Projections onto the spaces of Toeplitz operators

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Abstract. Projections onto the spaces of all Toeplitz operators on the $N$-torus and the unit sphere are constructed. The constructions are also extended to generalized Toeplitz operators and applied to show hyperreflexivity results.

1. Introduction. Arveson [1] defined a projection from the algebra $B(H^2(\mathbb{T}))$ of all bounded linear operators on the Hardy space on the unit circle onto the space of all Toeplitz operators on $H^2(\mathbb{T})$. He used the classical Banach limit. We construct a projection from the algebra $B(H^2(\mathbb{T}^N))$ of all bounded linear operators on the Hardy space on the $N$-torus onto the space of all Toeplitz operators on $H^2(\mathbb{T}^N)$. We use the extension of the Banach limit to $N$-parameter sequences, given in Section 2.

In Section 4 we will use the above projection to show that the subspace of all Toeplitz operators on the $N$-torus is 2-hyperreflexive (for definition see Section 4). The single variable case was considered in [7].

A natural generalization of the unit circle is not only the $N$-torus but also the unit sphere $\partial \mathbb{B}_N$. In Section 5 we construct a projection from the algebra $B(H^2(\partial \mathbb{B}_N))$ of all bounded linear operators on the Hardy space on the unit sphere onto the space of all Toeplitz operators on $H^2(\partial \mathbb{B}_N)$.

In Section 6 we extend the idea of such a projection to generalized Toeplitz operators which were introduced in [10], [11]. We consider both one and multi-variable cases. In Section 7 we give a hyperreflexivity result for generalized Toeplitz operators. The one variable case was considered in [8].

2. Multi-variable Banach limit. There is a functional on all bounded sequences in $\ell^\infty$, which to any convergent sequence $\{x(n)\}_{n \in \mathbb{N}}$ assigns its
limit (see e.g. [4]). It is called the Banach limit. We extend this idea to multi-variable bounded sequences in $\ell^\infty(\mathbb{N}^N)$.

**Theorem 2.1.** There is a linear functional $A: \ell^\infty(\mathbb{N}^N, \mathbb{C}) \to \mathbb{C}$ (resp. $A: \ell^\infty(\mathbb{N}^N, \mathbb{R}) \to \mathbb{R}$) such that

(a) $\|A\| = 1$ and $A(1) = 1$, where $1$ is the constantly $1$ sequence,

(b) if $x \in \ell^\infty(\mathbb{N}^N)$ converges, then $A(x) = \lim_{k \to \infty, k \in \mathbb{N}^N} x(k)$,

(c) if $x \in \ell^\infty(\mathbb{N}^N)$ is nonnegative, i.e. $x(k) \geq 0$ for all $k \in \mathbb{N}^N$, then $A(x) \geq 0$,

(d) for any sequence $x \in \ell^\infty(\mathbb{N}^N)$ and any $i = 1, \ldots, N$, let $x^{(i)}$ denote the sequence $x^{(i)}(k) = x(k + e_i)$, where $e_i = (0, \ldots, 1, \ldots, 0)$; then $A(x) = A(x^{(i)})$.

**Proof.** First we deal with the case of a real-valued functional, i.e. $A: \ell^\infty(\mathbb{N}^N, \mathbb{R}) \to \mathbb{R}$. For each $i$, set $\mathcal{M}_i = \{x - x^{(i)} : x \in \ell^\infty(\mathbb{N}^N, \mathbb{R})\}$, where $x^{(i)}$ is defined as in (d). Note that $\mathcal{M}_i$ is a linear manifold. Let $\mathcal{M}$ be the subspace spanned by all $\mathcal{M}_i$, $i = 1, \ldots, N$. We show first that

$$d(1, \mathcal{M}) = 1,$$

where $d$ denotes the distance from the sequence $1$ to the subspace $\mathcal{M}$. Since $0 \in \mathcal{M}$ we have $d(1, \mathcal{M}) \leq 1$. Assume that there are $\varepsilon > 0$ and $x_i \in \mathcal{M}_i$ and $\alpha_i \in \mathbb{R}$ with $\|\alpha_i x_i\|_\infty \leq M$, $i = 1, \ldots, N$, such that

$$\left\|\mathbf{1} - \sum_{i=1}^N \alpha_i (x_i - x_i^{(i)})\right\|_\infty < 1 - \varepsilon.$$

