f : i_1(M_1) \to i_1(M_2)$ covering f. The correspondence $i_1 : \mathcal{M}f \to \mathcal{PB}$ is a ppb-functor (with values in $\mathcal{PB}$). Moreover, i_1 is injective on objects and morphisms.

Example 2.22. Given a Lie group G we have the principal bundle $i_2(G) = (G \to \mathrm{pt})$ with the Lie group G. Any Lie group homomorphism $\varphi : G_1 \to G_2$ can be considered as a $\mathcal{PB}$-morphism $i_2(\varphi) = \varphi : i_2(G_1) \to i_2(G_2)$ with the Lie group homomorphism $\varphi : G_1 \to G_2$. The correspondence $i_2 : \mathcal{G}r \to \mathcal{PB}$ is a product preserving functor. Moreover, i_2 is injective on objects and morphisms.

Lemma 2.23. *Any trivial $\mathcal{PB}$-object $M \times G$ of M with the Lie group G can be written as the product $M \times G = i_1(M) \times i_2(G)$ (modulo the obvious identification $M \times \mathrm{pt} = M$).*

3. Admissible triples corresponding to ppgb-functors. We are going to describe the ppgb-functors $F : \mathcal{PB} \to \mathcal{FM}$ and their natural transformations by means of so-called admissible triples and their morphisms. The classification theorems for such functors F will be presented in Section 5. In this section we construct admissible triples from ppgb-functors and morphisms of admissible triples from natural transformations of ppgb-functors. We start with the concept of admissible triples.

Let (F_1, F_2, α) be a triple consisting of a product preserving bundle functor $F_1 : \mathcal{M}f \to \mathcal{FM}$, a product preserving regular functor $F_2 : \mathcal{G}r \to \mathcal{M}f$ and a functor transformation $\alpha : ((F_1)_{|\mathcal{G}r}, F_2) \to F_2$. More precisely, α is a family of mappings $\alpha_G : F_1(G) \times F_2(G) \to F_2(G)$ for any Lie group G such that $\alpha_{G_1} \circ (F_1(\nu) \times F_2(\nu)) = F_2(\nu) \circ \alpha_G$ for any Lie group morphism $\nu : G \to G_1$. The regularity of F_2 means that F_2 transforms smoothly parametrized families of Lie group homomorphisms into smoothly parametrized families of maps. We know that $F_1(G)$ is a Lie group if G is, and $F_1(\nu) : F_1(G) \to F_1(G_1)$ is a Lie group homomorphism if $\nu : G \to G_1$ is (see Introduction).

Definition 3.1. An *admissible triple* is a triple (F_1, F_2, α) as above such that $\alpha_G : F_1(G) \times F_2(G) \to F_2(G)$ is an action of the Lie group $F_1(G)$ on $F_2(G)$ for any Lie group G.

For example, a triple (Id, K, α) consisting of the "identity" ppb-functor $\mathrm{Id} : \mathcal{M}f \to \mathcal{FM}$ ($\mathrm{Id}(M) = (\mathrm{id}_M : M \to M)$, $\mathrm{Id}(f) = f$), a regular product preserving functor $K : \mathcal{G}r \to \mathcal{M}f$ and a $\mathcal{G}r$-invariant family $\alpha = \{\alpha_G\}$

of actions $\alpha_G : G \times K(G) \to K(G)$ (we considered several such (K, α) in Example 2.15) is an admissible triple.

Definition 3.2. Let (F_1, F_2, α) and (H_1, H_2, β) be admissible triples. A *morphism* $\nu : (F_1, F_2, \alpha) \to (H_1, H_2, \beta)$ of *admissible triples* is a pair $\nu = (\nu^1, \nu^2)$ of a natural transformation $\nu^1 : F_1 \to H_1$ of bundle functors and a functor transformation $\nu^2 : F_2 \to H_2$ such that $\nu^2_G \circ \alpha_G = \beta_G \circ (\nu^1_G \times \nu^2_G)$ for any Lie group G.

Any ppgb-functor $F : \mathcal{PB} \to \mathcal{FM}$ determines an admissible triple. Indeed, we have the following example.

Example 3.3. Consider a ppgb-functor $F : \mathcal{PB} \to \mathcal{FM}$. Composing F with the product preserving functors i_1 and i_2 from Examples 2.21 and 2.22, we obtain a ppb-functor $F^{(1)} = F \circ i_1 : \mathcal{M}f \to \mathcal{FM}$ and a product preserving functor $F^{(2)} = \tau \circ F \circ i_2 : \mathcal{G}r \to \mathcal{M}f$, where $\tau : \mathcal{FM} \to \mathcal{M}f$ is the (forgetting) total space functor. Given a Lie group G we define

$$\alpha^{(F)}_G := F(\mu_G) : F^{(1)}(G) \times F^{(2)}(G) = F(i_1(G) \times i_2(G)) \to F^{(2)}(G),$$

where the multiplication $\mu_G : G \times G \to G$ of G is considered as a $\mathcal{PB}$-map $\mu_G : i_1(G) \times i_2(G) \to i_2(G)$. Then $(F^{(1)}, F^{(2)}, \alpha^{(F)})$ is an admissible triple. Indeed, the associative principle $\mu_G \circ (\mathrm{id}_G \times \mu_G) = \mu_G \circ (\mu_G \times \mathrm{id}_G)$ of the multiplication map $\mu_G : G \times G \to G$ of G can be interpreted as the equality $\mu_G \circ (i_1(\mathrm{id}_G) \times \mu_G) = \mu_G \circ (i_1(\mu_G) \times i_2(\mathrm{id}_G))$, where μ_G is treated as a $\mathcal{PB}$-morphism $\mu_G : i_1(G) \times i_2(G) \to i_2(G)$. Then applying F to this equality we obtain the left action condition $\alpha^{(F)}_G \circ (\mathrm{id}_{F^{(1)}(G)} \times \alpha^{(F)}_G) = \alpha^{(F)}_G \circ (\mu_{F^{(1)}(G)} \times \mathrm{id}_{F^{(2)}(G)})$. Similarly, from $\mu_G \circ (i_1(e_G) \times i_2(\mathrm{id}_G)) = i_2(\mathrm{id}_G)$, we deduce $\alpha^{(F)}_G \circ (e_{F^{(1)}(G)} \times \mathrm{id}_{F^{(2)}(G)}) = \mathrm{id}_{F^{(2)}(G)}$. Thus $\alpha^{(F)}_G$ is a left action. Because of the canonical character of $\alpha^{(F)}$ we deduce that $\alpha^{(F)}$ is a functor transformation.

Definition 3.4. Let $F : \mathcal{PB} \to \mathcal{FM}$ be a ppgb-functor. The admissible triple $(F^{(1)}, F^{(2)}, \alpha^{(F)})$ described in Example 3.3 is called the *admissible triple corresponding to* F.

Natural transformations of ppgb-functors $\mathcal{PB} \to \mathcal{FM}$ induce morphisms of corresponding admissible triples. Indeed, we have the following example.

Example 3.5. Let $\eta = \{\eta_P\} : F \to H$ be a natural transformation between ppgb-functors $F, H : \mathcal{PB} \to \mathcal{FM}$. We have natural transformations $\eta^{(1)} := \{\eta_{i_1(M)}\} : F^{(1)} \to H^{(1)}$ and $\eta^{(2)} := \{\eta_{i_2(G)}\} : F^{(2)} \to H^{(2)}$, where $(F^{(1)}, F^{(2)}, \alpha^{(F)})$ and $(H^{(1)}, H^{(2)}, \alpha^{(H)})$ are the admissible triples corresponding to F and H respectively. The pair $\nu^{(\eta)} := (\eta^{(1)}, \eta^{(2)})$ is a morphism $(F^{(1)}, F^{(2)}, \alpha^{(F)}) \to (H^{(1)}, H^{(2)}, \alpha^{(H)})$ of admissible triples. Indeed, $\eta_{i_1(G) \times i_2(G)} = \eta_{i_1(G)} \times \eta_{i_2(G)} : F(i_1(G)) \times F(i_2(G)) \to H(i_1(G)) \times H(i_2(G))$

and the multiplication map $\mu_G : i_1(G) \times i_2(G) \to i_2(G)$ of G is a $\mathcal{PB}$-morphism. Then $\eta_{i_2(G)} \circ F(\mu_G) = H(\mu_G) \circ (\eta_{i_1(G)} \times \eta_{i_2(G)})$ as η is a natural transformation. So, $\eta^{(2)}_G \circ \alpha^{(F)}_G = \alpha^{(H)}_G \circ (\eta^{(1)}_G \times \eta^{(2)}_G)$ for any G.

DEFINITION 3.6. Let $\eta : F \to H$ be a natural transformation of ppgb-functors $F, H : \mathcal{PB} \to \mathcal{FM}$. The morphism $\nu^{(\eta)}$ described in Example 3.5 is called the *morphism of admissible triples corresponding to* η.

4. Ppgb-functors corresponding to admissible triples. In this section we construct ppgb-functors from admissible triples and natural transformations between ppgb-functors from morphisms of admissible triples as follows.

EXAMPLE 4.1. Suppose we have an admissible triple (F_1, F_2, α). We construct a gb-functor $F = F^{(F_1,F_2,\alpha)} : \mathcal{PB} \to \mathcal{FM}$ as follows. Let $p : P \to M$ be a $\mathcal{PB}$-object with the Lie group G. Since F_1 is a ppb-functor on $\mathcal{M}f$, $F_1(p) : F_1(P) \to F_1(M)$ is a principal bundle with the Lie group $F_1(G)$ (see Introduction). We define $F^{(F_1,F_2,\alpha)}(P) = F_1(P)[F_2(G), \alpha_G]$ to be the associated bundle with the standard fibre $F_2(G)$ (being a left $F_1(G)$-space by the action α_G). Then $F^{(F_1,F_2,\alpha)}(P)$ is a fibre bundle over $F_1(M)$ (and then over M). Let $f : P_1 \to P_2$ be a $\mathcal{PB}$-morphism of $\mathcal{PB}$-objects $p_1 : P_1 \to M_1$ and $p_2 : P_2 \to M_2$ covering $\underline{f} : M_1 \to M_2$ and with the corresponding Lie group homomorphism $\varphi_f : G_1 \to G_2$. Since $F_1 : \mathcal{M}f \to \mathcal{FM}$ is a ppb-functor, $F_1(f) : F_1(P_1) \to F_1(P_2)$ is a homomorphism of principal bundles covering $F_1(\underline{f}) : F_1(M_1) \to F_1(M_2)$ and with the Lie group homomorphism $F_1(\varphi_f) : F_1(G_1) \to F_1(G_2)$ (see Introduction). We define $F^{(F_1,F_2,\alpha)}(f) : F(P_1) \to F(P_2)$ by $F^{(F_1,F_2,\alpha)}(f)([y, v]) = [F_1(f)(y), F_2(\varphi_f)(v)]$ for any $[y, v] \in F^{(F_1,F_2,\alpha)}(P_1)$ with $y \in F_1(P_1)$ and $v \in F_2(G_1)$.

To see that $F^{(F_1,F_2,\alpha)}(f)$ is well-defined, we prove the implication "if $[y_1, v_1] = [y_2, v_2]$ then $[F_1(f)(y_1), F_2(\varphi_f)(v_1)] = [F_1(f)(y_2), F_2(\varphi_f)(v_2)]$" as follows. Let $[y_1, v_1] = [y_2, v_2]$. Then $y_2 = y_1 \cdot \xi^{-1}$ and $v_2 = \alpha_{G_1}(\xi, v_1)$ for some $\xi \in F_1(G)$, where the dot $\cdot$ denotes the right action of the principal bundle. We have $F_1(f)(y_2) = F_1(f)(y_1) \cdot (F_1(\varphi_f)(\xi))^{-1}$. So, it remains to see that $F_2(\varphi_f)(v_2) = \alpha_{G_2}(F_1(\varphi_f)(\xi), F_2(\varphi_f)(v_1))$. But the last equality is the invariance property of the functor transformation $\alpha = \{\alpha_G\}$ with respect to the Lie group homomorphism $\varphi_f : G_1 \to G_2$. Thus $F^{(F_1,F_2,\alpha)}(f) : F(P_1) \to F(P_2)$ is well-defined. The correspondence $F^{(F_1,F_2,\alpha)} : \mathcal{PB} \to \mathcal{FM}$ is a gb-functor.

LEMMA 4.2. *The gb-functor* $F^{(F_1,F_2,\alpha)} : \mathcal{PB} \to \mathcal{FM}$ *is product preserving.*

Proof. Put $\tilde{F} := F^{(F_1,F_2,\alpha)}$. Let P_1 and P_2 be $\mathcal{PB}$-objects. Let $\mathrm{pr}_i : P_1 \times P_2 \to P_i$ $(i = 1, 2)$ be the projections. We prove that $I_{P_1,P_2} :=$

$(\tilde{F}(\mathrm{pr}_1), \tilde{F}(\mathrm{pr}_2)) : \tilde{F}(P_1 \times P_2) \to \tilde{F}(P_1) \times \tilde{F}(P_2)$ is a diffeomorphism as follows. By a local trivialization argument we may assume that $P_i = i_1(\mathbb{R}^{m_i}) \times i_2(G_i)$, $i = 1, 2$, are trivial $\mathcal{PB}$-objects. Then $\tilde{F}(P_1 \times P_2) = \tilde{F}(i_1(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}) \times i_2(G_1 \times G_2)) = F_1(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}) \times F_2(G_1 \times G_2) = F_1(\mathbb{R}^{m_1}) \times F_1(\mathbb{R}^{m_2}) \times F_2(G_1) \times F_2(G_2)$ and $\tilde{F}(P_1) \times \tilde{F}(P_2) = F_1(\mathbb{R}^{m_1}) \times F_2(G_2) \times F_1(\mathbb{R}^{m_2}) \times F_2(G_2)$, as given $G \in \mathrm{Obj}(\mathcal{G}r)$ we have $\tilde{F}(i_1(\mathbb{R}^m) \times i_2(G)) = F_1(\mathbb{R}^m) \times F_2(G)$ modulo the usual identification $[(y, e_{F_1(G)}), \xi] = (y, \xi)$. Then it is easily seen that $I_{P_1,P_2} : F_1(\mathbb{R}^{m_1}) \times F_1(\mathbb{R}^{m_2}) \times F_2(G_1) \times F_2(G_2) \to F_1(\mathbb{R}^{m_1}) \times F_2(G_1) \times F_1(\mathbb{R}^{m_2}) \times F_2(G_2)$ satisfies $I_{P_1,P_2}(y_1, y_2, \xi_1, \xi_2) = (y_1, \xi_1, y_2, \xi_2)$ for all $(y_1, y_2, \xi_1, \xi_2) \in F_1(\mathbb{R}^{m_1}) \times F_1(\mathbb{R}^{m_2}) \times F_2(G_1) \times F_2(G_2)$, i.e. it is a diffeomorphism. ∎

Definition 4.3. Let (F_1, F_2, α) be an admissible triple. The ppgb-functor $F^{(F_1,F_2,\alpha)} : \mathcal{PB} \to \mathcal{FM}$ described in Example 4.1 is called the *ppgb-functor corresponding to* (F_1, F_2, α).

Example 4.4. Let $\nu = (\nu^1, \nu^2) : (F_1, F_2, \alpha) \to (H_1, H_2, \beta)$ be a morphism of admissible triples. Consider a $\mathcal{PB}$-object $p : P \to M$ with the standard Lie group G. We define $\eta_P^{(\nu)} : F^{(F_1,F_2,\alpha)}(P) \to F^{(H_1,H_2,\beta)}(P)$ by $\eta_P^{(\nu)}([y, \xi]) = [\nu_P^1(y), \nu_G^2(\xi)]$ for any $[y, \xi] \in F^{(F_1,F_2,\alpha)}(P) = F_1(P)[F_2(G), \alpha_G]$. One can standardly (as in Example 4.1) verify that the definition of $\eta_P^{(\nu)}$ is correct. The family $\eta^{(\nu)} = \{\eta_P^{(\nu)}\} : F^{(F_1,F_2,\alpha)} \to F^{(H_1,H_2,\beta)}$ is a natural transformation.

Definition 4.5. Let $\nu = (\nu_1, \nu_2) : (F_1, F_2, \alpha) \to (H_1, H_2, \beta)$ be a morphism of admissible triples. The natural transformation $\eta^{(\nu)} : F^{(F_1,F_2,\alpha)} \to F^{(H_1,H_2,\beta)}$ described in Example 4.4 is called the *natural transformation corresponding to* ν.