In particular, for fixed $n \in \mathbb{N}$, for all $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$ such that $|k|_\infty = \max |k_i| \leq n$, we have

$$1 - \sum_{i=1}^N \alpha_i (x_i(k) - x_i^{(i)}(k)) < 1 - \varepsilon.$$

Summation over $k$ gives

$$n^N - \sum_{i=1}^N \alpha_i \left( \sum_{|k|_\infty \leq n} x_i(k) - \sum_{|k|_\infty \leq n} x_i(k + e_i) \right) < n^N - n^N \varepsilon.$$

Thus

$$n^N \varepsilon < \sum_{i=1}^N |\alpha_i| \left( \sum_{|k|_\infty \leq n} |x_i(k)| + \sum_{|k|_\infty \leq n} |x_i(k + e_i)| \right) \leq 2n^{N-1} NM.$$

Hence $n \varepsilon < 2NM$ and we have a contradiction for $n$ large enough, so (1) follows.

The Hahn–Banach theorem yields a linear functional $A$ on $\ell^\infty(\mathbb{N}^N, \mathbb{R})$ such that $A(1) = 1$, $A(\mathcal{M}) = 0$ and $\|A\| = 1$. 
To see (b), for a given sequence $x \in \ell^\infty(\mathbb{N}^N, \mathbb{R})$ and for all multi-indices in $\mathbb{N}^N$, define by multi-induction the following sequences:

$$x_{(e_i)} = x^{(i)} \quad \text{and} \quad x_{(k+e_i)} = (x_{(k)})^{(i)}.$$ 

Note that the definition is correct since $(x_{(k-e_i)})^{(j)} = (x_{(k-e_j)})^{(i)}$. For fixed $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$ we have

$$x_{(k)} - x = (x_{(k)} - x_{(k-e_N)}) + \cdots + (x_{(k_1,\ldots,k_2,1)} - x_{(k_1,\ldots,k_2,0)}) + \cdots + (x_{(k_1,0,\ldots,0)} - x_{(k_1-1,0,\ldots,0)}) + \cdots + (x_{(1,0,\ldots,0)} - x).$$

Thus $A(x) = A(x_{(k)})$. If $x$ is convergent and $\alpha = \lim_{k' \to \infty, k' \in \mathbb{N}^N} x(k')$, then

$$|A(x) - \alpha| = |A(x_{(k)} - \alpha \cdot 1)| \leq \|x_{(k)} - \alpha \cdot 1\|_\infty \leq \sup\{|x(k_1', \ldots, k'_N) - \alpha| : k'_i > k_i \text{ for all } i\}.$$ 

Thus $|A(x) - \alpha|$ is arbitrarily small and we get (b).

Condition (c) and extension to the case of complex-valued sequences can be shown as in the single variable case (see for example [4]).

3. Projection onto Toeplitz operators on the $N$-torus. Let $\mathbb{T}$ be the unit circle on the complex plane $\mathbb{C}$. Set $L^2(\mathbb{T}) = L^2(\mathbb{T}, m)$ and $L^\infty(\mathbb{T}) = L^\infty(\mathbb{T}, m)$, where $m$ is the normalized Lebesgue measure on $\mathbb{T}$. Let $H^2(\mathbb{T})$ be the Hardy space corresponding to $L^2(\mathbb{T})$ and let $P_{H^2(\mathbb{T})}$ be the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. We denote by $H^\infty(\mathbb{T})$ the Hardy space corresponding to $L^\infty(\mathbb{T})$, i.e. the space of those functions from $L^\infty(\mathbb{T})$ which have an analytic extension to the whole unit disc $\mathbb{D}$.