5. Classification theorems. We have the following two important lemmas.

Lemma 5.1. *Let $F : \mathcal{PB} \to \mathcal{FM}$ be a ppgb-functor, $(F^{(1)}, F^{(2)}, \alpha^{(F)})$ be the admissible triple corresponding to F and $F^{(F^{(1)},F^{(2)},\alpha^{(F)})} : \mathcal{PB} \to \mathcal{FM}$ be the ppgb-functor corresponding to $(F^{(1)}, F^{(2)}, \alpha^{(F)})$. Then F and $F^{(F^{(1)},F^{(2)},\alpha^{(F)})}$ are isomorphic.*

Proof. Given a $\mathcal{PB}$-object $P = (p : P \to M)$ with the Lie group G we have a $\mathcal{PB}$-map $f_P : i_1(P) \times i_2(G) \to P$ defined by $f_P(p, \xi) = p \cdot \xi$, where the dot $\cdot$ denotes the right (principal bundle) action of G on P. Define

$$\tilde{\Theta}_P^F := F(f_P) : F^{(1)}(P) \times F^{(2)}(G) = F(i_1(P) \times i_2(G)) \to F(P),$$

where $F^{(1)}(P) \times F^{(2)}(G) = F(i_1(P) \times i_2(G))$ modulo the ppgb-functor identification. It remains to show that $\tilde{\Theta}_P^F$ can be factorized by means of the quotient projection $\Phi_P : F^{(1)}(P) \times F^{(2)}(G) \to F^{(1)}(P)[F^{(2)}(G), \alpha_G^{(F)}]$ and that the quotient map $\Theta_P^F : F^{(F^{(1)},F^{(2)},\alpha^{(F)})}(P) \to F(P)$ is a diffeomorphism. Because $\tilde{\Theta}_P^F$ is a functor, by a local trivialization argument we may assume that $P = i_1(\mathbb{R}^m) \times i_2(G)$. Then $F(i_1(\mathbb{R}^m) \times i_2(G)) = F^{(1)}(\mathbb{R}^m) \times F^{(2)}(G)$ (ppgb-functor identification) and $F^{(F^{(1)},F^{(2)},\alpha^{(F)})}(i_1(\mathbb{R}^m) \times i_2(G)) = F^{(1)}(\mathbb{R}^m) \times F^{(2)}(G)$ by the usual identification $[(y, e_{F^{(1)}(G)}), \xi] = (y, \xi)$. Since $f_{i_1(\mathbb{R}^m)\times i_2(G)} = \mathrm{id}_{\mathbb{R}^m} \times \mu_G : i_1(\mathbb{R}^m) \times i_1(G) \times i_2(G) \to i_1(\mathbb{R}^m) \times i_2(G)$, we see that

$$\begin{aligned}\widetilde{\Theta}^F_{i_1(\mathbb{R}^m)\times i_2(G)}(y,\eta,\xi) = (y, \alpha_G^{(F)}(\eta,\xi)) &= \Phi_{i_1(\mathbb{R}^m)\times i_2(G)}(y, e_{F^{(1)}(G)}, \alpha_G^{(F)}(\eta,\xi)) \\ &= \Phi_{i_1(\mathbb{R}^m)\times i_2(G)}(y,\eta,\xi)\end{aligned}$$

for any $(y,\eta,\xi) \in F^{(1)}(\mathbb{R}^m) \times F^{(1)}(G) \times F^{(2)}(G)$. Hence $\tilde{\Theta}^F_{i_1(\mathbb{R}^m)\times i_2(G)}$ induces the identity map $F^{(1)}(\mathbb{R}^m) \times F^{(2)}(G) \to F^{(1)}(\mathbb{R}^m) \times F^{(2)}(G)$. ∎

LEMMA 5.2. *Let (F_1, F_2, α) be an admissible triple. Let $\tilde{F} = F^{(F_1,F_2,\alpha)}$ be its corresponding ppgb-functor. Let $(\tilde{F}^{(1)}, \tilde{F}^{(2)}, \alpha^{(\tilde{F})})$ be the admissible triple corresponding to $\tilde{F}$. Then (F_1, F_2, α) is isomorphic to $(\tilde{F}^{(1)}, \tilde{F}^{(2)}, \alpha^{(\tilde{F})})$.*

Proof. Since $F_1 : \mathcal{M}f \to \mathcal{FM}$ is product preserving, $F_1(\mathrm{pt}) = \{e_1\}$ is a one-point manifold. Similarly, $F_2(\mathrm{pt}) = \{e_2\}$. Then $\tilde{F}^{(1)}(M) = \tilde{F}(i_1(M)) = F_1(M) \times_{\alpha_{\mathrm{pt}}} F_2(\mathrm{pt}) \cong F_1(M)$. So, we have a natural isomorphism $\mathcal{V}^1 : \tilde{F}^{(1)} \to F_1$ given by $\mathcal{V}^1_M([\xi, e_2]) = \xi$. Similarly, $\tilde{F}^{(2)}(G) = \tilde{F}(i_2(G)) = F_1(G) \times_{\alpha_G} F_2(G) \cong F_2(G)$. So we have a functor isomorphism $\mathcal{V}^2 : \tilde{F}^{(2)} \to F_2$ given by $\mathcal{V}^2_G([e_{F_1(G)}, \eta]) = \eta$. We prove that $\mathcal{V}^{(F_1,F_2,\alpha)} := (\mathcal{V}^1, \mathcal{V}^2) : (\tilde{F}^{(1)}, \tilde{F}^{(2)}, \alpha^{(\tilde{F})}) \to (F_1, F_2, \alpha)$ is an isomorphism of admissible triples. It remains to verify that $\alpha_G \circ (\mathcal{V}^1_G \times \mathcal{V}^2_G) = \mathcal{V}^2_G \circ \alpha_G^{(\tilde{F})}$. Let $\mu_G : i_1(G) \times i_2(G) \to i_2(G)$ be the $\mathcal{PB}$-map given by the multiplication of G. Then (by definition) $\alpha_G^{(\tilde{F})} = \tilde{F}(\mu_G) : \tilde{F}^{(1)}(G) \times \tilde{F}^{(2)}(G) = \tilde{F}(i_1(G) \times i_2(G)) \to \tilde{F}^{(2)}(G)$. Consider arbitrary elements $[\xi, e_2] \in \tilde{F}^{(1)}(G)$ and $[e_{F_1(G)}, \eta] \in \tilde{F}^{(2)}(G)$. Then

$$\begin{aligned}&\mathcal{V}^2_G \circ \alpha_G^{(\tilde{F})}([\xi, e_2], [e_{F_1(G)}, \eta]) = \mathcal{V}^2_G \circ \tilde{F}(\mu_G)([\xi, e_2], [e_{F_1(G)}, \eta]) \\ &\quad = \mathcal{V}^2_G([e_{F_1(G)}, \alpha_G(\xi,\eta)]) = \alpha_G(\xi,\eta) = \alpha_G \circ (\mathcal{V}^1_G \times \mathcal{V}^2_G)([\xi, e_2], [e_{F_1(G)}, \eta]).\end{aligned}$$ ∎

Summing up we get the following classification theorems.

THEOREM 5.3. *The correspondence $F \mapsto (F^{(1)}, F^{(2)}, \alpha^{(F)})$ induces a bijection between the equivalence classes of ppgb-functors on $\mathcal{PB}$ and the equivalence classes of admissible triples. The inverse bijection is induced by $(F_1, F_2, \alpha) \mapsto F^{(F_1,F_2,\alpha)}$.*

Proof. If $\eta : F \to H$ is a natural isomorphism of ppgb-functors then $\nu^{(\eta)} : (F^{(1)}, F^{(2)}, \alpha^{(F)}) \to (H^{(1)}, H^{(2)}, \alpha^{(H)})$ is an isomorphism of admissible triples. Thus the correspondence $[F] \mapsto [(F^{(1)}, F^{(2)}, \alpha^{(F)})]$ is well-defined. Similarly, if $\nu : (F_1, F_2, \alpha) \to (H_1, H_2, \beta)$ is an isomorphism of admissible triples, then $\eta^{(\nu)} : F^{(F_1,F_2,\alpha)} \to F^{(H_1,H_2,\beta)}$ is a natural isomorphism of ppgb-functors. Hence the correspondence $[(F_1, F_2, \alpha)] \mapsto [F^{(F_1,F_2,\alpha)}]$ is also well-defined. From Lemmas 5.1 and 5.2 it follows that the correspondences are mutually inverse. ∎

THEOREM 5.4. *Let $F, H : \mathcal{PB} \to \mathcal{FM}$ be ppgb-functors. Let $(F^{(1)}, F^{(2)}, \alpha^{(F)})$ and $(H^{(1)}, H^{(2)}, \alpha^{(H)})$ be the admissible triples corresponding to F and H. The correspondence $\eta \mapsto \nu^{(\eta)}$ is a bijection between the natural transformations $F \to H$ and the morphisms $(F^{(1)}, F^{(2)}, \alpha^{(F)}) \to (H^{(1)}, H^{(2)}, \alpha^{(H)})$ of admissible triples. The inverse bijection is given by the correspondence $\nu \mapsto \eta^{[\nu]}$ described in Lemma* 5.5 *below.*

Proof. This follows immediately from Lemma 5.5. ∎

LEMMA 5.5. *Let $F, H : \mathcal{PB} \to \mathcal{FM}$ be ppgb-functors and $(F^{(1)}, F^{(2)}, \alpha^{(F)})$ and $(H^{(1)}, H^{(2)}, \alpha^{(H)})$ be the admissible triples corresponding to F and H. Let $\nu = (\nu^1, \nu^2) : (F^{(1)}, F^{(2)}, \alpha^{(F)}) \to (H^{(1)}, H^{(2)}, \alpha^{(H)})$ be a morphism of admissible triples. Let $\Theta^F : F^{(F^{(1)},F^{(2)},\alpha^{(F)})} \to F$ and $\Theta^H : F^{(H^{(1)},H^{(2)},\alpha^{(H)})} \to H$ be the isomorphisms from the proof of Lemma* 5.1. *Let $\eta^{(\nu)} : F^{(F^{(1)},F^{(2)},\alpha^{(F)})} \to F^{(H^{(1)},H^{(2)},\alpha^{(H)})}$ be the natural transformation corresponding to ν. Define a natural transformation $\eta^{[\nu]} : F \to H$ by $\eta^{[\nu]}_P = \Theta^H_P \circ \eta^{(\nu)}_P \circ (\Theta^F_P)^{-1}$ for any principal bundle P. Let $\tilde{\nu} : (F^{(1)}, F^{(2)}, \alpha^{(F)}) \to (H^{(1)}, H^{(2)}, \alpha^{(H)})$ be the morphism of admissible triples corresponding to $\eta^{[\nu]}$. Then $\tilde{\nu} = \nu$. If $\mathcal{E} : F \to H$ is another natural transformation such that the corresponding morphism $\nu^{(\mathcal{E})} : (F^{(1)}, F^{(2)}, \alpha^{(F)}) \to (H^{(1)}, H^{(2)}, \alpha^{(H)})$ of admissible triples is such that $\nu^{(\mathcal{E})} = \nu$ then $\mathcal{E} = \eta^{[\nu]}$.*

Proof. Clearly, if natural transformations $\eta', \eta'' : F \to H$ are such that $\nu^{(\eta')} = \nu^{(\eta'')}$ then $\eta' = \eta''$. So, it remains to show that $\tilde{\nu} = \nu$, i.e.

$$\nu^1_M \circ \Theta^F_{i_1(M)} = \Theta^H_{i_1(M)} \circ \eta^{(\nu)}_{i_1(M)} \quad \text{and} \quad \nu^2_G \circ \Theta^F_{i_2(G)} = \Theta^H_{i_2(G)} \circ \eta^{(\nu)}_{i_2(G)}$$

for any manifold M and any Lie group G. We see that $\Theta^F_{i_1(M)}$ and $\Theta^F_{i_2(G)}$ are the maps $\mathcal{V}^1$ and $\mathcal{V}^2$ from the proof of Lemma 5.2 for $(F^{(1)}, F^{(2)}, \alpha^{(F)})$ in place of (F_1, F_2, α), and $\Theta^H_{i_1(M)}$ and $\Theta^H_{i_2(M)}$ are the maps $\mathcal{V}^1$ and $\mathcal{V}^2$ from the proof of Lemma 5.2 for $(H^{(1)}, H^{(2)}, \alpha^{(H)})$ in place of (F_1, F_2, α). Then $\nu^1_M \circ \Theta^F_{i_1(M)}([\xi, e_2]) = \nu^1_M(\xi) = \Theta^H_{i_1(M)}([\nu^1_M(\xi), e_2]) = \Theta^H_{i_1(M)} \circ \eta^{(\nu)}_{i_1(M)}([\xi, e_2])$ (where e_2 is as in the proof of Lemma 5.2 for $(F^{(1)}, F^{(2)}, \alpha^{(F)})$ (or $(H^{(1)}, H^{(2)}, \alpha^{(H)})$) in place of (F_1, F_2, α)), and similarly for G in place of M. ∎

REMARK 5.6. Let $\mathcal{G} \subset \mathcal{G}r$ be a subcategory in the category of all Lie groups and their morphisms. Denote by $\mathcal{PB}(\mathcal{G})$ the category of all principal bundles with Lie structure groups as $\mathcal{G}$-objects and all principal bundle maps with Lie group homomorphisms as $\mathcal{G}$-morphisms. Suppose $\mathcal{G}$ satisfies the following conditions (i)–(iv):

(i) The trivial Lie group pt is a $\mathcal{G}$-object.
(ii) $\mathcal{G}$ is closed with respect to taking products (i.e. if G_1 and G_2 are $\mathcal{G}$-objects, then so is $G_1 \times G_2$).
(iii) Any Lie group $\mathcal{G}r$-isomorphic to a $\mathcal{G}$-object is a $\mathcal{G}$-object.
(iv) Any Lie group homomorphism of $\mathcal{G}$-objects is a $\mathcal{G}$-morphism.

Then (as is easily seen) the obvious versions of Theorems 5.3 and 5.4 for ppgb-functors on $\mathcal{PB}(\mathcal{G})$ instead of $\mathcal{PB} = \mathcal{PB}(\mathcal{G}r)$ hold.

The following categories satisfy conditions (i)–(iv): the category of abelian Lie groups, the category of nilpotent Lie groups, the category of solvable Lie groups, the category of compact Lie groups and the category of trivial Lie groups.

Theorems 5.3 and 5.4 for trivial Lie groups are in fact the description (mentioned in Introduction) of ppb-functors on $\mathcal{M}f$ in terms of Weil algebras.

Remark 5.7. In [Mi5], we presented a full description of ppgb-functors $F : \mathcal{VB} \to \mathcal{FM}$ on the category $\mathcal{VB}$ of vector bundles and vector bundle maps in terms of Weil modules. On the other hand the category $\mathcal{VB}$ is not of the form $\mathcal{VB} \cong \mathcal{PB}(\mathcal{G})$ for any subcategory $\mathcal{G}$ satisfying conditions (i)–(iv) of Remark 5.6. [Indeed, suppose that $F : \mathcal{VB} \to \mathcal{FB}$ is a gb-functor with the point property $F(\{\text{point}\}) = \{\text{point}\}$ with values in $\mathcal{PB}$. Let $E \to M$ be a $\mathcal{VB}$-object such that M is connected. $F(E)$ is a principal bundle with the Lie group G_E. Let $f : E \to E$ be a $\mathcal{VB}$-map. $F(f)$ is a principal bundle map with the Lie group homomorphism $\nu_f : G_E \to G_E$. There is a $\mathcal{VB}$-map $g : E \to E$ such that $g = f$ over some open set $U \subset M$ and constant over some open set $V \subset M$. By locality, we have $F(g) = F(f)$ over U and $F(g) = \text{const}$ over V (as F has the point property). Then ν_g is trivial and $\nu_g = \nu_f$. Consequently, for every $\mathcal{VB}$-map $f : E \to E$, $F(f)$ is a principal bundle map with the trivial Lie group homomorphism. Putting $f = \mathrm{id}_E$, we deduce that $\nu_{\mathrm{id}_E} = \mathrm{id}_{G_E}$ is the trivial Lie group homomorphism. Then G_E is a trivial Lie group for any $\mathcal{VB}$-object $E \to M$. Consequently, $F(E) = M = (\mathrm{id}_M : M \to M) \in \mathcal{PB}$ with the trivial Lie group for any $\mathcal{VB}$-object $E \to M$.] So, the result from [Mi5] cannot be a consequence (version) of Theorems 5.3 and 5.4.

6. The local trivialization expression. Roughly speaking, Theorems 5.3 and 5.4 say that a ppgb-functor $F : \mathcal{PB} \to \mathcal{FM}$ is determined by the admissible triple $(F^{(1)}, F^{(2)}, \alpha^{(F)})$ corresponding to F. In particular, the extension $F(f) : F(P) \to F(P_1)$ of a $\mathcal{PB}$-map $f : P \to P_1$ is determined by f and $(F^{(1)}, F^{(2)}, \alpha^{(F)})$. Below, we present the local trivialization expression of $F(f) : F(P) \to F(P_1)$ by means of $(F^{(1)}, F^{(2)}, \alpha^{(F)})$.

Proposition 6.1. *Let $F : \mathcal{PB} \to \mathcal{FM}$ be a ppgb-functor; and let $(F^{(1)}, F^{(2)}, \alpha^{(F)})$ be the admissible triple corresponding to F. Let $f : i_1(\mathbb{R}^{m_1}) \times i_2(G_1) \to i_1(\mathbb{R}^{m_2}) \times i_2(G_2)$ be a $\mathcal{PB}$-map between trivial $\mathcal{PB}$-objects with the group homomorphism $\varphi : G_1 \to G_2$. We can write $f(x, \xi) = (\underline{f}(x), h(x)\cdot\varphi(\xi))$ for some maps $\underline{f} : \mathbb{R}^{m_1} \to \mathbb{R}^{m_2}$ and $h : \mathbb{R}^{m_1} \to G_2$. Then $F(f) : F^{(1)}(\mathbb{R}^{m_1}) \times F^{(2)}(G_1) \to F^{(1)}(\mathbb{R}^{m_2}) \times F^{(2)}(G_2)$ is given by*

$$F(f)(y, \eta) = (F^{(1)}(\underline{f})(y), \alpha^{(F)}_{G_2}(F^{(1)}(h)(y), F^{(2)}(\varphi)(\eta)))$$

for any $y \in F^{(1)}(\mathbb{R}^{m_1})$ and $\eta \in F^{(2)}(G_1)$. In particular, $F : \mathcal{PB} \to \mathcal{FM}$ is of finite order (the same as $F^{(1)}$).

Proof. Clearly $\mathrm{pr}_1 \circ f = i_1(\underline{f}) \circ \mathrm{pr}_1$ and $\mathrm{pr}_2 \circ f = \mu_{G_2} \circ (i_1(h) \times i_2(\mathrm{id}_G))$, where pr_i are the projections. Applying F, we get

$$F(f) = (F(i_1(\underline{f})) \circ \mathrm{Pr}_1, F(\mu_{G_2}) \circ (F(i_1(h)) \times F(i_2(\varphi)))),$$

where $\mathrm{Pr}_1 : F(i_1(\mathbb{R}^{m_1})) \times F(i_2(G)) \to F(i_1(\mathbb{R}^{m_1}))$ is the usual projection. Then $F(f) = (F^{(1)}(\underline{f}) \circ \mathrm{Pr}_1, \alpha^{(F)}_{G_2} \circ (F^{(1)}(h) \times F^{(2)}(\varphi)))$. The particular case of the proposition is clear because of the above local trivialization expression (as $\mathrm{j}^r_x(f_1) = \mathrm{j}^r_x(f_2)$ iff $\mathrm{j}^r_x(\underline{f}_1) = \mathrm{j}^r_x(\underline{f}_2)$ and $\mathrm{j}^r_x(h_1) = \mathrm{j}^r_x(h_2)$ and $\varphi_1 = \varphi_2$). ■

7. The natural transformations $F_{|\mathcal{PB}_m(G)} \to H_{|\mathcal{PB}_m(G)}$. Let $F, H : \mathcal{PB} \to \mathcal{FM}$ be ppgb-functors. We fix a Lie group G and a natural number m. In this section we describe all natural transformations $F_{|\mathcal{PB}_m(G)} \to H_{|\mathcal{PB}_m(G)}$ between the gauge-natural bundles $F_{|\mathcal{PB}_m(G)}$ and $H_{|\mathcal{PB}_m(G)}$ (where $F_{|\mathcal{PB}_m(G)} : \mathcal{PB}_m(G) \to \mathcal{FM}$ is the restriction of F to $\mathcal{PB}_m(G) \subset \mathcal{PB}$). We start with the following definition.

Definition 7.1. Let F, H and G be as above. Let $(F^{(1)}, F^{(2)}, \alpha^{(F)})$ and $(H^{(1)}, H^{(2)}, \alpha^{(H)})$ be the admissible triples corresponding to F and H, and $A^{(F)} = F^{(1)}(\mathbb{R})$ and $A^{(H)} = H^{(1)}(\mathbb{R})$ be the Weil algebras corresponding to $F^{(1)}$ and $H^{(1)}$. An (F, H, G)-*admissible pair* is a pair (ρ, σ) of mappings $\rho : F^{(2)}(G) \times A^{(F)} \to A^{(H)}$ and $\sigma : F^{(2)}(G) \to H^{(2)}(G)$ with the following three properties:

(a) Given $y \in F^{(2)}(G)$, $c \in F^{(1)}(G)$ and $a \in A^{(F)}$, we have $\rho(y, a) = \rho(\alpha^{(F)}_G(c, y), a)$.
(b) Given $y \in F^{(2)}(G)$, the map $\rho_y : A^{(F)} \to A^{(H)}$ defined by $\rho_y(a) = \rho(y, a)$ for all $a \in A^{(F)}$ is a Weil algebra homomorphism.
(c) Given $c \in F^{(1)}(G)$ and $y \in F^{(2)}(G)$, $\sigma \circ \alpha^{(F)}_G(c, y) = \alpha^{(H)}_G(\tilde{\rho_y}(c), \sigma(y))$, where $\tilde{\varphi} : F^{(1)} \to H^{(1)}$ is the natural transformation corresponding to a Weil algebra homomorphism $\varphi : A^{(F)} \to A^{(H)}$.

Any natural transformation $\eta : F_{|\mathcal{PB}_m(G)} \to H_{|\mathcal{PB}_m(G)}$ between gauge-natural bundles determines an (F, H, G)-admissible pair. Namely, we have the following example.

Example 7.2. Using η, we define $\rho^{(\eta)} : F^{(2)}(G) \times A^{(F)} \to A^{(H)}$ by

$$\rho^{(\eta)}(y, a) = H(p_1) \circ \eta_{i_1(\mathbb{R}^m) \times i_2(G)}(F^{(1)}(j)(a), y),$$

where $j : i_1(\mathbb{R}) \to i_1(\mathbb{R}^m)$, $j(t) = (t, 0)$, $t \in \mathbb{R}$ and $p_1 : i_1(\mathbb{R}^m) \times i_2(G) \to i_1(\mathbb{R})$, $p_1(x, g) = x^1$, $x = (x^1, \ldots, x^m) \in \mathbb{R}^m$, $g \in G$, are $\mathcal{PB}$-maps. We also define $\sigma^{(\eta)} : F^{(2)}(G) \to H^{(2)}(G)$ by

$$\sigma^{(\eta)}(v) = H(\mathrm{pr}_2) \circ \eta_{i_1(\mathbb{R}^m) \times i_2(G)}(0, v),$$

where $0 \in F^{(1)}(\mathbb{R}^m)$ and $\mathrm{pr}_2 : i_1(\mathbb{R}^m) \times i_2(G) \to i_2(G)$ is the projection.

Lemma 7.3. *If $m \geq 2$, then $(\rho^{(\eta)}, \sigma^{(\eta)})$ is an (F, H, G)-admissible pair.*

First we prove the following technical fact.

Lemma 7.4. *Let $F = T^A : \mathcal{M}f \to \mathcal{FM}$ be a ppb-functor. Let G be a Lie group and $m \geq 1$ be an integer. Then the elements $F(h)(d) \in F(G)$ for $h : \mathbb{R}^m \to G$ and $d \in F(\mathbb{R}^m)$ generate the group $F(G)$.*

Proof. Clearly, in any connected component of $F(G)$ there is an element of the form $F(h)(d)$ for some $h : \mathbb{R}^m \to G$ and $d \in F(\mathbb{R}^m)$. So, without loss of generality, we can assume that $F(G)$ (or equivalently G) is connected. Clearly, $\mathcal{L}(G)$ is the set of all $\tilde{h}(c)$ for linear maps $\tilde{h} : \mathbb{R}^m \to \mathcal{L}(G)$ and $c \in \mathbb{R}^m$. Then $\mathcal{L}(T^A(G)) = \mathcal{L}(G) \otimes A$ is generated (over $\mathbb{R}$) by the elements $T^A(\tilde{h})(c \otimes a) = \tilde{h}(c) \otimes a$ for linear $\tilde{h} : \mathbb{R}^m \to \mathcal{L}(G)$, $a \in A$ and $c \in \mathbb{R}^m$. Then we can write $\mathcal{L}(F(G)) = V_1 \oplus \cdots \oplus V_k$, where $V_i = \mathrm{span}\{F(\tilde{h}_i)(d_i)\}$, $\tilde{h}_i : \mathbb{R}^m \to \mathcal{L}(G)$ is linear and $d_i \in F(\mathbb{R}^m)$ for $i = 1, \ldots, k = \dim(\mathcal{L}(F(G)))$. By the linearity of $\tilde{h}_i$, $tF(\tilde{h}_i)(d_i) = F(t\tilde{h}_i)(d) = F(\tilde{h}_i)(td_i)$ for all $t \in \mathbb{R}$. By the general Lie group theory, the map $\Phi : \mathcal{L}(F(G)) \to F(G)$ given by $\Phi(v) = \mathrm{Exp}_{F(G)}(v_1) \cdot \ldots \cdot \mathrm{Exp}_{F(G)}(v_k)$, $v = (v_1, \ldots, v_k) \in \mathcal{L}(F(G)) = V_1 \oplus \cdots \oplus V_k$ is a diffeomorphism from some neighbourhood of $0 \in \mathcal{L}(F(G))$ onto some neighbourhood of $e_{F(G)} \in F(G)$. Then the formula $F(\mathrm{Exp}_G) = \mathrm{Exp}_{F(G)}$ (see Introduction) ends the proof. ■

Proof of Lemma 7.3. Using Lemma 7.4 and Proposition 6.1 we verify that $(\rho^{(\eta)}, \sigma^{(\eta)})$ from Example 7.2 satisfies conditions (a)–(c) in Definition 7.1.

(a) Using the invariance $H(\psi_\tau) \circ \eta_{i_1(\mathbb{R}^m) \times i_2(G)} = \eta_{i_1(\mathbb{R}^m) \times i_2(G)} \circ F(\psi_\tau)$ of η with respect to the $\mathcal{PB}_m(G)$-morphisms $\psi_\tau = i_1(\mathrm{id}_\mathbb{R} \times \tau\, \mathrm{id}_{\mathbb{R}^{m-1}}) \times i_2(\mathrm{id}_G) : i_1(\mathbb{R}^m) \times i_2(G) \to i_1(\mathbb{R}^m) \times i_2(G)$ for $\tau > 0$, we get

$$H(p_1) \circ \eta_{i_1(\mathbb{R}^m) \times i_2(G)}((b^1, \tau b^2, \ldots, \tau b^m), y) = H(p_1) \circ \eta_{i_1(\mathbb{R}^m) \times i_2(G)}(b, y)$$

for any $b = (b^1, \ldots, b^m) \in (A^{(F)})^m = F^{(1)}(\mathbb{R}^m)$ and any $y \in F^{(2)}(G)$. Letting $\tau \to 0$ we get $\rho^{(\eta)}(y, b^1) = H(p_1) \circ \eta_{i_1(\mathbb{R}^m) \times i_2(G)}(b, y)$ for any b and y as above. Then using the invariance of η with respect to the $\mathcal{PB}_m(G)$-maps $f : i_1(\mathbb{R}^m) \times i_2(G) \to i_1(\mathbb{R}^m) \times i_2(G)$ of the form $f(x, g) = (x, h(x^2, \ldots, x^m)g)$, $x \in \mathbb{R}^m$, $g \in G$, where $h : \mathbb{R}^{m-1} \to G$, we get $\rho^{(\eta)}(y, b^1) = \rho^{(\eta)}(\alpha_G^{(F)}(F^{(1)}(h)(d), y), b^1)$

for any $b = (b^1, d)$ and y as above. Using Lemma 7.4 (with $m-1 \ge 1$ instead of m) we get $\rho^{(\eta)}(y, b^1) = \rho^{(\eta)}(\alpha_G^{(F)}(c, y), b^1)$ for any $y \in F^{(2)}(G)$, $c \in F^{(1)}(G)$ and $b_1 \in A^{(F)}$.

(b) Using a similar technique to the proof of [KMS, Lemma 42.7], we show that $\rho_y^{(\eta)} : A^{(F)} \to A^{(H)}$ is an algebra homomorphism (i.e. that $(\rho^{(\eta)}, \sigma^{(\eta)})$ satisfies condition (b) in Definition 7.1).

1. *Linearity.* Using the invariance of η with respect to the $\mathcal{PB}_m(G)$-maps $i_1(\tau\,\mathrm{id}_{\mathbb{R}^m}) \times i_2(\mathrm{id}_G) : i_1(\mathbb{R}^m) \times i_2(G) \to i_1(\mathbb{R}^m) \times i_2(G)$ for $\tau > 0$ we get the homogeneity condition $\rho_y^{(\eta)}(\tau a) = \tau \rho_y^{(\eta)}(a)$. Then $\rho_y^{(\eta)} : A^{(F)} \to A^{(H)}$ is linear by the homogeneous function theorem.

2. *Multiplicity.* Let $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ be given by $\varphi(x^1, \dots, x^m) = (x^1 + (x^1)^2, x^2, \dots, x^m)$. Using the invariance of η with respect to the map $i_1(\varphi) \times i_2(\mathrm{id}_G)$ (it is a $\mathcal{PB}_m(G)$-map over some neighbourhood of $0 \in \mathbb{R}^m$) we get $\rho_y^{(\eta)}(a+a^2) = \rho_y^{(\eta)}(a) + (\rho_y^{(\eta)}(a))^2$. Then $\rho_y^{(\eta)}(a^2) = (\rho_y^{(\eta)}(a))^2$ for any $a \in A^{(F)}$. Hence $\rho_y^{(\eta)}(ab) = \rho_y^{(\eta)}(a)\rho_y^{(\eta)}(b)$ for any $a, b \in A^{(F)}$ because of the polarization formula.

3. *Unity preservation.* Using the invariance of η with respect to the $\mathcal{PB}_m(G)$-map $i_1(\tau_{(1,0,\dots,0)}) \times i_2(\mathrm{id}_G) : i_1(\mathbb{R}^m) \times i_2(G) \to i_1(\mathbb{R}^m) \times i_2(G)$, where $\tau_{(1,\dots,0)} : \mathbb{R}^m \to \mathbb{R}^m$ is the translation by $(1, 0, \dots, 0)$, we get $\rho_y^{(\eta)}(1 + a) = 1 + \rho_y^{(\eta)}(a)$. Then $\rho_y^{(\eta)}(1) = 1$.

(c) Using the invariance of η with respect to the $\mathcal{PB}_m(G)$-maps $i_1(\tau\,\mathrm{id}_{\mathbb{R}^m}) \times i_2(G) : i_1(\mathbb{R}^m) \times i_2(G) \to i_1(\mathbb{R}^m) \times i_2(G)$ for $\tau > 0$ we get $H(\mathrm{pr}_2) \circ \eta_{i_1(\mathbb{R}^m)\times i_2(G)}(\tau b, y) = H(\mathrm{pr}_2) \circ \eta_{i_1(\mathbb{R}^m)\times i_2(G)}(b, y)$ for any $b \in F^{(1)}(\mathbb{R}^m)$ and $y \in F^{(2)}(G)$. Putting $\tau \to 0$ we get $\sigma^{(\eta)}(y) = H(\mathrm{pr}_2) \circ \eta_{i_1(\mathbb{R}^m)\times i_2(G)}(b, y)$ for any y and b as above. Then using the invariance of η with respect to the $\mathcal{PB}_m(G)$-morphisms $f : i_1(\mathbb{R}^m) \times i_2(G) \to i_1(\mathbb{R}^m) \times i_2(G)$ of the form $f(x, g) = (x, h(x)g)$, where $h : \mathbb{R}^m \to G$, we obtain

$$\sigma^{(\eta)} \circ \alpha_G^{(F)}(F^{(1)}(h)(b), y) = \alpha_G^{(H)}(\tilde{\rho}_y^{(\eta)}(F^{(1)}(h)(b)), \sigma^{(\eta)}(y))$$

for any b and y as above. Then using Lemma 7.4 we have

$$\sigma^{(\eta)} \circ \alpha_G^{(F)}(d, y) = \alpha_G^{(H)}(\tilde{\rho}_y^{(\eta)}(d), \sigma^{(\eta)}(y))$$

for any $y \in F^{(2)}(G)$ and any d from some subset generating $F^{(1)}(G)$. But (see Introduction) $\tilde{\rho}_y^{(\eta)} : F^{(1)}(G) \to H^{(1)}(G)$ is a Lie group homomorphism because $\tilde{\rho}_y^{(\eta)} : F^{(1)} \to H^{(1)}$ is a natural transformation. Moreover, we have proved the invariance of $\rho^{(\eta)}$. So, the last equality holds for all $d \in F^{(1)}(G)$. ∎

EXAMPLE 7.5. Let (ρ, σ) be an (F, H, G)-admissible pair. Given a principal bundle $P \in \mathrm{Obj}(\mathcal{PB}_m(G))$, we define $\eta_P^{(\rho,\sigma)} : F^{(F^{(1)}, F^{(2)}, \alpha^{(F)})}(P) \to$

$F^{(H^{(1)},H^{(2)},\alpha^{(H)})}(P)$ by

$$\eta_P^{(\rho,\sigma)}([\xi,y]) = [\tilde{\rho}_y(\xi),\sigma(y)]$$

for any $\xi \in F^{(1)}(P)$ and any $y \in F^{(2)}(G)$, where $\tilde{\rho}_y : F^{(1)}(P) \to H^{(1)}(P)$ is the natural transformation corresponding to the algebra homomorphism $\rho_y : A^{(F)} \to A^{(H)}$. Using properties (a)–(c) of Definition 7.1 one can easily show that $\eta_P^{(\rho,\sigma)}$ is well-defined. Clearly, $\eta^{(\rho,\sigma)} = \{\eta_P^{(\rho,\sigma)}\} : F^{(F^{(1)},F^{(2)},\alpha^{(F)})}_{|\mathcal{PB}_m(G)} \to F^{(H^{(1)},H^{(2)},\alpha^{(H)})}_{|\mathcal{PB}_m(G)}$ is a natural transformation.

Summing up we obtain the following classification theorem.