For each $\varphi \in L^\infty(\mathbb{T})$ we define $T_\varphi \in B(H^2(\mathbb{T}))$ by $T_\varphi f = P_{H^2(\mathbb{T})}(\varphi f)$ for $f \in H^2(\mathbb{T})$. The operator $T_\varphi$ is called a Toeplitz operator with symbol $\varphi$. Let $T(\mathbb{T})$ denote the space of all Toeplitz operators, and $A(\mathbb{T})$ the space of Toeplitz operators with symbols from $H^\infty(\mathbb{T})$. We have ([6, Corollary to Problem 194])

$$T(\mathbb{T}) = \{ A \in B(H^2(\mathbb{T})) : A = T_z^* AT_z \},$$

and by [6, Problem 116],

$$A(\mathbb{T}) = \{ A \in B(H^2(\mathbb{T})) : AT_z = T_z A \}.$$ 

Similarly we denote the corresponding spaces on the $N$-torus, $L^2(\mathbb{T}^N)$, $L^\infty(\mathbb{T}^N)$, $H^2(\mathbb{T}^N)$, $H^\infty(\mathbb{T}^N)$ and the projection $P_{H^2(\mathbb{T}^N)} : L^2(\mathbb{T}^N) \to H^2(\mathbb{T}^N)$. For each $\varphi \in L^\infty(\mathbb{T}^N)$ we define the Toeplitz operator $T_\varphi \in B(H^2(\mathbb{T}^N))$ by $T_\varphi f = P_{H^2(\mathbb{T}^N)}(\varphi f)$. We denote by $T(\mathbb{T}^N)$ the space of all Toeplitz operators and by $A(\mathbb{T}^N)$ the space of all Toeplitz operators with symbols from $H^\infty(\mathbb{T}^N)$. By $T_{z_i}$, $i = 1, \ldots, N$, we denote the multiplication operators by the independent variables. Since the operators $T_{z_i}$ commute we can set $T_{z^k} = T_{z_1}^{k_1} \cdots T_{z_N}^{k_N}$ for $k = (k_1, \ldots, k_n) \in \mathbb{N}^N$ ($z^k = z_1^{k_1} \cdots z_N^{k_N}$).
Similarly to the one variable case we have the following characterizations (see [9, Proposition 3.3]):

\[ T(T^N) = \{ A \in B(H^2(T^N)) : A = T^*_z A T_z, \ i = 1, \ldots, N \}, \]

\[ A(T^N) = \{ A \in B(H^2(T^N)) : AT_z = T_z A, \ i = 1, \ldots, N \}. \]

We will construct a projection onto the space of all Toeplitz operators on the \( N \)-torus.

**Theorem 3.1.** There is a positive linear projection \( \pi : B(H^2(T^N)) \to T(T^N) \) such that

(a) \( \pi(I) = I, \ ||\pi|| = 1, \)

(b) \( \pi(T) = T \) for \( T \in T(T^N), \)

(c) \( \pi(AT_\varphi) = \pi(A)T_\varphi \) for \( A \in B(H^2(T^N)) \) and \( T_\varphi \in A(T^N), \)

(d) \( \pi(A) \) belongs to the weakly-closed convex hull of \( \{ T^*_z A T_z : k \in N^N \} \)

for \( A \in B(H^2(T^N)), \)

(e) \( \pi(P_k) = 1, \) where \( P_k \) is the orthogonal projection on the range of \( T_z^* \).

**Proof.** For \( A \in B(H^2(T^N)) \) and \( x, y \in H^2(T^N) \) we define

\[ [x, y] = A(\{ (T^*_z A T_z, x, y) \} ) \]

where \( A \) denotes the multi-variable Banach limit given in Theorem 2.1. Since \( (x, y) \mapsto [x, y] \) is a bounded sesquilinear form, there is an operator \( \pi(A) \in B(H^2(T^N)) \) such that

\[ (\pi(A)x, y) = A(\{ (T^*_z A T_z, x, y) \} ). \]

From the definition it is easy to see that \( \pi(I) = I. \) Note that for any \( i, \) by Theorem 2.1(d),

\[ (T^*_z, \pi(A)T_z, x, y) = (\pi(A)T_z, x, T_z, y) = A(\{ (T^*_z A T_z, x, T_z, y) \} k \in N^N ) \]

\[ = A(\{ (T^*_z A T_z, x, y) \} k \in N^N ) = (\pi(A)x, y). \]