THEOREM 7.6. *Let $F,H : \mathcal{PB} \to \mathcal{FM}$ be ppgb-functors. Let $m \geq 2$ and G be a Lie group. Then the correspondence $\eta \mapsto (\rho^{(\eta)},\sigma^{(\eta)})$ (see Example 7.2) is a bijection between natural transformations $F_{|\mathcal{PB}_m(G)} \to H_{|\mathcal{PB}_m(G)}$ and (F,H,G)-admissible pairs. The inverse bijection is given by $(\rho,\sigma) \to \Theta^H \circ \eta^{(\rho,\sigma)} \circ (\Theta^F)^{-1}$, where $\eta^{(\rho,\sigma)}$ is defined in Example 7.5 and Θ^F is the natural isomorphism from the proof of Lemma 5.1.*

8. Natural transformations between extended Weil functors and between vertical Weil functors. As a simple application of Theorem 5.4 we determine explicitly all natural transformations $T^A \to T^B$ between the extended Weil functors on $\mathcal{PB}$.

COROLLARY 8.1. *Any natural transformation $\eta : T^A \to T^B$ between the extended Weil functors $T^A, T^B : \mathcal{PB} \to \mathcal{FM}$ is of the form*

$$\tilde{\varphi}_P : T^A(P) \to T^B(P), \qquad P \in \mathrm{Obj}(\mathcal{PB}),$$

for some uniquely determined (by η) Weil algebra homomorphism $\varphi : A \to B$, where $\tilde{\varphi}_N : T^A(N) \to T^B(N)$, $N \in \mathrm{Obj}(\mathcal{M}f)$, is the natural transformation of the usual Weil functors on $\mathcal{M}f$ corresponding to φ.

Proof. Clearly, the admissible triple corresponding to $F = T^A : \mathcal{PB} \to \mathcal{FM}$ is $(F^{(1)},F^{(2)},\alpha^{(F)}) = (T^A, T^A_{|\mathcal{G}r}, \mu^A)$, where $T^A : \mathcal{M}f \to \mathcal{FM}$ is the (usual) Weil functor corresponding to A and $\mu^A_G = \mu_{T^A(G)}$ is the multiplication of the Lie group $T^A(G)$, treated (in the obvious way) as an action of $T^A(G)$ on $T^A(G)$. Let $\nu = (\nu^1,\nu^2) : (T^A, T^A_{|\mathcal{G}r}, \mu^A) \to (T^B, T^B_{|\mathcal{G}r}, \mu^B)$ be a morphism of admissible triples. Then (by Definition 3.2 of morphisms of admissible triples) $\mu^B_G(\nu^1_G(g), \nu^2_G(e_{T^A(G)})) = \nu^2_G(\mu^A_G(g, e_{T^A(G)})) = \nu^2_G(g)$ for all $g \in T^A(G)$. For the trivial morphism $\tilde{e}_G : G \to G$ we have $\nu^2_G \circ T^A(\tilde{e}_G)(e_{T^A(G)}) = T^B(\tilde{e}_G) \circ \nu^2_G(e_{T^A(G)})$, and then $\nu^2_G(e_{T^A(G)}) = e_{T^B(G)}$. Hence $\nu^1_G = \nu^2_G$. But $\nu^1 = \tilde{\varphi} : T^A \to T^B$ for some Weil algebra homomorphism $\varphi : A \to B$. Thus the morphisms between the admissible triples $(T^A, T^A_{|\mathcal{G}r}, \mu^A)$ and $(T^B, T^B_{|\mathcal{G}r}, \mu^B)$ corresponding to T^A and T^B are in bijection with the algebra homomorphisms between A and B. By Theorem 5.4, the morphisms between the admissible triples $(T^A, T^A_{|\mathcal{G}r}, \mu^A)$ and $(T^B, T^B_{|\mathcal{G}r}, \mu^B)$ corresponding to T^A and T^B are in bijection with the natural transformations between the ppgb-functors $T^A : \mathcal{PB} \to \mathcal{FM}$ and $T^B : \mathcal{PB} \to \mathcal{FM}$. By the theory of ppb-functors on manifolds, the algebra homomorphisms between A and B are in bijection with the natural transformations of the usual Weil functors $T^A : \mathcal{M}f \to \mathcal{FM}$ and $T^B : \mathcal{M}f \to \mathcal{FM}$. The above facts complete the proof. ∎

As a simple application of Theorem 7.6 we describe explicitly all natural transformations $T^A_{|\mathcal{PB}_m(G)} \to T^B_{|\mathcal{PB}_m(G)}$.

Corollary 8.2. *If $m \geq 2$ and G is a Lie group, then any natural transformation $\eta : T^A_{|\mathcal{PB}_m(G)} \to T^B_{|\mathcal{PB}_m(G)}$ is of the form*

$$R_\xi \circ \tilde{\varphi}_P : T^A(P) \to T^B(P), \qquad P \in \mathrm{Obj}(\mathcal{PB}_m(G)),$$

for a uniquely determined (by η) Weil algebra homomorphism $\varphi : A \to B$ and a uniquely determined (by η) $\xi \in T^B(G)$, where $R : T^B(P) \times T^B(G) \to T^B(P)$ is the right action of $T^B(G)$ on the principal bundle $T^B(p) : T^B(P) \to T^B(M)$.

Proof. The admissible triple corresponding to $F = T^A : \mathcal{PB} \to \mathcal{FM}$ is $(F^{(1)}, F^{(2)}, \alpha^{(F)}) = (T^A, T^A_{|\mathcal{G}r}, \mu^A)$ (as in the proof of Corollary 8.1). Let (ρ, σ) be a (T^A, T^B, G)-admissible pair. From condition (a) in Definition 7.1 for $y = e_{T^A(G)} \in T^A(G) = F^{(2)}(G)$ we get $\rho(c, a) = \rho(e_{T^A(G)}, a)$, $c \in A = A^{(F)}$. From condition (c) in Definition 7.1 for $y = e_{T^A(G)}$ we see that $\sigma(c) = \tilde{\rho}_{e_{T^A(G)}}(c) \cdot \sigma(e_{T^A(G)})$. Then (ρ, σ) is determined by the Weil algebra homomorphism $\rho_{e_{T^A(G)}} : A \to B$ and the value $\sigma(e_{T^A(G)}) \in T^B(G)$. Hence Corollary 8.2 is a simple consequence of Theorem 7.6. ∎

Let $V^A, V^B : \mathcal{PB} \to \mathcal{FM}$ be the vertical Weil functors (Example 2.14) corresponding to the Weil algebras A and B. As a next simple application of Theorem 7.6 we can determine all natural transformations $V^A_{|\mathcal{PB}_m(G)} \to V^B_{|\mathcal{PB}_m(G)}$ as follows.

Example 8.3. Let $k : T^A_{e_G}(G) \to T^B_{e_G}(G)$ be a map. Given a $\mathcal{PB}_m(G)$-object $P \to M$ define $\eta^{[k]}_P : V^A(P) \to V^B(P)$ as follows. Let $v \in T^A_p(P_x)$, $p \in P_x$, $x \in M$. Choose a trivialization $\psi : P_{|U} \to \mathbb{R}^m \times G$ such that $\psi(p) = (0, e_G)$. We put $\eta^{[k]}_P(v) = (T^B(\psi_x))^{-1}(k(T^A(\psi_x)(v)))$, where $\psi_x : P_x \to \{0\} \times G = G$ is the restriction of ψ. If ψ' is another such trivialization then $\psi_x = \psi'_x$. Thus the definition of $\eta^{[k]}_P : V^A(P) \to V^B(P)$ is correct. The correspondence $\eta^{[k]} : V^A_{|\mathcal{PB}_m(G)} \to V^B_{|\mathcal{PB}_m(G)}$ is a natural transformation.

Example 8.4. Let $l : T^A_{e_G}(G) \to G$ be a map. Given a $\mathcal{PB}_m(G)$-object $P \to M$ define $\eta^{(l)}_P : V^A(P) \to V^A(P)$ (now $A = B$) as follows. Let $v \in T^A_p(P_x)$, $p \in P_x$, $x \in M$. Similarly to Example 8.3, choose a trivialization $\psi : P_{|U} \to \mathbb{R}^m \times G$ such that $\psi(p) = (0, e_G)$ and put $\eta^{(l)}_P(v) = (T^A(\psi_x))^{-1}(T^A(r_{l(T^A(\psi_x)(v))})(T^A(\psi_x)(v)))$, where r_g is the right translation on G by $g \in G$. The definition of $\eta^{(l)}_P : V^A(P) \to V^A(P)$ is correct because ψ_x is uniquely determined. In particular, if $l = g \in G$ is a constant map, then $\eta^{(g)}_P = V^A(r_g) : V^A(P) \to V^A(P)$, where V^A is treated as a functor on fibred manifolds (r_g is not a $\mathcal{PB}_m(G)$-map, but it is a fibred map). The correspondence $\eta^{(l)} : V^A_{|\mathcal{PB}_m(G)} \to V^A_{|\mathcal{PB}_m(G)}$ is a natural transformation.

Corollary 8.5. *If $m \geq 2$, then any natural transformation $\eta : V^A_{|\mathcal{PB}_m(G)} \to V^B_{|\mathcal{PB}_m(G)}$ is of the form*

$$\eta_P = \eta^{[k]}_P \circ \eta^{(l)}_P : V^A(P) \to V^B(P)$$

for some uniquely determined (by η) $k : T^A_{e_G}(G) \to T^B_{e_G}(G)$ and some $l : T^A_{e_G}(G) \to G$. In the special case $A = B = \mathbb{D}$, we get a full description of all natural transformations $\eta : V_{|\mathcal{PB}_m(G)} \to V_{|\mathcal{PB}_m(G)}$ in terms of pairs (k, l) of maps $k : \mathcal{L}(G) \to \mathcal{L}(G)$ and $l : \mathcal{L}(G) \to G$.

Proof. The admissible triple corresponding to $F = V^A$ is $(F^{(1)}, F^{(2)}, \alpha^{(F)}) = (T^{\mathbb{R}}, T^A_{|\mathcal{G}r}, \tilde{\mu}^A)$, where $\tilde{\mu}^A_G : G \times T^A(G) \to T^A(G)$, $\tilde{\mu}^A_G(g, \xi) = T^A(L_g)(\xi)$, $L_g : G \to G$, $L_g(g_1) = gg_1$, $g, g_1 \in G$, $\xi \in T^A(G)$. Then any (V^A, V^B, G)-admissible pair (ρ, σ), $\rho : T^A(G) \times \mathbb{R} \to \mathbb{R}$, $\sigma : T^A(G) \to T^B(G)$, is determined by $(l, k) := \sigma_{|T^A_{e_G}(G)} : T^A_{e_G}(G) \to T^B(G) \cong G \times T^B_{e_G}(G)$. Theorem 7.6 ends the proof. ∎

9. Prolongation of principal connections. By definition (see [KMS]), a *general connection* on a fibred manifold $p : Y \to M$ is a section $\Gamma : Y \to J^1(Y)$ of the first jet prolongation $J^1(Y) \to Y$ of $Y \to M$, which can be (equivalently) considered as the corresponding lifting map $\Gamma : Y \times_M T(M) \to T(Y)$. A *principal connection* on a principal bundle $p : P \to M$ is a right invariant general connection Γ on $p : P \to M$.

It is rather well-known that if $\Gamma : P \to J^1P$ is a principal connection on a principal bundle $p : P \to M$ with the Lie group G and $Y = P[S, \mu] \to M$ is the associated bundle to P with a standard fibre S and a left action $\mu : G \times S \to S$, then one can well-define a general connection $\Gamma^{[S,\mu]} : Y \to J^1Y$ on $Y \to M$ by $\Gamma^{[S,\mu]}([q, s]) := \mathrm{j}^1_x([\sigma, s])$, $[q, s] \in P[S, \mu]$, where $\mathrm{j}^1_x(\sigma) = \Gamma(q) \in J^1(P)$ ($\sigma : M \to P$ is a local section near $x = p(q)$).

The following example shows that Theorem 5.3 can be applied to prolongation of principal connections.

EXAMPLE 9.1. Let $F : \mathcal{PB} \to \mathcal{FM}$ be a ppgb-functor. Let $\Gamma : P \to J^1P$ be a principal connection on a principal bundle $p : P \to M$ with the Lie group G. We can construct a general connection $\mathcal{F}(\Gamma)$ on $F(p) : F(P) \to F(M)$ as follows. By Theorem 5.3, we can assume that $F(P) = F_1(P)[F_2(G), \alpha_G]$, where (F_1, F_2, α) is an admissible triple. Now, by the theory of prolongation of principal connections to ppb-functors on $\mathcal{M}f$, we have the principal connection $\mathcal{F}_1(\Gamma)$ on the principal bundle $F_1(p) : F_1(P) \to F_1(M)$ (see Introduction). This principal connection $\mathcal{F}_1(\Gamma)$ induces a general connection $\mathcal{F}(\Gamma) := \mathcal{F}_1(\Gamma)^{[F_2(G),\alpha_G]}$ on the associated bundle $F(p) : F(P) = F_1(P)[F_2(G), \alpha_G] \to F_1(M) = F(M)$.

Clearly, one can also use other principal connections $B(\Gamma)$ on $F_1(P) \to F_1(M)$ canonically depending on Γ instead of $\mathcal{F}_1(\Gamma)$ and obtain general connections $B(\Gamma)^{[S,\alpha_G]}$ on $F(p) : F(P) \to F(M)$.

10. A "reduction" theorem for gauge-natural operators lifting principal connections to Weil bundles. Let A be a Weil algebra, m be a natural number and G be a Lie group. Let $\mathcal{L}(G)$ be the Lie algebra of G. In accordance with the last sentence of Section 9, we try to describe the $\mathcal{PB}_m(G)$-gauge-natural operators $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ (lifting principal connections Γ on $\mathcal{PB}_m(G)$-objects $p : P \to M$ to principal connections $B(\Gamma)$ on $T^A(p) : T^A(P) \to T^A(M)$). More precisely, in this section we reduce the classification of all $\mathcal{PB}_m(G)$-gauge-natural operators $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ to the classification of all $\mathcal{M}f_m$-natural operators $C : T^* \otimes \mathcal{L}(G)_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$ (lifting $\mathcal{L}(G)$-valued 1-forms $\omega \in \Omega^1(M, \mathcal{L}(G))$ on m-manifolds M to $\mathcal{L}(T^A(G))$-valued 1-forms $C(\omega) \in \Omega^1(T^A(M), \mathcal{L}(T^A(G)))$ on $T^A(M)$) satisfying the so-called Ad-invariance condition.

We start with the following two definitions (particular cases of the general definition of (gauge) natural operators in the sense of [KMS]).

DEFINITION 10.1. Let A, m, G be as above. A $\mathcal{PB}_m(G)$-*gauge-natural operator* $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ is a $\mathcal{PB}_m(G)$-invariant family of regular operators (functions)

$$B_P : \mathrm{Con}_{\mathrm{princ}}(P) \to \mathrm{Con}_{\mathrm{princ}}(T^A(P))$$

for any $\mathcal{PB}_m(G)$-object $p : P \to M$, where $\mathrm{Con}_{\mathrm{princ}}(P)$ is the set of all principal connections on $p : P \to M$ and (similarly) $\mathrm{Con}_{\mathrm{princ}}(T^A(P))$ is the set of all principal connections on $T^A(p) : T^A(P) \to T^A(M)$. The $\mathcal{PB}_m(G)$-invariance of B means that if principal connections $\Gamma \in \mathrm{Con}_{\mathrm{princ}}(P)$ and $\Gamma_1 \in \mathrm{Con}_{\mathrm{princ}}(P_1)$ are f-related by a $\mathcal{PB}_m(G)$-map $f : P \to P_1$ (i.e. $\Gamma_1 \circ f = J^1(f) \circ \Gamma$), then the principal connections $B_P(\Gamma)$ and $B_{P_1}(\Gamma_1)$ are $T^A(f)$-related. The regularity means that B transforms smoothly parametrized families of principal connections into smoothly parametrized ones. A $\mathcal{PB}_m(G)$-gauge-natural operator $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ is *affine* if B_P is an affine map for any $\mathcal{PB}_m(G)$-object $p : P \to M$ (i.e. $B_P(t\Gamma + (1-t)\Gamma_1) = tB_P(\Gamma) + (1-t)B_P(\Gamma_1)$ for any $\Gamma, \Gamma_1 \in \mathrm{Con}_{\mathrm{princ}}(P)$ and any $t \in \mathbb{R}$).

For example, the family $\mathcal{T}^A : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ given by $\mathcal{T}^A_P(\Gamma) = \mathcal{T}^A(\Gamma)$, the principal connection as in Introduction for $F = T^A$, is an affine $\mathcal{PB}_m(G)$-gauge-natural operator.

The space of all $\mathcal{PB}_m(G)$-gauge-natural operators $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ is (in the obvious way) an affine space. Actually, for $\mathcal{PB}_m(G)$-gauge-natural operators $B, B^1 : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ and a real number $t \in \mathbb{R}$, the $\mathcal{PB}_m(G)$-gauge-natural operator $tB + (1-t)B^1 : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ is given by $(tB + (1-t)B^1)_P(\Gamma) = tB_P(\Gamma) + (1-t)B^1_P(\Gamma)$ for any $\Gamma \in \mathrm{Con}_{\mathrm{princ}}(P)$ and any $\mathcal{PB}_m(G)$-object $p : P \to M$.

Definition 10.2. Let m be a natural number, and V and W be finite-dimensional real vector spaces. An *$\mathcal{M}f_m$-natural operator* $C : T^* \otimes W_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes V)T^A$ is an $\mathcal{M}f_m$-invariant family of regular operators (functions)

$$C_M : \Omega^1(M, W) \to \Omega^1(T^A(M), V)$$

for any m-manifold (i.e. any $\mathcal{M}f_m$-object) M, where $\Omega^1(M, W)$ is the space of all W-valued 1-forms on M. The $\mathcal{M}f_m$-invariance and the regularity mean almost the same as in Definition 10.1. An $\mathcal{M}f_m$-natural operator $C : T^* \otimes W_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes V)T^A$ is *linear* if C_M is an $\mathbb{R}$-linear map for any $\mathcal{M}f_m$-object M.

The space of all $\mathcal{M}f_m$-natural operators $C : T^* \otimes W_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes V)T^A$ is (in the obvious way) a vector space over $\mathbb{R}$. A full description of all $\mathcal{M}f_m$-natural operators $C : T^* \otimes W_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes V)T^A$ can be found in Appendix.

We need the following definition.

Definition 10.3. Let A, m, G be as in Definition 10.1. We say that an $\mathcal{M}f_m$-natural operator $C : T^* \otimes \mathcal{L}(G)_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$ (as in Definition 10.2 for $W = \mathcal{L}(G)$ and $V = \mathcal{L}(T^A(G))$) satisfies the Ad-*invariance condition* if

$$C_M(\mathrm{Ad}_{h^{-1}} \cdot \omega + h^*\Theta_G) = \mathrm{Ad}_{T^A(h)^{-1}} \cdot C_M(\omega) + T^A(h)^*\Theta_{T^A(G)}$$

for any map $h : M \to G$ and any $\omega \in \Omega^1(M, \mathcal{L}(G))$, where Θ_G is the Maurer–Cartan form of G, Ad denotes the adjoint representation and (of course) $h^*\Theta$ is the pull-back of Θ_G with respect to h. Moreover, $\mathrm{Ad}_{h^{-1}} \cdot \omega \in \Omega^1(M, \mathcal{L}(G))$ is defined by $(\mathrm{Ad}_{h^{-1}} \cdot \omega)_x(X) = \mathrm{Ad}_{h(x)^{-1}}(\omega_x(X))$ for $X \in T_x(M)$, $x \in M$. We say that C satisfies the *reduced* Ad-*invariance condition* if

$$C_M(\mathrm{Ad}_{h^{-1}} \cdot \omega + h^*\Theta_G) = \mathrm{Ad}_{T^A(h)^{-1}} \cdot C_M(\omega)$$

for any $\omega \in \Omega^1(M, \mathcal{L}(G))$, $h : M \to G$, $M \in \mathrm{Obj}(\mathcal{M}f_m)$.

LEMMA 10.4. *If G is commutative, then the reduced* Ad*-invariance condition is equivalent to*

$$C_M(\omega + df) = C_M(\omega)$$

for any $\omega \in \Omega^1(M, \mathcal{L}(G))$, $f : M \to \mathcal{L}(G)$, $M \in \mathrm{Obj}(\mathcal{M}f_m)$.

Proof. The lemma is true for $G = (\mathbb{R}^n, +)$ because in this case $\Theta_G = (dx^1, \dots, dx^n)$ (so $h^*\Theta_G = dh$) and the adjoint representations Ad are trivial. Consequently, the above fact is true for any commutative Lie group of dimension n because commutative Lie groups of dimension n are locally Lie group isomorphic to $(\mathbb{R}^n, +)$ and $\mathcal{M}f_m$-natural operators are local. ∎

The space of $\mathcal{M}f_m$-natural operators $C : T^* \otimes \mathcal{L}(G)_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$ satisfying the Ad-invariance condition is (in the obvious way) an affine space over the vector space of $\mathcal{M}f_m$-natural operators $C : T^* \otimes \mathcal{L}(G)_{\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$ satisfying the reduced Ad-invariance condition.

The following example shows that a $\mathcal{PB}_m(G)$-gauge-natural operator $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ induces an $\mathcal{M}f_m$-natural operator $C^B : T^* \otimes \mathcal{L}(G)_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$ satisfying the Ad-invariance condition.

EXAMPLE 10.5. Let $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ be a $\mathcal{PB}_m(G)$-gauge-natural operator. Given an $\mathcal{L}(G)$-valued 1-form $\omega \in \Omega^1(M, \mathcal{L}(G))$ on an m-manifold M we define a $\mathcal{L}(T^A(G))$-valued 1-form $C^B_M(\omega) \in \Omega^1(T^AM, \mathcal{L}(T^A(G)))$ on $T^A(M)$ as follows. Let $\sigma_G : M \to M \times G$ be the section $x \mapsto (x, e_G)$ of the trivial $\mathcal{PB}_m(G)$-object $M \times G \to M$. There exists a unique principal connection Γ on the trivial principal bundle $M \times G \to M$ such that $\sigma^*_G\omega_\Gamma = \omega$, where $\omega_\Gamma : T(M \times G) \to \mathcal{L}(G)$ is the connection form of Γ. We put $C^B_M(\omega) := T^A(\sigma_G)^*\omega_{B_{M\times G}(\Gamma)}$. The family $C^B = \{C^B_M\} : T^* \otimes \mathcal{L}(G)_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$ of functions $C^B_M : \Omega^1(M, \mathcal{L}(G)) \to \Omega^1(T^A(M), \mathcal{L}(T^A(G)))$ for any m-manifold M is an $\mathcal{M}f_m$-natural operator.

LEMMA 10.6. *The $\mathcal{M}f_m$-natural operator C^B satisfies the* Ad*-invariance condition.*

Proof. Consider a vector-valued form $\omega \in \Omega^1(M, \mathcal{L}(G))$ and a map $h : M \to G$. Let Γ be the unique principal connection on $M \times G \to M$ such that $\sigma^*_G\omega_\Gamma = \omega$, where (as above) $\sigma_G : M \to M \times G$ is defined by $\sigma_G(x) = (x, e_G)$ and ω_Γ is the connection form of Γ. Let $\sigma_h : M \to M \times G$ be a section of $M \times G \to M$ defined by $\sigma_h = \sigma_G \cdot h$. Then according to the general theory of principal connections (see [KN])

$$\sigma^*_h\omega_\Gamma = \mathrm{Ad}_{h^{-1}} \cdot \sigma^*_G\omega_\Gamma + h^*\Theta_G, \tag{10.1}$$

where Θ_G is the Maurer–Cartan 1-form on G. Let $\Psi_h : M \times G \to M \times G$ be defined by $\Psi_h(x, g) = (x, h(x) \cdot g)$. Let Γ_h be the image of Γ under Ψ^{-1}_h.

Then (as is readily seen)

$$\sigma_G^*\omega_{\Gamma_h} = \sigma_h^*\omega_\Gamma. \tag{10.2}$$

Using Proposition 6.1 we easily deduce that $T^A(\sigma_G) = \sigma_{T^A(G)} : T^A(M) \to T^A(M)\times T^A(G)$ (i.e. $T^A(\sigma_G)(\tilde{x}) = (\tilde{x}, e_{T^A(G)})$), $T^A(\sigma_h) = \sigma_{T^A(h)} : T^A(M) \to T^A(M) \times T^A(G)$ (i.e. $T^A(\sigma_h) = T^A(\sigma_G) \cdot T^A(h) = \sigma_{T^A(G)} \cdot T^A(h)$) and $T^A(\Psi_h) = \Psi_{T^A(h)} : T^A(M)\times T^A(G) \to T^A(M)\times T^A(G)$ (i.e. $T^A(\Psi_h)(\tilde{x}, \tilde{g}) = (\tilde{x}, T^A(h)(\tilde{x}) \cdot \tilde{g})$). By the invariance of B with respect to Ψ_h, $B_{M\times G}(\Gamma_h)$ is the image of $B_{M\times G}(\Gamma)$ under $T^A(\Psi_h)^{-1}$. Then (just as (10.1) and (10.2)) we have

$$T^A(\sigma_h)^*\omega_{B_{M\times G}(\Gamma)} = \mathrm{Ad}_{T^A(h)^{-1}} \cdot T^A(\sigma_G)^*\omega_{B_{M\times G}(\Gamma)} + T^A(h)^*\Theta_{T^A(G)} \tag{10.3}$$

(where $\omega_{B_{M\times G}(\Gamma)}$ is the connection form of $B_{M\times G}(\Gamma)$) and

$$T^A(\sigma_G)^*\omega_{B_{M\times G}(\Gamma_h)} = T^A(\sigma_h)^*\omega_{B_{M\times G}(\Gamma)}. \tag{10.4}$$

By the definition of C^B (see Example 10.5) we immediately get

$$C^B_M(\sigma_G^*\omega_{\Gamma_h}) = T^A(\sigma_G)^*\omega_{B_{M\times G}(\Gamma_h)}. \tag{10.5}$$

Consequently,

$$\begin{aligned}
C^B_M(\mathrm{Ad}_{h^{-1}} \cdot \omega + h^*\Theta_G) &= C^B_M(\sigma_h^*\omega_\Gamma) = C^B_M(\sigma_G^*\omega_{\Gamma_h}) \\
&= T^A(\sigma_G)^*\omega_{B_{M\times G}(\Gamma_h)} = T^A(\sigma_h)^*\omega_{B_{M\times G}(\Gamma)} \\
&= \mathrm{Ad}_{T^A(h)^{-1}} \cdot T^A(\sigma_G)^*\omega_{B_{M\times G}(\Gamma)} + T^A(h)^*\Theta_{T^A(G)} \\
&= \mathrm{Ad}_{T^A(h)^{-1}} \cdot C^B_M(\omega) + T^A(h)^*\Theta_{T^A(G)}. \ \blacksquare
\end{aligned}$$

Conversely, we have the following construction.

Example 10.7. Let $C : T^* \otimes \mathcal{L}(G)_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$ be an $\mathcal{M}f_m$-natural operator satisfying the Ad-invariance condition. Let Γ be a principal connection on a $\mathcal{PB}_m(G)$-object $p : P \to M$, and let $\omega_\Gamma : TP \to \mathcal{L}(G)$ be its connection form. We define a principal connection $B^C_P(\Gamma)$ on $T^A(p) : T^A(P) \to T^A(M)$ by

$$T^A(\sigma)^*\omega_{B^C_P(\Gamma)} = C_M(\sigma^*\omega_\Gamma)$$

for any (local) section $\sigma : M \to P$. If $\sigma_1 = \sigma \cdot h$ is another local section, where $h : M \to G$ is a (local) map, then $\sigma_1^*\omega_\Gamma = \mathrm{Ad}_{h^{-1}} \cdot \sigma^*\omega_\Gamma + h^*\Theta_G$. Consequently, using the Ad-invariance condition of C we get

$$T^A(\sigma_1)^*\omega_{B^C_P(\Gamma)} = \mathrm{Ad}_{T^A(h)^{-1}} \cdot T^A(\sigma)^*\omega_{B^C_P(\Gamma)} + T^A(h)^*\Theta_{T^AG}.$$

Therefore the definition of $B^C_P(\Gamma)$ is correct. Then for any $\mathcal{PB}_m(G)$-object P we have $B^C_P : \mathrm{Con}_{\mathrm{princ}}(P) \to \mathrm{Con}_{\mathrm{princ}}(T^A(P))$. The family $B^C = \{B^C_P\} : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ is a $\mathcal{PB}_m(G)$-gauge-natural operator.

Summing up, we obtain the following "reduction" theorem.

THEOREM 10.8. *Let G be a Lie group, A be a Weil algebra and m be a natural number. The correspondence $B \mapsto C^B$ is an affine isomorphism between the affine space of $\mathcal{PB}_m(G)$-gauge-natural operators $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ and the affine space of $\mathcal{M}f_m$-natural operators $C : T^* \otimes \mathcal{L}(G)_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$ satisfying the* Ad*-invariance condition* (*the inverse isomorphism is given by the correspondence $C \to B^C$*). *In other words, the correspondence $B \mapsto C^B - C^{\mathcal{T}^A}$ is a bijection between the $\mathcal{PB}_m(G)$-gauge-natural operators $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ and the $\mathcal{M}f_m$-natural operators $C : T^* \otimes \mathcal{L}(G)_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$ satisfying the reduced* Ad*-invariance condition. This bijection restricts to the one between the affine $\mathcal{PB}_m(G)$-gauge-natural operators $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ and the linear $\mathcal{M}f_m$-natural operators $C : T^* \otimes \mathcal{L}(G) \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$ satisfying the reduced* Ad*-invariance condition.*

For a commutative Lie group G we have the following corollary of the above reduction theorem.

COROLLARY 10.9. *Let G, A and m be as in the above theorem. If G is commutative, then the $\mathcal{PB}_m(G)$-gauge-natural operators $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ are in bijection with the $\mathcal{M}f_m$-natural operators $C : T^* \otimes \mathcal{L}(G)_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$ satisfying the condition $C_M(\omega + df) = C_M(\omega)$ for any $\omega \in \Omega^1(M, \mathcal{L}(G))$, $f : M \to \mathcal{L}(G)$ and $M \in \mathrm{Obj}(\mathcal{M}f_m)$. Moreover, if G is commutative, then the affine $\mathcal{PB}_m(G)$-gauge-natural operators $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$ are in bijection with the linear $\mathcal{M}f_m$-natural operators $C : T^* \otimes \mathcal{L}(G)_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$ satisfying the condition $C_M(df) = 0$ for any $f : M \to \mathcal{L}(G)$ and $M \in \mathrm{Obj}(\mathcal{M}f_m)$.*

11. An estimate. The following estimate shows that if $A = \mathbb{R} \oplus N_A$ is a Weil algebra with $\mathrm{width}(A) := \dim(N_A/N_A^2) \geq 1$, m is an integer with $m \geq \mathrm{width}(A) + 2$ and G is a commutative Lie group with $\dim(G) \geq 1$, then there exist many affine $\mathcal{PB}_m(G)$-gauge-natural operators $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$.

THEOREM 11.1. *Let A be a Weil algebra with* $\mathrm{width}(A) = p$ *and G be a commutative Lie group. If $m \geq p + 2$, then*

$$\dim(\mathrm{Op}(A, G, m)) \geq p \cdot (\dim(G))^2 \cdot \dim(A),$$

where $\mathrm{Op}(A, G, m)$ *denotes the affine space of all affine $\mathcal{PB}_m(G)$-gauge-natural operators $B : Q_{|\mathcal{PB}_m(G)} \rightsquigarrow Q(T^A \to T^A\mathcal{B})$.*

Proof. Let Op^1 be the vector space of all $\mathcal{M}f_m$-natural linear operators $C : T^* \otimes \mathcal{L}(G)_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$ and Op^2 be the vector space of all $\mathcal{M}f_m$-natural linear operators $D : T^{(0,0)} \otimes \mathcal{L}(G)_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(T^A(G)))T^A$

lifting maps $f : M \to \mathcal{L}(G)$ into $\mathcal{L}(T^A(G))$-valued 1-forms on $T^A(M)$ (the definition of $\mathcal{M}f_m$-natural operators $D : T^{(0,0)} \otimes W_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes V)T^A$ is similar to Definition 10.2). By Lemma 12.3 (see Appendix),

$$\dim(Q^{A,\mathcal{L}(G)}) \geq p \cdot \dim(G),$$

where $Q^{A,V}$ is the space from Definition 12.2. From Corollaries 12.15 and 13.12 we immediately get

$$\dim(\mathrm{Op}^1) = (\dim(Q^{A,\mathcal{L}(G)}) + \dim(A) \cdot \dim(G)) \cdot \dim(A) \cdot \dim(G),$$
$$\dim(\mathrm{Op}^2) = (\dim(A) \cdot \dim(G))^2.$$

We can define a linear map $\Phi : \mathrm{Op}^1 \to \mathrm{Op}^2$ by $\Phi(C)_M(f) = C_M(df)$ for any $C = \{C_M\} \in \mathrm{Op}^1$, $M \in \mathrm{Obj}(\mathcal{M}f_m)$, $f : M \to \mathcal{L}(G)$, where d is the usual exterior differentiation. The kernel $\ker(\Phi)$ is exactly the vector space of all linear natural operators $C : T^* \otimes \mathcal{L}(G)_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes \mathcal{L}(G) \otimes A)T^A$ with $C_M(df) = 0$ for any $f : M \to \mathcal{L}(G)$. Then

$$\dim(\mathrm{Op}(A, G, m)) = \dim(\ker(\Phi))$$

by the last sentence of Corollary 10.9. Therefore

$$\begin{aligned}\dim(\mathrm{Op}(A, G, m)) &= \dim(\ker(\Phi)) \geq \dim(\mathrm{Op}^1) - \dim(\mathrm{Op}^2)\\ &= \dim(Q^{A,\mathcal{L}(G)}) \cdot \dim(A) \cdot \dim(G) \geq p \cdot (\dim(G))^2 \cdot \dim(A)\end{aligned}$$

by obvious linear algebra. ∎

APPENDIX

In this Appendix (which consists of Sections 12 and 13), we generalize the results from [Mi2]. More precisely, given a ppb-functor $H : \mathcal{M}f \to \mathcal{FM}$ and finite-dimensional real vector spaces V and W we classify all $\mathcal{M}f_m$-natural operators $T^* \otimes V_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes W)H$ and $T^{(0,0)} \otimes V_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes W)H$ lifting V-valued 1-forms or V-valued maps on m-manifolds M to W-valued 1-forms on $H(M)$ (for $V = W = \mathbb{R}$ we recover the result from [Mi2]). Without loss of generality we assume that $H = J_A : \mathcal{M}f \to \mathcal{FM}$ is the Weil functor of A-velocities in the sense of A. Morimoto [Mo], where $A = \mathcal{C}^\infty_0(\mathbb{R}^p)/\underline{A}$ is a Weil algebra with $\mathrm{width}(A) = p$ (here $\mathcal{C}^\infty_0(\mathbb{R}^p)$ is the local algebra of germs at 0 of maps $\mathbb{R}^p \to \mathbb{R}$ with the maximal ideal $\underline{m}$ and $\underline{A}$ is a finite codimension ideal in $\mathcal{C}^\infty_0(\mathbb{R}^p)$, i.e. such that $\underline{A} \supset \underline{m}^{r+1}$ for some finite r). The equality $\mathrm{width}(A) = p$ is equivalent to the inclusion $\underline{m}^2 \supset \underline{A}$. The precise definition of J_A can also be found in [KMS]. The content of Sections 12 and 13 is (in fact) a suitably modified and extended material from [Mi2].