Thus \( T^*_z, \pi(A)T_z = \pi(A) \) and, by the characterization (4) of Toeplitz operators, we see that \( \pi(A) \in T(T^N). \)

If \( A \in T(T^N) \) then, by (4), \( \{ (T^*_z A T_z, x, y) \} k \in N^N = \{ (Ax, y) \} k \in N^N \) is a constant sequence for all \( x, y \) and \( (\pi(A)x, y) = (Ax, y) \) by Theorem 2.1(b), thus \( \pi(A) = A. \)

Formula (6) also implies that \( \pi \) is positive. If (d) is not satisfied then for a given operator \( A \in B(H^2(T^N)) \) there are \( x, y \in H^2(T^N) \) such that \( (\pi(A)x, y) \neq 0, \) but \( (Bx, y) = 0 \) for all \( B \) in the weakly-closed convex hull of \( \{ (T^*_z A T_z, x, y) : k \in N^N \}. \) This contradicts Theorem 2.1(c). The remaining properties follow from formula (6). □
4. 2-hyperreflexivity of Toeplitz operators on the $N$-torus. As before, for a given complex separable Hilbert space $\mathcal{H}$ we denote by $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. It is well known that the space of trace class operators $\tau c$ is predual to $B(\mathcal{H})$ with the dual action $\langle A, f \rangle = \text{tr}(Af)$ for $A \in B(\mathcal{H})$ and $f \in \tau c$. The trace norm in $\tau c$ will be denoted by $\| \cdot \|_1$. Denote by $F_k$ the set of operators of rank at most $k$. Every rank-one operator may be written as $x \otimes y$ for some $x, y \in \mathcal{H}$, where $(x \otimes y)z = (z, y)x$ for $z \in \mathcal{H}$. Moreover, $\text{tr}(T(x \otimes y)) = (Tx, y)$.

Let $S \subset B(\mathcal{H})$ be a norm-closed subspace. We denote by $d(T, S)$ the standard distance from an operator $T$ to the subspace $S$. It is known that when $S$ is weak$^*$-closed, $d(T, S) = \sup \{|\text{tr}(Tf)| : f \in S_\perp, \|f\|_1 \leq 1\}$, where $S_\perp$ denotes the preannihilator of $S$.

Recall that if $S$ is a weak$^*$-closed subspace of $B(\mathcal{H})$, then $S$ is reflexive if and only if $S_\perp$ is a closed linear span of rank-one operators contained in $S_\perp$ (i.e., $S_\perp = \text{span}(S_\perp \cap F_1)$). At the other extreme, if $S_\perp \cap F_1 = \{0\}$ then we call $S$ transitive. A weak$^*$-closed subspace $S \subset B(\mathcal{H})$ is called $k$-reflexive if $S_\perp = \text{span}(S_\perp \cap F_k)$. We also have a stronger property: $S$ is called hyperreflexive if there is a constant $a$ such that

\[
d(T, S) \leq a \sup \{|\langle T, x \otimes y \rangle| : x \otimes y \in S_\perp, \|x \otimes y\|_1 \leq 1\}
\]

for all $T \in B(\mathcal{H})$, and $k$-hyperreflexive if there is a such that for any $T \in B(\mathcal{H})$,

\[
d(T, S) \leq a \sup \{|\text{tr}(Tf)| : f \in S_\perp \cap F_k, \|f\|_1 \leq 1\}.
\]

The distance on the right hand side will be denoted by $\alpha_k(T, S)$. Let $\kappa_k(S)$ be the infimum of the constants $a$ in (8); we call it the $k$-hyperreflexivity constant. For further properties of $k$-reflexivity and $k$-hyperreflexivity the reader is referred to [3] and [7].

Analyzing the spaces of all Toeplitz operators on the unit circle $T(\mathbb{T})$ and on the $N$-torus $T(\mathbb{T}^N)$ from the reflexivity point of view, note first that the characterizations (2) and (4) allow us to see that both spaces are weak$^*$-closed. The space $T(\mathbb{T})$ is transitive, but 2-reflexive (see [2]) and even 2-hyperreflexive (see [7]).