12. The $\mathcal{M}f_m$-natural operators $T^* \otimes V_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes W)J_A$. We have the following example of $\mathcal{M}f_m$-natural operators $T^* \otimes V_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes W)J_A$ (in the sense of Definition 10.2).

EXAMPLE 12.1 (*(λ)-lift*). Let $\lambda : V \otimes A \to W$ be an $\mathbb{R}$-linear map. Consider a V-valued 1-form $\omega \in \Omega^1(M, V)$ on an m-manifold M. Let $\omega : T(M) \to V$ be the fibre linear map corresponding to ω. Let $\eta_M : T(J_A(M)) \to J_A(T(M))$ be the exchange equivalence. Define $\omega^{(\lambda)} : T(J_A(M)) \to W$ by

$$\omega^{(\lambda)} := \lambda \circ J_A(\omega) \circ \eta_M, \tag{12.1}$$

where the identification $J_A(V) = V \otimes A$ is the usual one. It is clear that $\alpha \cdot \frac{d}{dt}\big|_{t=0}(j_A(\gamma(t, \cdot))) = \frac{d}{dt}\big|_{t=0}(j_A(\gamma(\alpha t, \cdot)))$ for any $\gamma : \mathbb{R} \times \mathbb{R}^p \to M$ and any $\alpha \in \mathbb{R}$, where $\cdot$ is the fibre multiplication of the tangent bundle $T(J_A(M))$. Using this fact one can easily show that $\omega^{(\lambda)}$ is homogeneous on each fibre of the tangent bundle $T(J_A(M))$. Hence it is linear on each fibre because of the homogeneous function theorem. Therefore $\omega^{(\lambda)}$ is a W-valued 1-form on $J_A(M)$. We put $B^{(\lambda)}_M(\omega) = \omega^{(\lambda)}$. The family $B^{(\lambda)} : T^* \otimes V_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes W)J_A$ of functions $B^{(\lambda)}_M : \Omega^1(M, V) \to \Omega^1(J_A(M), W)$ for m-manifolds M is an $\mathcal{M}f_m$-natural operator.

To present the next example we need the following preparation.

DEFINITION 12.2. Let $A = \mathcal{C}^\infty_0(\mathbb{R}^p)/\underline{A}$, V and W be as above. Denote by $\Omega^1_0(\mathbb{R}^p, V)$ the $\mathcal{C}^\infty_0(\mathbb{R}^p)$-module of all germs at 0 of V-valued 1-forms on $\mathbb{R}^p$. Let

$$Q^{A,V} = \Omega^1_0(\mathbb{R}^p, V)/(\underline{A} \cdot \Omega^1_0(\mathbb{R}^p, V) + \mathcal{C}^\infty_0(\mathbb{R}^p, V) \cdot d\underline{A}) \tag{12.2}$$

be the factor module, where $\underline{A} \cdot \Omega^1_0(\mathbb{R}^p, V)$ is the product of $\Omega^1_0(\mathbb{R}^p, V)$ by $\underline{A}$, $\mathcal{C}^\infty_0(\mathbb{R}^p, V) \cdot d\underline{A}$ is the submodule in $\Omega^1_0(\mathbb{R}^p, V)$ spanned by $d\eta \otimes v$ for all $\eta \in \underline{A}$ (df denotes the differential of f) and $v \in V$, and $\underline{A} \cdot \Omega^1_0(\mathbb{R}^p, V) + \mathcal{C}^\infty_0(\mathbb{R}^p, V) \cdot d\underline{A} \subset \Omega^1_0(\mathbb{R}^p, V)$ is the algebraic sum of the modules $\underline{A} \cdot \Omega^1_0(\mathbb{R}^p, V)$ and $\mathcal{C}^\infty_0(\mathbb{R}^p, V) \cdot d\underline{A}$. Given a V-valued 1-form ω on $\mathbb{R}^p$, the equivalence class of $\mathrm{germ}_0(\omega)$ modulo $\underline{A} \cdot \Omega^1_0(\mathbb{R}^p, V) + \mathcal{C}^\infty_0(\mathbb{R}^p, V) \cdot d\underline{A}$ will be denoted by $[\omega]_A$, i.e.

$$[\omega]_A = \mathrm{germ}_0(\omega)_{\mathrm{mod}(\underline{A} \cdot \Omega^1_0(\mathbb{R}^p, V) + \mathcal{C}^\infty_0(\mathbb{R}^p, V) \cdot d\underline{A})} \in Q^{A,V}. \tag{12.3}$$

We will keep this notation throughout the rest of Appendix.

LEMMA 12.3. *$Q^{A,V}$ is a finite-dimensional vector space over $\mathbb{R}$. Moreover,* $\dim(Q^{A,V}) \geq p \cdot \dim(V)$.

Proof. It is a simple observation that $Q^{A,V}$ is finite-dimensional over $\mathbb{R}$. The inequality holds because we have the linear epimorphism $Q^{A,V} \to T^*_0\mathbb{R}^p \otimes V$ given by $[\omega]_A \mapsto \omega(0)$ (it is well-defined as $\underline{m}^2 \supset \underline{A}$). ∎

EXAMPLE 12.4 (*$\langle\varphi\rangle$-lift*). Let $\varphi : Q^{A,V} \to W$ be an $\mathbb{R}$-linear map. Let ω be a V-valued 1-form on an m-manifold M. Define $\omega^{[\varphi]} : J_A(M) \to W$ by

$$\omega^{[\varphi]}(j_A(\gamma)) = \varphi([\gamma^*\omega]_A) \tag{12.4}$$

for any $\gamma : \mathbb{R}^p \to M$, where $\gamma^*\omega$ is the pull-back of ω with respect to γ. If $\eta : \mathbb{R}^p \to M$ is another map such that $j_A(\gamma) = j_A(\eta)$, then $[\gamma^*\omega]_A = [\eta^*\omega]_A$. To see this one can assume that $M = \mathbb{R}^m$ and $\eta(0) = \gamma(0) = 0$. Let $\omega = \sum_{i=1}^m a_i dx^i$, $\eta = (\eta^1, \dots, \eta^m)$, $\gamma = (\gamma^1, \dots, \gamma^m)$, $a_i : \mathbb{R}^m \to V$. Then $\mathrm{germ}_0(\eta^j - \gamma^j) \in \underline{A}$ and $\mathrm{germ}_0(a_i \circ \gamma - a_i \circ \eta) \in \mathcal{C}^\infty_0(\mathbb{R}^p, V)\underline{A}$ for any $i, j = 1, \dots, m$. So

$$\mathrm{germ}_0(\gamma^*\omega - \eta^*\omega) = \sum_{i=1}^m \mathrm{germ}_0((a_i \circ \gamma - a_i \circ \eta)d(\gamma^i)) + \sum_{i=1}^m \mathrm{germ}_0((a_i \circ \eta)d(\gamma^i - \eta^i)) \in \underline{A} \cdot \Omega^1_0(\mathbb{R}^p, V) + \mathcal{C}^\infty_0(\mathbb{R}^p, V) \cdot d\underline{A},$$

as well. Therefore $\omega^{[\varphi]}$ is well-defined. Define

$$\omega^{\langle\varphi\rangle} = d(\omega^{[\varphi]}), \tag{12.5}$$

the differential of $\omega^{[\varphi]}$. We put $B^{\langle\varphi\rangle}_M(\omega) = \omega^{\langle\varphi\rangle}$. The family $B^{\langle\varphi\rangle} : T^* \otimes V_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes W)J_A$ of functions $B^{\langle\varphi\rangle}_M : \Omega^1(M, V) \to \Omega^1(J_A(M), W)$ for m-manifolds M is an $\mathcal{M}f_m$-natural operator.

Lemma 12.5. *The set $\mathcal{T}(A, V, W, m)$ of all $\mathcal{M}f_m$-natural operators $B : T^* \otimes V_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes W)J_A$ is a $\mathcal{C}^\infty(Q^{A,V})$-module.*

Proof. For any $B, C \in \mathcal{T}(A, V, W, m)$ and $f, g \in \mathcal{C}^\infty(Q^{A,V})$ we define

$$(fB + gC)_M(\omega)(j_A(\gamma)) = f([\gamma^*\omega]_A) \cdot B_M(\omega)(j_A(\gamma)) + g([\gamma^*\omega]_A) \cdot C_M(\omega)(j_A(\gamma)), \tag{12.6}$$

where $\omega \in \Omega^1(M, V)$, $\gamma : \mathbb{R}^p \to M$, $M \in \mathrm{Obj}(\mathcal{M}f_m)$. ∎

The main result of this section is the following classification theorem.

Theorem 12.6. *Let m be a natural number and $A = \mathcal{C}^\infty_0(\mathbb{R}^p)/\underline{A}$ be a Weil algebra with* $\mathrm{width}(A) = p$. *Let V and W be finite-dimensional vector spaces over $\mathbb{R}$. Let $q_1, \dots, q_K$ be a basis of the vector space $Q^{A,V}$, $u_1, \dots, u_L$ be a basis of $V \otimes A$ and $w_1, \dots, w_Q$ be a basis of W. Let $\varphi_{ij} = q^*_i \otimes w_j$ for $i = 1, \dots, K$ and $j = 1, \dots, Q$ be the corresponding basis of the vector space* $\mathrm{Hom}(Q^{A,V}, W)$, *and $\lambda_{kl} = u^*_k \otimes w_l$ for $k = 1, \dots, L$ and $l = 1, \dots, Q$ be the corresponding basis of the vector space* $\mathrm{Hom}(V \otimes A, W)$. *If $m \ge p + 2$, then the $\mathcal{M}f_m$-natural operators (described in Examples* 12.4 *and* 12.1) *$B^{\langle\varphi_{ij}\rangle}$ and $B^{(\lambda_{kl})}$ for $i = 1, \dots, K$, $k = 1, \dots, L$ and $j, l = 1, \dots, Q$ form a basis of the $\mathcal{C}^\infty(Q^{A,V})$-module $\mathcal{T}(A, V, W, m)$ of all $\mathcal{M}f_m$-natural operators $T^* \otimes V_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes W)J_A$.*

The proof of Theorem 12.6 will occupy the rest of this section.

Definition 12.7. Let $t^1, \dots, t^p$ be the coordinates on $\mathbb{R}^p$ and $x^1, \dots, x^m$ be the coordinates on $\mathbb{R}^m$. If $m \geq p+1$, let $e := j_A(t^1, \dots, t^p, 0, \dots, 0) \in J_A(\mathbb{R}^m)$. Given $B \in \mathcal{T}(A, V, W, m)$ we define $\Phi_B : \Omega^1(\mathbb{R}^m, V) \to W$ by

$$\Phi_B(\omega) := \left\langle (B_{\mathbb{R}^m}(\omega))(e), \mathcal{J}_A\left(\frac{\partial}{\partial x^m}\right)(e)\right\rangle, \tag{12.7}$$

where $\mathcal{J}_A(X)$ is the flow lift of a vector field X on M to $J_A(M)$.

Using the invariance of B we obtain

Lemma 12.8. *If $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ is an embedding preserving $\mathcal{J}_A(\frac{\partial}{\partial x^m})(e)$, then $\Phi_B(\omega) = \Phi_B(\varphi^*\omega)$ for any $B \in \mathcal{T}(A, V, W, m)$ and any V-valued 1-form ω on $\mathbb{R}^m$.*

The main part of the proof of Theorem 12.6 is to show the following proposition.

Proposition 12.9. *Assume $m \geq p+2$. For any $B \in \mathcal{T}(A, V, W, m)$ there exists a (well-defined) map $G_B : Q^{A,V} \times Q^{A,V} \times V \otimes A \to W$ such that*

$$G_B([\omega_0]_A, [\omega_1]_A, j_A(H)) = \Phi_B(q^*(\omega_0) + x^m q^*(\omega_1) + (H \circ q)dx^m) \tag{12.8}$$

for any V-valued 1-forms ω_0, ω_1 on $\mathbb{R}^p$ and any map $H : \mathbb{R}^p \to V$, where $q : \mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^{m-p} \to \mathbb{R}^p$ is the usual projection, q^* denotes the pull-back with respect to q and Φ_B is as in Definition 12.7. The function $G : \mathcal{T}(A, V, W, m) \to \mathcal{C}^\infty(Q^{A,V} \times Q^{A,V} \times V \otimes A, W)$ given by $G(B) = G_B$ is a monomorphism of $\mathcal{C}^\infty(Q^{A,V})$-modules, provided the $\mathcal{C}^\infty(Q^{A,V})$-module structure in $\mathcal{C}^\infty(Q^{A,V} \times Q^{A,V} \times V \otimes A, W)$ is given by $(\lambda f)(a, b, c) = \lambda(a) f(a, b, c)$, where $\lambda \in \mathcal{C}^\infty(Q^{A,V})$, $f \in \mathcal{C}^\infty(Q^{A,V} \times Q^{A,V} \times V \otimes A, W)$ and $(a, b, c) \in Q^{A,V} \times Q^{A,V} \times V \otimes A$.

To prove the proposition we need some lemmas. We start with

Lemma 12.10. *Let $B \in \mathcal{T}(A, V, W, m)$. Assume that $m \geq p+1$ and $\Phi_B = 0$. Then $B = 0$.*

Proof. It is clear that $\{(x^1, \dots, x^{m-1}, x^m + t)\}_{t \in \mathbb{R}}$ is the flow of $\frac{\partial}{\partial x^m}$. Then $\{J_A(x^1, \dots, x^{m-1}, x^m + t)\}_{t \in \mathbb{R}}$ is the flow of $\mathcal{J}_A(\frac{\partial}{\partial x^m})$. Then $\mathcal{J}_A(\frac{\partial}{\partial x^m})(e)$ is the velocity at $0 \in \mathbb{R}$ of the curve $t \mapsto j_A(t^1, \dots, t^p, 0, \dots, 0, t)$, i.e.

$$\mathcal{J}_A\left(\frac{\partial}{\partial x^m}\right)(e) = \frac{d}{dt}\bigg|_{t=0}(j_A(t^1, \dots, t^p, 0, \dots, 0, t)).$$

An arbitrary element of $T(J_A(\mathbb{R}^m))$ is of the form $\frac{d}{dt}|_{t=0}(j_A(\gamma(t, \cdot)))$ for some $\gamma : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^m$. If γ is of maximal rank at $0 \in \mathbb{R}^{p+1}$, then (by the rank theorem) there is an embedding $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ such that $(\varphi \circ \gamma)(t, \tau) = (\tau, 0, t) \in \mathbb{R}^m$ for any (t, τ) in some neighbourhood of $(0, 0) \in \mathbb{R} \times \mathbb{R}^p$. Consequently, the $\mathcal{M}f_m$-orbit of $\mathcal{J}_A(\frac{\partial}{\partial x_m})(e)$ is dense in $T(J_A(\mathbb{R}^m))$. Then, using the invariance of B and the assumption $\Phi_B = 0$, we derive that $\langle B_{\mathbb{R}^m}(\omega), v\rangle = 0$ for any V-valued 1-form ω on $\mathbb{R}^m$ and any v in some dense subset in $T(J_A(\mathbb{R}^m))$. Then $B_{\mathbb{R}^m}(\omega)=0$ for any V-valued 1-form ω on $\mathbb{R}^m$, i.e. $B_{\mathbb{R}^m}=0$. Hence $B=0$, as any natural operator B is uniquely determined by $B_{\mathbb{R}^m}$. ∎

Using Lemma 12.10 we prove

LEMMA 12.11. *Let $B\in\mathcal{T}(A,V,W,m)$. Assume that $m\geq p+1$ and $\Phi_B(\omega)=0$ for any $\omega\in\Omega^1(\mathbb{R}^m,V)$ of the form*

$$\omega=(f_1\circ(x^1,\dots,x^p,x^m))dx^1+\dots+(f_p\circ(x^1,\dots,x^p,x^m))dx^p + (g\circ(x^1,\dots,x^p,x^m))dx^m, \tag{12.9}$$

where $f_1,\dots,f_p,g:\mathbb{R}^{p+1}\to V$ are some maps. Then $B=0$.

Proof. Let ω be a V-valued 1-form on $\mathbb{R}^m$. The embedding $\varphi_t=(x^1,\dots,x^p, tx^{p+1},\dots,tx^{m-1},x^m):\mathbb{R}^m\to\mathbb{R}^m$, $t\neq 0$, preserves e and $\frac{\partial}{\partial x^m}$. Then, by Lemma 12.8 for φ_t, we get $\Phi_B((\varphi_t)^*\omega)=\Phi_B(\omega)$ for any $t\neq 0$. If $t\to 0$, we obtain

$$\Phi_B(\omega)=\Phi_B((\varphi_0)^*\omega)=0$$

because $(\varphi_0)^*\omega$ is of the form (12.9). Now, the lemma is a consequence of Lemma 12.10. ∎

LEMMA 12.12. *If $m\geq p+2$, then $\Phi_B(\omega)=\Phi_B(\omega+hdx^{p+1})$ for any natural operator $B\in\mathcal{T}(A,V,W,m)$, any V-valued 1-form ω on $\mathbb{R}^m$ and any map $h:\mathbb{R}^m\to V$.*

Proof. Let $\varphi_0=(x^1,\dots,x^p,0,\dots,0,x^m):\mathbb{R}^m\to\mathbb{R}^m$. It is clear that $(\varphi_0)^*\omega=(\varphi_0)^*(\omega+hdx^{p+1})$ (as $m\geq p+2$). Now, by the proof of Lemma 12.11, we have

$$\Phi_B(\omega)=\Phi_B((\varphi_0)^*\omega)=\Phi_B((\varphi_0)^*(\omega+hdx^{p+1}))=\Phi_B(\omega+hdx^{p+1}).\ \blacksquare$$

LEMMA 12.13. *Let $B\in\mathcal{T}(A,V,W,m)$. Assume that $m\geq p+1$ and $\Phi_B(\tilde\omega)=0$ for any $\tilde\omega\in\Omega^1(\mathbb{R}^m,V)$ of the form*

$$\tilde\omega=q^*(\omega_0)+x^mq^*(\omega_1)+(H\circ q)dx^m, \tag{12.10}$$

where ω_0,ω_1 are V-valued 1-forms on $\mathbb{R}^m$, $H:\mathbb{R}^p\to V$ and $q:\mathbb{R}^m=\mathbb{R}^p\times\mathbb{R}^{m-p}\to\mathbb{R}^p$ is the projection. Then $B=0$.

Proof. Let ω be the V-valued 1-form as in formula (12.9). By Lemma 12.11 it is sufficient to show that $\Phi_B(\omega)=0$. By [KMS, Corollary 19.8] (of the non-linear Peetre theorem) for the $(\pi:J^A(\mathbb{R}^m)\to\mathbb{R}^m)$-local operator $D=\left\langle B_{\mathbb{R}^m},\mathcal{J}_A\left(\frac{\partial}{\partial x^m}\right)\right\rangle:\mathcal{C}^\infty(\mathbb{R}^m,T^*(\mathbb{R}^m)\otimes V)\supset\Omega^1(\mathbb{R}^m,V)\to\mathcal{C}^\infty(J_A(\mathbb{R}^m),W)$ with Whitney-extendible domain $E=\Omega^1(\mathbb{R}^m,V)$, $f=\omega\in\Omega^1(\mathbb{R}^m,V)$ and a compact set $K=\{e\}\subset J_A(\mathbb{R}^m)$, there is a natural number $r=r(\omega)$ such that

$$\Phi_B(\overline{\omega})=\Phi_B(\omega)$$

for any V-valued 1-form $\overline{\omega}$ on $\mathbb{R}^m$ with $\mathrm{j}_0^r(\overline{\omega}) = \mathrm{j}_0^r(\omega)$. So, one can assume that ω is as in (12.9) with $f_1, \dots, f_p, g : \mathbb{R}^{p+1} \to V$ being polynomials of degree $\leq r$. Denote by Φ_B^r the restriction of Φ_B to the finite-dimensional vector space of all forms as in (12.9) with $f_1, \dots, f_p, g$ being polynomials of degree $\leq r$. Since B satisfies the regularity condition, Φ_B^r is smooth. Using the invariance of B with respect to the embedding $\eta_t = (x^1, \dots, x^{m-1}, tx^m) : \mathbb{R}^m \to \mathbb{R}^m$, $t \neq 0$, preserving e and sending $\frac{\partial}{\partial x^m}$ to $t\frac{\partial}{\partial x^m}$ we deduce (as in the proof of Lemma 12.11) that

$$t\Phi_B^r(\omega) = \Phi_B^r(\eta_t^*\omega) = \Phi_B^r\big((f_1 \circ (x^1, \dots, x^p, tx^n))dx^1 + \cdots$$
$$+ (f_p \circ (x^1, \dots, x^p, tx^m))dx^p + t(g \circ (x^1, \dots, x^p, tx^m))dx^m\big).$$

Differentiating both sides of this formula and of a similar formula with

$$\tilde{\omega} = \left((f_1 \circ (x^1, \dots, x^p, 0)) + \left(\frac{\partial f_1}{\partial x^m} \circ (x^1, \dots, x^p, 0)\right)x^m\right)dx^1 + \cdots$$
$$+ \left((f_p \circ (x^1, \dots, x^p, 0)) + \left(\frac{\partial f_p}{\partial x^m} \circ (x^1, \dots, x^p, 0)\right)x^m\right)dx^p$$
$$+ (g \circ (x^1, \dots, x^p, 0))dx^m$$

instead of ω with respect to t and then putting $t = 0$ we deduce that $\Phi_B^r(\omega) = \Phi_B^r(\tilde{\omega})$, i.e. $\Phi_B(\omega) = \Phi_B(\tilde{\omega})$. Of course, $\tilde{\omega}$ is as in (12.10). Thus $\Phi_B(\tilde{\omega}) = 0$. So, $\Phi_B(\omega) = 0$. ∎

Lemma 12.14. *Let $B \in \mathcal{T}(A, V, W, n)$. Let $\omega_0, \omega_1, \overline{\omega}_0, \overline{\omega}_1$ be V-valued 1-forms on $\mathbb{R}^p$ and $H, \overline{H} : \mathbb{R}^p \to V$ be mappings such that*

$$j_A(H) = j_A(\overline{H}), \qquad [\omega_0]_A = [\overline{\omega}_0]_A, \qquad [\omega_1]_A = [\overline{\omega}_1]_A,$$

where $[\ ]_A$ is as in Definition 12.2. Write $\omega = q^*(\omega_0) + x^m q^*(\omega_1) + (H \circ q)dx^m$ and $\overline{\omega} = q^*(\overline{\omega}_0) + x^m q^*(\overline{\omega}_1) + (\overline{H} \circ q)dx^m$, where $q : \mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^{m-p} \to \mathbb{R}^p$ is the projection. If $m \geq p + 2$, then $\Phi_B(\omega) = \Phi_B(\overline{\omega})$.

Proof. The proof will be completed once we show that

$$\Phi_B(\omega) = \Phi_B(\omega + q^*(F d\eta)), \tag{12.11}$$
$$\Phi_B(\omega) = \Phi_B(\omega + x^m q^*(F d\eta)), \tag{12.12}$$
$$\Phi_B(\omega) = \Phi_B(\omega + ((\eta G) \circ q)dx^j), \tag{12.13}$$
$$\Phi_B(\omega) = \Phi_B(\omega + ((\eta G) \circ q)dx^m), \tag{12.14}$$
$$\Phi_B(\omega) = \Phi_B(\omega + ((\eta G) \circ q)x^m dx^j) \tag{12.15}$$

for any V-valued 1-forms ω_0, ω_1 on $\mathbb{R}^p$, $j \in \{1, \dots, p\}$ and maps $H, F, G : \mathbb{R}^p \to V$, $\eta : \mathbb{R}^p \to \mathbb{R}$ with $\mathrm{germ}_0(\eta) \in \underline{A}$ and $G = \mathrm{const}$, where $\omega = q^*(\omega_0) + x^m q^*(\omega_1) + (H \circ q)dx^m$.

Let $\psi_1 = (x^1, \dots, x^p, x^{p+1} + (\eta \circ q), x^{p+2}, \dots, x^m) : \mathbb{R}^m \to \mathbb{R}^m$. This is a diffeomorphism. Of course, it preserves e and $\frac{\partial}{\partial x^m}$. Then using Lemma 12.12

and Lemma 12.8 for ψ_1 in place of φ and again Lemma 12.12, we have

$$\begin{aligned}\Phi_B(\omega) &= \Phi_B(\omega + (F\circ q)dx^{p+1}) = \Phi_B(\psi_1^*(\omega + (F\circ q)dx^{p+1}))\\ &= \Phi_B(\omega + q^*(Fd\eta) + (F\circ q)dx^{p+1}) = \Phi_B(\omega + q^*(Fd\eta)).\end{aligned}$$

Formula (12.11) is verified.

Replacing $F\circ q$ by $(F\circ q)x^m$ in the proof of (12.11) we obtain (12.12).

Let $\psi_2 = (x^1,\dots,x^p,x^{p+1}+(\eta\circ q)x^j,x^{p+2},\dots,x^m):\mathbb{R}^m\to\mathbb{R}^m$. This is a diffeomorphism. It preserves e and $\frac{\partial}{\partial x^m}$. Then using Lemma 12.12, Lemma 12.8 for ψ_2 in place of φ, Lemma 12.12 and formula (12.11) with $\omega+(G\eta\circ q)dx^j$ playing the role of ω, we have

$$\begin{aligned}\Phi_B(\omega) &= \Phi_B(\omega + Gdx^{p+1}) = \Phi_B(\psi_2^*(\omega + Gdx^{p+1}))\\ &= \Phi_B(\omega + (G\eta\circ q)dx^j + Gx^jq^*(d\eta) + Gdx^{p+1})\\ &= \Phi_B(\omega + (G\eta\circ q)dx^j + Gx^jq^*(d\eta))\\ &= \Phi_B(\omega + (G\eta\circ q)dx^j + q^*(t^jGd\eta)) = \Phi_B(\omega + (G\eta\circ q)dx^j).\end{aligned}$$

Formula (12.13) is proved.

Let $\psi_3 = (x^1,\dots,x^p,x^{p+1}+(\eta\circ q)x^m,x^{p+2},\dots,x^m):\mathbb{R}^m\to\mathbb{R}^m$. This is a diffeomorphism that preserves $\mathcal{J}_A\big(\frac{\partial}{\partial x^m}\big)(e) = \frac{d}{dt}\big|_{t=0}(j_A(t^1,\dots,t^p,0,\dots,0,t))$. Then using Lemma 12.12, Lemma 12.8 for ψ_3 in place of φ, Lemma 12.12 and formula (12.12) with $\omega+((G\eta)\circ q)dx^m$ playing the role of ω, we have

$$\begin{aligned}\Phi_B(\omega) &= \Phi_B(\omega + Gdx^{p+1}) = \Phi_B(\psi_3^*(\omega + Gdx^{p+1}))\\ &= \Phi_B(\omega + x^mq^*(Gd\eta) + ((G\eta)\circ q)dx^m + Gdx^{p+1})\\ &= \Phi_B(\omega + ((G\eta)\circ q)dx^m + x^mq^*(Gd\eta))\\ &= \Phi_B(\omega + ((G\eta)\circ q)dx^m).\end{aligned}$$

Formula (12.14) is proved.

Let $\psi_4 = (x^1,\dots,x^p,x^{p+1}+(\eta\circ q)x^jx^m,x^{p+2},\dots,x^m):\mathbb{R}^m\to\mathbb{R}^m$. This is a diffeomorphism that preserves $\mathcal{J}_A\big(\frac{\partial}{\partial x^m}\big)(e)$. Using Lemma 12.12, Lemma 12.8 for ψ_4 in place of φ, Lemma 12.12, formula (12.14) with $\omega+((G\eta)\circ q)x^mdx^j+x^mx^jq^*(Gd\eta)$ playing the role of ω and with $t^j\eta$ playing the role of η and formula (12.12) with $\omega+((G\eta)\circ q)x^mdx^j$ playing the role of ω, we have

$$\begin{aligned}\Phi_B(\omega) &= \Phi_B(\omega + Gdx^{p+1}) = \Phi_B(\psi_4^*(\omega + Gdx^{p+1}))\\ &= \Phi_B(\omega + ((G\eta)\circ q)x^jdx^m + x^jx^mq^*(Gd\eta)\\ &\quad + ((G\eta)\circ q)x^mdx^j + Gdx^{p+1})\\ &= \Phi_B(\omega + x^jx^mq^*(Gd\eta) + ((G\eta)\circ q)x^mdx^j + ((G\eta)\circ q)x^jdx^m)\\ &= \Phi_B(\omega + x^jx^mq^*(Gd\eta) + ((G\eta)\circ q)x^mdx^j + ((Gt^j\eta)\circ q)dx^m)\\ &= \Phi_B(\omega + ((G\eta)\circ q)x^mdx^j + x^jx^mq^*(Gd\eta))\end{aligned}$$

$$= \Phi_B(\omega + ((G\eta)\circ q)x^m dx^j + x^m q^*(Gt^j d\eta))$$
$$= \Phi_B(\omega + ((G\eta)\circ q)x^m dx^j).$$

The proof of the lemma is complete. ∎

Proof of Proposition 12.9. By Lemma 12.14, G_B is well-defined. It is smooth because of the regularity condition on B. Directly from the definitions of the module structures it is easy to verify that G is a homomorphism of $\mathcal{C}^\infty(Q^{A,V})$-modules. By Lemma 12.13, G is injective. ∎

Using Proposition 12.9 we can prove Theorem 12.6 as follows.

Proof of Theorem 2.6. We fix bases of the vector spaces $Q^{A,V}$, W and $J_A(V) = V\otimes A$. Let $B \in \mathcal{T}(A,V,W,m)$. Let $G_B : Q^{A,V}\times Q^{A,V}\times V\otimes A \to W$ be as in Proposition 12.9. From the invariance of B with respect to the diffeomorphisms $(x^1,\ldots,x^{m-1},tx^m) : \mathbb{R}^m \to \mathbb{R}^m$ for $t\neq 0$ it follows that

$$tG_B(a,b,c) = G_B(a,tb,tc) \tag{12.16}$$

for any $(a,b,c) \in Q^{A,V}\times Q^{A,V}\times V\otimes A$ and any $t\in\mathbb{R}\setminus\{0\}$ (because each diffeomorphism $(x^1,\ldots,x^{m-1},tx^m)$ preserves e and sends $\frac{\partial}{\partial x^m}$ to $t\frac{\partial}{\partial x^m}$). Then, by the homogeneous function theorem, G_B is a linear combination of the coordinates of b and c with respect to the bases with coefficients being $\mathcal{C}^\infty$-maps $Q^{A,V}\to W$ depending on a. Thus owing to Proposition 12.9 we see that the proof of Theorem 12.6 will be complete once we show that

$$G_{B\langle\lambda\rangle}(a,b,c) = \lambda(c) \quad\text{and}\quad G_{B\langle\varphi\rangle}(a,b,c) = \varphi(b) \tag{12.17}$$

for any $\lambda\in\mathrm{Hom}(V\otimes A,W)$ and any $\varphi\in\mathrm{Hom}(Q^{A,V},W)$.