In [9] it was shown that $T(\mathbb{T}^N)$ is transitive, thus not reflexive, but that it is 2-reflexive. Now we will show the stronger condition: 2-hyperreflexivity.

**Theorem 4.1.** The space of all Toeplitz operators on the torus $T(\mathbb{T}^N)$ is 2-hyperreflexive and $\kappa_2(T(\mathbb{T}^N)) \leq 2$.

**Proof.** Let $A \in B(H^2(\mathbb{T}^N))$. Since $\pi(A)$ belongs to the weakly-closed convex hull of the set $\{T^*_z \cdot AT_z : k \in \mathbb{N}^N\}$, we have
\[ d(A, T(T^N)) \leq \|A - \pi(A)\| \leq \sup_{k \in \mathbb{N}} \|A - T_{z_k}^* AT_{z_k}\| \]

\[ \leq \sup_{k \in \mathbb{N}} \sup \{ |((A - T_{z_k}^* AT_{z_k})x, y)| : x, y \in H^2(T^N), \|x \otimes y\|_1 = 1 \} \]

\[ \leq \sup_{k \in \mathbb{N}} \sup \{ |(Ax, y) - (Az^k x, z^k y)| : x, y \in H^2(T^N), \|x \otimes y\|_1 = 1 \} \]

\[ \leq \sup_{k \in \mathbb{N}} \sup \{ |\text{tr}(A(x \otimes y - z^k x \otimes z^k y))| : x, y \in H^2(T^N), \|x \otimes y\|_1 = 1 \}. \]

Since \( \text{rank}(x \otimes y - z^k x \otimes z^k y) \leq 2 \) and \( \|x \otimes y - z^k x \otimes z^k y\|_1 \leq 2 \) if \( \|x \otimes y\|_1 = 1 \), it follows that \( d(A, T(T^N)) \leq 2 \alpha_2(A, T(T^N)) \).

5. Projection onto Toeplitz operators on the unit ball. Let \( \mathbb{B}_N \) be the unit ball in \( \mathbb{C}^N \) and denote by \( \sigma \) the normalized surface measure on the unit sphere \( \partial \mathbb{B}_N \). We set \( L^2(\partial \mathbb{B}_N) = L^2(\partial \mathbb{B}_N, \sigma) \) and \( L^\infty(\partial \mathbb{B}_N, \sigma) \) denote by \( H^2(\partial \mathbb{B}_N) \), \( P_{H^2}(\partial \mathbb{B}_N) \) etc. the corresponding spaces and operators on \( \partial \mathbb{B}_N \). Also the symbols \( T_{z_i} \) and \( T_{z_k} \) have the same meaning as before.

In [5] it was shown that

\[ T(\partial \mathbb{B}_N) = \left\{ A \in B(H^2(\partial \mathbb{B}_N)) : A = \sum_{i=1}^N T_{z_i}^* AT_{z_i} \right\}, \]

\[ A(\partial \mathbb{B}_N) = \{ A \in B(H^2(\partial \mathbb{B}_N)) : AT_{z_i} = T_{z_i}A, i = 1, \ldots, N \}. \]

For a given operator \( A \in B(H^2(\partial \mathbb{B}_N)) \) we define by induction a sequence \( \{A^{(n)}\}_{n \in \mathbb{N}} \) in \( B(H^2(\partial \mathbb{B}_N)) \):

\[ A^{(0)} = A, \quad A^{(n+1)} = \sum_{i=1}^N T_{z_i}^* A^{(n)} T_{z_i}. \]

Note that \( I^{(n)} = I \) and if \( T \in T(\partial \mathbb{B}_N) \), then \( T^{(n)} = T \) by (9). Moreover, by (10),

\[ (AT_{z_i})^{(n)} = A^{(n)} T_{z_i} \quad \text{for} \quad A \in B(H^2(\partial \mathbb{B}_N)) \quad \text{and} \quad T_{z_i} \in A(\partial \mathbb{B}_N). \]