To prove (12.17) write $i = (t^1,\ldots,t^p,0) : \mathbb{R}^p \to \mathbb{R}^m$ and let $q = (x^1,\ldots,x^p) : \mathbb{R}^m = \mathbb{R}^p\times\mathbb{R}^{m-p} \to \mathbb{R}^p$ be the usual projection. Let ω_0,ω_1 be V-valued 1-forms on $\mathbb{R}^p$ and let $H:\mathbb{R}^p\to V$ be a mapping. Let $\lambda\in\mathrm{Hom}(V\otimes A,W)$ and $\varphi\in\mathrm{Hom}(Q^{A,V},W)$. We have

$$\begin{aligned}
&G_{B\langle\lambda\rangle}([\omega_0]_A,[\omega_1]_A,j_A(H))\\
&\quad= \left\langle B^{\langle\lambda\rangle}_{\mathbb{R}^m}(q^*(\omega_0)+x^m q^*(\omega_1)+(H\circ q)dx^m)(e), \mathcal{J}_A\left(\frac{\partial}{\partial x^m}\right)(e)\right\rangle\\
&\quad= (\lambda\circ J_A(q^*(\omega_0)+x^m q^*(\omega_1)+(H\circ q)dx^m)\circ\eta_{\mathbb{R}^m})\left(J_A\left(\frac{\partial}{\partial x^m}\right)(e)\right)\\
&\quad= \left(\lambda\circ J_A\left(\left\langle q^*(\omega_0)+x^m q^*(\omega_1)+(H\circ q)dx^m, \frac{\partial}{\partial x^m}\right\rangle\right)\right)(e) = \lambda(j_A(H)).
\end{aligned}$$

Similarly, since $\gamma_t = (x^1,\ldots,x^{m-1},x^m+t)$ is the flow of $\frac{\partial}{\partial x^m}$ and $e = j_A(i)$,

$$\begin{aligned}
&G_{B^{\langle\varphi\rangle}}([\omega_0]_A,[\omega_1]_A,j_A(H))\\
&\quad=\left\langle B^{\langle\varphi\rangle}_{\mathbb{R}^m}(q^*(\omega_0)+x^mq^*(\omega_1)+(H\circ q)dx^m),\mathcal{J}_A\left(\frac{\partial}{\partial x^m}\right)(e)\right\rangle\\
&\quad=\left\langle d((q^*(\omega_0)+x^mq^*(\omega_1)+(H\circ q)dx^m)^{[\varphi]}),\mathcal{J}_A\left(\frac{\partial}{\partial x^m}\right)(e)\right\rangle\\
&\quad=\frac{d}{dt}\bigg|_{t=0}((q^*(\omega_0)+x^mq^*(\omega_1)+(H\circ q)dx^m)^{[\varphi]}(j_A(\gamma_t\circ i)))\\
&\quad=\frac{d}{dt}\bigg|_{t=0}(\varphi([(\gamma_t\circ i)^*(q^*(\omega_0)+x^mq^*(\omega_1)+(H\circ q)dx^m)]_A))\\
&\quad=\frac{d}{dt}\bigg|_{t=0}(\varphi([\omega_0+t\omega_1]_A))=\varphi([\omega_1]_A).\ \blacksquare
\end{aligned}$$

COROLLARY 12.15. *Let m be a natural number and $A=\mathcal{C}^\infty_0(\mathbb{R}^p)/\underline{A}$ be a Weil algebra with* $\mathrm{width}(A)=p$. *Let V and W be finite-dimensional vector spaces over $\mathbb{R}$. Let $q_1,\ldots,q_K$ be a basis of the vector space $(Q^{A,V})^*$, $u_1,\ldots,u_L$ be a basis of $V\otimes A$ and $w_1,\ldots,w_Q$ be a basis of W. Let $\varphi_{ij}=q^*_i\otimes w_j$ for $i=1,\ldots,K$ and $j=1,\ldots,Q$ be the corresponding basis of* $\mathrm{Hom}(Q^{A,V},W)$, *and $\lambda_{kl}=u^*_k\otimes w_l$ for $k=1,\ldots,L$ and $l=1,\ldots,Q$ be the corresponding basis of* $\mathrm{Hom}(V\otimes A,W)$. *If $m\geq p+2$, then the linear $\mathcal{M}f_m$-natural operators $B^{\langle\varphi_{ij}\rangle}$ and $B^{(\lambda_{kl})}$ for i,j,k,l as above form a basis* (*over* $\mathbb{R}$) *of the vector space of all linear $\mathcal{M}f_m$-natural operators $T^*\otimes V_{|\mathcal{M}f_m}\rightsquigarrow(T^*\otimes W)J_A$.*

Proof. Let $B:T^*\otimes V\rightsquigarrow(T^*\otimes W)J_A$ be a linear $\mathcal{M}f_m$-natural operator. By Theorem 12.6 we can write $B=\sum f_{ij}B^{\langle\varphi_{ij}\rangle}+\sum g_{kl}B^{(\lambda_{kl})}$ for some uniquely determined maps $f_{ij},g_{kl}:Q^{A,V}\to\mathbb{R}$. Using the linearity of B we have $B_M(t\omega)=tB_M(\omega)$ for any $\omega\in\Omega^1(M,V)$ and any $t\in\mathbb{R}$. This gives the homogeneity conditions $f_{ij}(tu)=f_{ij}(u)$ and $g_{kl}(tu)=g_{kl}(u)$ for any $u=[\gamma^*\omega]_A\in Q^{A,V}$ and any $t\in\mathbb{R}$. Then $f_{ij}=\mathrm{const}$ and $g_{kl}=\mathrm{const}$. ∎

13. The $\mathcal{M}f_m$-natural operators $T^{(0,0)}\otimes V_{|\mathcal{M}f_m}\rightsquigarrow(T^*\otimes W)J_A$. We have the following $\mathcal{M}f_m$-natural operators $T^{(0,0)}\otimes V_{|\mathcal{M}f_m}\rightsquigarrow(T^*\otimes W)J_A$.

EXAMPLE 13.1 ((λ)-lift). Let $\lambda:V\otimes A\to W$ be an $\mathbb{R}$-linear map. Consider a map $f:M\to V$ on an m-manifold M. Define $f^{(\lambda)}:J_A(M)\to W$ by

$$f^{(\lambda)}:=\lambda\circ J_A(f),\tag{13.1}$$

where the identification $J_A(V)=V\otimes A$ is the usual one. Let

$$B^{(\lambda)}(f)=df^{(\lambda)}.\tag{13.2}$$

The family $B^{(\lambda)}:T^{(0,0)}\otimes V_{|\mathcal{M}f_m}\rightsquigarrow(T^*\otimes W)J_A$ of functions $B^{(\lambda)}_M:\mathcal{C}^\infty(M,V)\to\Omega^1(J_A(M),W)$ for m-manifolds M is a natural operator.

LEMMA 13.2. *The set* $\mathcal{T}_o(A,V,W,m)$ *of all natural operators* $B : T^{(0,0)} \otimes V_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes W)J_A$ *is a* $\mathcal{C}^\infty(A \otimes V)$*-module.*

Proof. For any $B, C \in \mathcal{T}_o(A,V,W,m)$ and $g, h \in \mathcal{C}^\infty(A \otimes V)$ we define

$$(gB + hC)_M(f)(v) = g(J_A(f)(v)) \cdot B_M(f)(v) + h(J_A(f)(v)) \cdot C_M(f)(v), \tag{13.3}$$

where $f : M \to V$, $v \in J_A(M)$, $M \in \mathrm{Obj}(\mathcal{M}f_m)$. ∎

The main result of this section is the following classification theorem.

THEOREM 13.3. *Let* m *be a natural number and* $A = \mathcal{C}^\infty_0(\mathbb{R}^p)/\underline{A}$ *be a Weil algebra with* $\mathrm{width}(A) = p$. *Let* V *and* W *be finite-dimensional vector spaces over* $\mathbb{R}$. *Let* $u_1, \ldots, u_L$ *be a basis of* $V \otimes A$ *and* $w_1, \ldots, w_Q$ *be a basis of* W. *Let* $\lambda_{kl} = u_k^* \otimes w_l$ *for* $k = 1, \ldots, L$ *and* $l = 1, \ldots, Q$ *be the corresponding basis of the vector space* $\mathrm{Hom}(V \otimes A, W)$. *If* $m \geq p + 2$, *then the* $\mathcal{M}f_m$*-natural operators* (*described in Example* 13.1) $B^{(\lambda_{kl})}$ *for* $k = 1, \ldots, L$ *and* $l = 1, \ldots, Q$ *form a basis of the* $\mathcal{C}^\infty(V \otimes A)$*-module* $\mathcal{T}_o(A,V,W,m)$ *of all* $\mathcal{M}f_m$*-natural operators* $T^{(0,0)} \otimes V_{|\mathcal{M}f_m} \rightsquigarrow (T^* \otimes W)J_A$.

The proof of Theorem 13.3, which is a modification of the one of Theorem 12.6, will occupy the rest of this section.

DEFINITION 13.4. Let $t^1, \ldots, t^p$ be the coordinates on $\mathbb{R}^p$ and $x^1, \ldots, x^m$ be the coordinates on $\mathbb{R}^m$. If $m \geq p + 1$, let $e := j_A(t^1, \ldots, t^p, 0, \ldots, 0) \in J_A(\mathbb{R}^m)$. Given $B \in \mathcal{T}_o(A,V,W,m)$ we define $\Phi_B : \mathcal{C}^\infty(\mathbb{R}^m, V) \to W$ by

$$\Phi_B(f) := \left\langle (B_{\mathbb{R}^m}(f))(e), \mathcal{J}_A\left(\frac{\partial}{\partial x^m}\right)(e) \right\rangle, \tag{13.4}$$

where $\mathcal{J}_A(X)$ is the flow lift of a vector field X on M to $J_A(M)$.

Using the invariance of B we obtain

LEMMA 13.5. *If* $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ *is an embedding preserving* $\mathcal{J}_A\big(\frac{\partial}{\partial x^m}\big)(e)$, *then* $\Phi_B(f) = \Phi_B(\varphi^* f)$ *for any* $B \in \mathcal{T}_o(A,V,W,m)$ *and any* $f : \mathbb{R}^m \to V$.

The main part of the proof of Theorem 13.3 is to show the following proposition.

PROPOSITION 13.6. *Assume* $m \geq p + 2$. *For any* $B \in \mathcal{T}_o(A,V,W,m)$ *there exists a (well-defined) map* $G_B : (V \otimes A) \times (V \otimes A) \to W$ *such that*

$$G_B(j_A(h), j_A(H)) = \Phi_B(h \circ q + (H \circ q)x^m) \tag{13.5}$$

for any maps $h, H : \mathbb{R}^p \to V$, *where* $q : \mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^{m-p} \to \mathbb{R}^p$ *is the usual projection and* Φ_B *is as in Definition* 13.4. *The function* $G : \mathcal{T}_o(A,V,W,m) \to \mathcal{C}^\infty((V \otimes A) \times (V \otimes A), W)$ *given by* $G(B) = G_B$ *is a monomorphism of* $\mathcal{C}^\infty(V \otimes A)$*-modules, provided the* $\mathcal{C}^\infty(V \otimes A)$*-module structure in* $\mathcal{C}^\infty((V \otimes A) \times (V \otimes A), W)$ *is given by* $(\lambda f)(a,b) = \lambda(a) f(a,b)$, *where* $\lambda \in \mathcal{C}^\infty(V \otimes A)$, $f \in \mathcal{C}^\infty((V \otimes A) \times (V \otimes A), W)$ *and* $(a,b) \in (V \otimes A) \times (V \otimes A)$.

To prove the proposition we need some lemmas.

LEMMA 13.7. *Let $B \in \mathcal{T}_o(A,V,W,m)$. Assume that $m \geq p+1$. If $\Phi_B = 0$, then $B = 0$.*

Proof. The proof is similar to the one of Lemma 12.10. ∎

LEMMA 13.8. *Let $B \in \mathcal{T}_o(A,V,W,m)$. Assume that $m \geq p+1$. If $\Phi_B(f) = 0$ for any $f : \mathbb{R}^m \to V$ of the form*

$$f = h \circ (x^1, \ldots, x^p, x^m), \tag{13.6}$$

where $h : \mathbb{R}^{p+1} \to V$, then $B = 0$.

Proof. The proof is similar to the one of Lemma 12.11. We use Lemma 13.7 instead of Lemma 12.10. ∎

LEMMA 13.9. *If $m \geq p+2$, then $\Phi_B(f) = \Phi_B(f + Fx^{p+1})$ for any natural operator $B \in \mathcal{T}_o(A,V,W,m)$ and any $f, F : \mathbb{R}^m \to V$.*

Proof. The proof is similar to the one of Lemma 12.12. ∎

LEMMA 13.10. *Let $B \in \mathcal{T}_o(A,V,W,m)$. Assume that $m \geq p+1$. If $\Phi_B(\tilde{f}) = 0$ for any $\tilde{f} : \mathbb{R}^m \to V$ of the form*

$$\tilde{f} = h \circ q + x^m (H \circ q), \tag{13.7}$$

where $h, H : \mathbb{R}^p \to V$ and $q : \mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^{m-p} \to \mathbb{R}^p$ is the projection, then $B = 0$.

Proof. The proof is similar to the one of Lemma 12.13. We use Lemma 13.8 instead of Lemma 12.11. ∎

LEMMA 13.11. *Let $B \in \mathcal{T}_o(A,V,W,n)$. Let $h, \overline{h}, H, \overline{H} : \mathbb{R}^p \to V$ be mappings such that*

$$j_A(h) = j_A(\overline{h}), \qquad j_A(H) = j_A(\overline{H}).$$

Define $f = h \circ q + x^m(H \circ q)$ and $\overline{f} = \overline{h} \circ q + x^m(\overline{H} \circ q)$, where $q : \mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^{m-p} \to \mathbb{R}^p$ is the projection. If $m \geq p+2$, then $\Phi_B(f) = \Phi_B(\overline{f})$.

Proof. The proof is similar to the one of Lemma 12.14. It suffices to show that

$$\Phi_B(f) = \Phi_B(f + G\eta \circ q), \tag{13.8}$$

$$\Phi_B(f) = \Phi_B(f + Gx^m(\eta \circ q)), \tag{13.9}$$

for all $h, H : \mathbb{R}^p \to V$ and $\eta : \mathbb{R}^p \to \mathbb{R}$ with $\mathrm{germ}_0(\eta) \in \underline{A}$ and $G = \mathrm{const} \in V$, where $f = h \circ q + x^m(H \circ q)$.

Let $\psi_1 = (x^1, \ldots, x^p, x^{p+1} + (\eta \circ q), x^{p+2}, \ldots, x^m) : \mathbb{R}^m \to \mathbb{R}^m$. This is a diffeomorphism that preserves e and $\frac{\partial}{\partial x^m}$. Using Lemma 13.9 and Lemma

13.5 for ψ_1 in place of φ and again Lemma 13.9, we have

$$\begin{aligned}\Phi_B(f) &= \Phi_B(f + Gx^{p+1}) = \Phi_B(\psi_1^*(f + Gx^{p+1})) \\ &= \Phi_B(f + G\eta\circ q + Gx^{p+1}) = \Phi_B(f + G\eta\circ q).\end{aligned}$$

Formula (13.8) is verified. Replacing G by Gx^m in the proof of (13.8) we obtain (13.9). ∎

Proof of Proposition 13.6. By Lemma 13.11, G_B is well-defined. It is smooth because of the regularity condition on B. Directly from the definitions of the module structures it is easy to verify that G is a homomorphism of $\mathcal{C}^\infty(V\otimes A)$-modules. From Lemma 13.10 it follows that G is injective. ∎

Proof of Theorem 13.3. We fix bases of the vector spaces $V\otimes A$ and W. Let $B\in\mathcal{T}_o(A,V,W,m)$. Let $G_B:(V\otimes A)\times(V\otimes A)\to W$ be as in Proposition 13.6. From the invariance of B with respect to the diffeomorphisms $(x^1,\dots,x^{m-1},tx^m):\mathbb{R}^m\to\mathbb{R}^m$ for $t\neq 0$ it follows that

$$G_B(a,tb) = tG_B(a,b) \tag{13.10}$$

for any $(a,b)\in(V\otimes A)\times(V\otimes A)$ and any $t\in\mathbb{R}\setminus\{0\}$ (because each diffeomorphism $(x^1,\dots,x^{m-1},tx^m)$ preserves e and sends $\frac{\partial}{\partial x^m}$ to $t\frac{\partial}{\partial x^m}$). Then, by the homogeneous function theorem, G_B is a linear combination of the coordinates of b with respect to the bases, with coefficients being $\mathcal{C}^\infty$-maps $V\otimes A\to W$ depending on a. Thus owing to Proposition 13.6 it suffices to show that

$$G_{B^{(\lambda)}}(a,b) = \lambda(b) \tag{13.11}$$

for any $\lambda\in\mathrm{Hom}(V\otimes A,W)$. But (13.11) can be proved just as (12.17). ∎

COROLLARY 13.12. *Let V and W be finite-dimensional vector spaces over $\mathbb{R}$, let m be a natural number and let $A=\mathcal{C}_0^\infty(\mathbb{R}^m)/\underline{A}$ be a Weil algebra with* $\mathrm{width}(A)=p$. *Let $u_1,\dots,u_L$ be a basis of the vector space $V\otimes A$ and $w_1,\dots,w_Q$ be a basis of W. Let $\lambda_{kl}=u_k^*\otimes w_l$ for $k=1,\dots,L$ and $l=1,\dots,Q$ be the corresponding basis of* $\mathrm{Hom}(V\otimes A,W)$. *If $m\geq p+2$, then the linear natural operators $B^{(\lambda_{kl})}$ for k,l as above form a basis of the vector space of all linear $\mathcal{M}f_m$-natural operators $T^{(0,0)}\otimes V_{|\mathcal{M}f_m}\to(T^*\otimes W)J_A$.*

Proof. The proof is similar to the one of Corollary 12.15. ∎

REMARK 13.13. (a) In [D1], J. Dębecki essentially generalized the linear part of results from [Mi2] and obtained a full description of linear liftings of p-forms to q-forms on Weil bundles for almost all non-negative integers p and q. So, Corollaries 12.15 and 13.12 for $V=W=\mathbb{R}$ also recover the results from [D1] for $(p,q)=(1,1)$ and $(p,q)=(0,1)$.

(b) Corollaries 12.15 and 13.12 for $V=W=\mathbb{R}$ (or the results from [D1] for $(p,q)=(1,1)$ and $(p,q)=(0,1)$) imply Corollaries 12.15 and 13.13 because (for example) any linear $\mathcal{M}f_m$-natural operator $B:T^*\otimes\mathbb{R}^K_{|\mathcal{M}f_m}\rightsquigarrow$

$(T^* \otimes \mathbb{R}^L)T^A$ is determined by the system of linear $\mathcal{M}f_m$-natural operators $B^{kl} : T^*_{|\mathcal{M}f_m} \rightsquigarrow T^*T^A$ given by $(0,\ldots,B^{kl}_M(\omega),\ldots,0) := B_M(0,\ldots,\omega,\ldots,0)$, with $B^{kl}_M(\omega)$ at position l and ω at position k, $k = 1,\ldots,K$, $l = 1,\ldots,L$.

(c) Theorems 12.6 and 13.3 are not consequences of the same theorems for $V = W = \mathbb{R}$ (or the results from [Mi2]) because a trick similar to that in (b) is not available for arbitrary (not necessarily linear) $\mathcal{M}f_m$-natural operators.

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Włodzimierz M. Mikulski
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: Wlodzimierz.Mikulski@im.uj.edu.pl

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