**Lemma 5.1.** If \( A \in B(H^2(\partial \mathbb{B}_N)) \), then \( \|A^{(n)}\| \leq 2\|A\| \).

**Proof.** For \( k = (k_1, \ldots, k_N) \in \mathbb{N}^N \) we write \( k! = k_1! \cdots k_N! \) and \( |k| = k_1 + \cdots + k_N \). One can easily note that

\[ A^{(n)} = \sum_{|k| = n} \frac{n!}{k!} T_{z_k}^* AT_{z_k}. \]

If \( x \in H^2(\partial \mathbb{B}_N) \), then

\[ |(A^{(n)}x, x)| \leq \sum_{|k| = n} \frac{n!}{k!} |(Az^k x, z^k x)| \leq \|A\| \sum_{|k| = n} \frac{n!}{k!} \|z^k x\|^2 \]
where Λ is a bounded sesquilinear form, there is an operator π such that

\[ N(3.1(d)) \text{ is a consequence of (12), and the proof of (d) is similar to that of Theorem 5.2.} \]

Thus the numerical range satisfies \( w(A(n)) \leq \|A\| \) and \( \|A(n)\| \leq 2\|A\| \) by [6].

**Theorem 5.2.** There is a positive linear projection \( \pi : B(H^2(\partial \mathbb{B}_N)) \to T(\partial \mathbb{B}_N) \) such that

(a) \( \pi(I) = I, \|\pi\| \leq 2 \),
(b) \( \pi(T) = T \) for \( T \in T(\partial \mathbb{B}_N) \),
(c) \( \pi(\lambda T) = \pi(\lambda)T \) for \( \lambda \in \mathbb{C} \),
(d) \( \pi(A) \) belongs to the weakly-closed convex hull of \( \{A(n) : n \in \mathbb{N}\} \) for \( A \in B(H^2(\partial \mathbb{B}_N)) \).

**Proof.** For \( A \in B(H^2(\partial \mathbb{B}_N)) \) and \( x, y \in H^2(\partial \mathbb{B}_N) \) we define

\[ [x, y] = A(\{(A(n)x, y)\}_{n \in \mathbb{N}}) \]

where \( A \) denotes the one-dimensional Banach limit (see Theorem 2.1). Note that \( \{(A(n)x, y)\}_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathbb{C}) \) by Lemma 5.1. Since \( (x, y) \mapsto [x, y] \) is a bounded sesquilinear form, there is an operator \( \pi(A) \in B(H^2(\partial \mathbb{B}_N)) \) such that

\[ (\pi(A)x, y) = A(\{(A(n)x, y)\}_{n \in \mathbb{N}}). \]

Since \( I(n) = I \), we have \( \pi(I) = I \).

Now, by Theorem 2.1(d), we get

\[ \left( \sum_{i=1}^N T_{zi}^* \pi(A)T_{zi}x, y \right) = \sum_{i=1}^N (\pi(A)zi x, z_i y) = \sum_{i=1}^N A(\{(A(n)zi x, z_i y)\}_{n \in \mathbb{N}}) = A(\left\{ \left( \sum_{i=1}^N T_{zi}^* A(n)T_{zi}x, y \right) \right\}_{n \in \mathbb{N}}) = \pi(A)x, y. \]

Thus \( \pi(A) \in T(\partial \mathbb{B}_N) \) by (9).

If \( A \in T(\partial \mathbb{B}_N) \) then \( A(n) = A \) for all \( n \), and thus \( \pi(A) = A \). Property (c) is a consequence of (12), and the proof of (d) is similar to that of Theorem 3.1(d).

**6. Projection onto generalized Toeplitz operators.** The idea of generalized Toeplitz operators is to replace in the characterization (2) the backward shift \( T_z^* \) by any contraction. Precisely, for given contractions \( S, T \)
in $B(\mathcal{H})$, an operator $X \in B(\mathcal{H})$ is called a generalized Toeplitz operator with respect to $S$ and $T$ if $X = SXT^*$. These operators were investigated in [11]. The space of all such operators is denoted by $T(S, T)$. Note that this definition implies weak*-closedness of $T(S, T)$.

In [10] this idea was extended to two variables. It is easy to extend it to the multi-variable case. Having in mind the characterization (4) of Toeplitz operators on the $N$-torus we make the following definition. For given $N$-tuples $S = (S_1, \ldots, S_N)$ and $T = (T_1, \ldots, T_N)$ of commuting contractions on $\mathcal{H}$, an operator $X \in B(\mathcal{H})$ is called a generalized Toeplitz operator with respect to $S$ and $T$ if $X = S_iX T_i^*$ for $i = 1, \ldots, N$. The space of all such operators is denoted by $T(S, T)$. It is also weak*-closed. For a given commuting $N$-tuple $S = (S_1, \ldots, S_N)$ we set $S^k = S_1^{k_1} \cdots S_N^{k_N}$ for $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$.

Now we extend the definition of the projection considered in Section 3 to generalized Toeplitz operators. We formulate the theorem for arbitrary $N$, but even the case $N = 1$ is worth noting.

**Theorem 6.1.** Let $S$ and $T$ be two $N$-tuples of commuting contractions on $\mathcal{H}$. There is a linear projection $\pi: B(\mathcal{H}) \to T(S, T)$ such that

(a) $\|\pi\| \leq 1$,
(b) $\pi(X) = X$ for $X \in T(S, T)$,
(c) if $A \in B(\mathcal{H})$ then $\pi(A)$ belongs to the weakly-closed convex hull of $\{S^k A T^{*k} : k \in \mathbb{N}^N\}$.

**Proof.** Let $\Lambda$ be the functional from Theorem 2.1. For $A \in B(\mathcal{H})$ and $x, y \in \mathcal{H}$ we define

$$(\pi(A)x, y) = \Lambda(\{(S^k A T^{*k} x, y)\}_{k \in \mathbb{N}^N}).$$

To check the details, one can follow the proof of Theorem 3.1. □

**7. 2-hyperreflexivity of generalized Toeplitz operators.** The reflexive behavior of the space $T(S, T)$ of generalized Toeplitz operators depends on the contractions $S, T$. For example if the underlying Hilbert space is the Hardy space on the unit circle and $S = T = T_z^*$ then $T(T_z^*, T_z^*) = T(\mathbb{T})$ is transitive. On the other hand, the space $T(S, T)$ might be even (hyper)reflexive. For example, if $S = T = I_H$ then $T(I_H, I_H) = B(\mathcal{H})$, which is (hyper)reflexive. However, we can estimate the reflexive behavior even for arbitrary $N$ by

**Theorem 7.1.** Let $S$ and $T$ be two $N$-tuples of commuting contractions on $\mathcal{H}$. Then $T(S, T)$ is 2-hyperreflexive.

**Proof.** By Theorem 6.1(c), for any $A \in B(\mathcal{H})$, $\pi(A)$ belongs to the weakly-closed convex hull of $\{S^k A T^{*k} : k \in \mathbb{N}^N\}$. As in the proof of Theorem
4.1 we can show that
\[ d(A, T(S, T)) \leq \|A - \pi(A)\| \leq \sup_{k \in \mathbb{N}} \|A - S^k A T^{*k}\| \]
\[ \leq \sup_{k \in \mathbb{N}} \sup \{ |\text{tr}(A(x \otimes y - T^{*k}x \otimes S^{*k}y))| : \|x \otimes y\|_1 = 1 \}. \]
Since \(\text{rank}(x \otimes y - T^{*k}x \otimes S^{*k}y) \leq 2\) and \(\|x \otimes y - T^{*k}x \otimes S^{*k}y\|_1 \leq 2\) for \(\|x \otimes y\|_1 = 1\), we have
\[ d(A, T(S, T)) \leq 2\alpha_2(A, T(S, T)). \]

Theorem 7.1 for \(N = 1\) is also a consequence of [8].

**Added in proof.** D. Timotin (private communication) has shown that the norm of the projection in Theorem 5.2 is equal to 1, \(\|\pi\| = 1\).

**References**


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Reçu par la Rédaction le 27.1.2005
Révisé le 22.7.2005