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# On the Kantorovich–Rubinstein maximum principle for the Fortet–Mourier norm

by HENRYK GACKI (Katowice)

**Abstract.** A new version of the maximum principle is presented. The classical Kantorovich–Rubinstein principle gives necessary conditions for the maxima of a linear functional acting on the space of Lipschitzian functions. The maximum value of this functional defines the Hutchinson metric on the space of probability measures. We show an analogous result for the Fortet–Mourier metric. This principle is then applied in the stability theory of Markov–Feller semigroups.

**1. Introduction.** We study semigroups of Markov–Feller operators acting on the space $\mathcal{M}_{\text{sig}}$ of signed measures. Our goal is to show the utility of the maximum principle technique in proving the asymptotic stability for this class of semigroups. The classical Kantorovich–Rubinstein maximum principle for the Hutchinson metric was already used to prove the stability of stochastically perturbed dynamical systems with discrete time (see [4]), stochastic semigroups generated by the Tjon–Wu equation (see [11]) and semigroups generated by Poisson driven stochastic differential equations (see [5]).

In the present paper we formulate the maximum principle for the Fortet–Mourier metric. Our proof is based on a theorem concerning local changes of Lipschitzian functions (see [6]). This version of maximum principle will be formulated precisely in Section 3. In Section 4 we will show its applications in the stability theory of Markov–Feller semigroups. In particular we will discuss the problem of the asymptotic stability of a Markov operator appearing in the theory of the cell cycle [10]. Some recent results in this area were obtained by P. Janoska [8].

**2. Preliminaries.** Let $(X, \varrho)$ be a metric space. We denote by $\mathbb{R}$ the real line, $\mathbb{R}_+ = [0, \infty)$ and by $\mathbb{N}$ the set of positive integers. Further $\mathcal{B}$

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denotes the $\sigma$-algebra of Borel subset of $X$ and $\mathcal{M}$ the family of all finite Borel measures on $X$. Let

$$\mathcal{M}_{\rm sig} = \{\, \mu_1 - \mu_2 : \ \mu_1, \mu_2 \in \mathcal{M} \,\}$$

be the space of finite signed measures. By $\mathcal{M}_1$ we denote the subset of $\mathcal{M}$ such that $\mu(X) = 1$ for $\mu \in \mathcal{M}_1$. The elements of $\mathcal{M}_1$ will be called *distributions*. For arbitrary $\mu \in \mathcal{M}_{\rm sig}$ we denote by $\mu_+$ and $\mu_-$ the positive and negative parts of $\mu$. Then $\mu = \mu_+ - \mu_-$, and $|\mu| = \mu_+ + \mu_-$ is the total variation of $\mu$. As usual, $B(X)$ denotes the space of all bounded Borel measurable functions $f : X \to \mathbb{R}$, and $C(X)$ the subspace of all bounded continuous functions. Both spaces are considered with the supremum norm

$$\|f\| = \sup_{x\in X} |f(x)|.$$

For every $f : X \to \mathbb{R}$ and $\mu \in \mathcal{M}_{\rm sig}$ we write

$$\langle f, \mu\rangle = \int_X f(x)\, \mu(dx), \tag{1}$$

whenever this integral exists. In the space $\mathcal{M}_1$ we introduce the *Fortet–Mourier metric* by the formula

$$\|\mu_1 - \mu_2\|_{\mathcal{F}} = \sup\{|\langle f, \mu_1 - \mu_2\rangle| : f \in \mathcal{F}\}, \tag{2}$$

where $\mathcal{F}$ is the set of functions $f : X \to \mathbb{R}$ satisfying

$$\|f\| \le 1 \quad \text{and} \quad |f(x) - f(y)| \le \varrho(x, y) \quad \text{for } x, y \in X.$$

We say that a sequence $(\mu_n)$, $\mu_n \in \mathcal{M}_1$, converges weakly to a measure $\mu \in \mathcal{M}_1$ if

$$\lim_{n\to\infty} \langle f, \mu_n\rangle = \langle f, \mu\rangle \quad \text{for } f \in C(X). \tag{3}$$

If $X$ is a Polish space, condition (3) is equivalent to

$$\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathcal{F}} = 0.$$

Moreover, $\mathcal{M}_1$ with the distance given by (2) is a complete metric space (see [3]).

Denote by $B(x, r)$ the closed ball in $X$ with centre $x \in X$ and radius $r$. Let $\mu \in \mathcal{M}_1$. We define the *support* of $\mu$ to be

$$\operatorname{supp} \mu = \{x \in X : \mu(B(x, \varepsilon)) > 0 \text{ for every } \varepsilon > 0\}.$$

REMARK 1. It is easy to see that for every nontrivial measure $\mu \in \mathcal{M}_{\rm sig}$ such that $\mu(X) = 0$ the sets $\operatorname{supp} \mu_+$ and $\operatorname{supp} \mu_-$ are nonempty.

A metric space $(X, \varrho)$ is called *boundedly compact* if every closed and bounded subset of $X$ is a compact set (see [13]). This condition implies that $(X, \varrho)$ is a Polish space.

**3. Maximum principle.** A function $f : X \to \mathbb{R}$ defined on a metric space $(X, \varrho)$ will be called *contractive* if

$$|f(x) - f(y)| < \varrho(x, y) \quad \text{for } x, y \in X,\ x \neq y. \tag{4}$$

In the proof of our main result, Theorem 2, we will use the following property of contractive functions (see [6]):

THEOREM 1. *Let $(X, \varrho)$ be a boundedly compact metric space and let $f : X \to \mathbb{R}$ be a contractive function satisfying*

$$\inf f > -\infty. \tag{5}$$

*Further let an open set $G \subset X$ and a compact set $K \subset G$ be given. Then there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exists a contractive function $\widetilde{f} : X \to \mathbb{R}$ satisfying the following conditions*:

$$\widetilde{f}(x) = f(x) \quad \textit{for } x \in X \setminus G, \qquad \widetilde{f}(x) = f(x) + \varepsilon \quad \textit{for } x \in K. \tag{6}$$

$$f(x) \le \widetilde{f}(x) \le f(x) + \varepsilon \quad \textit{for } x \in G \setminus K. \tag{7}$$

Replacing $f$ by $-f$ we obtain from Theorem 1 the following result:

REMARK 2. If $\sup f < \infty$ and $f : X \to \mathbb{R}$ is a contractive function then there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (-\varepsilon_0, 0)$ there is a contractive function $\widetilde{f} : X \to \mathbb{R}$ satisfying conditions (6) and the inequality

$$f(x) + \varepsilon \le \widetilde{f}(x) \le f(x) \quad \text{for } x \in G \setminus K. \tag{8}$$

In order to formulate the main result of our paper we introduce a functional of the form

$$\varphi(f) = \langle f, \mu \rangle, \qquad f \in \mathcal{F}, \tag{9}$$

where $\mu$ is a given signed measure satisfying

$$\mu = \mu_1 - \mu_2, \qquad \mu_1 \neq \mu_2,\ \mu_1, \mu_2 \in \mathcal{M}_1. \tag{10}$$

The following theorem extends the Kantorovich–Rubinstein principle to the Fortet–Mourier norm.

THEOREM 2. *Assume that $(X, \varrho)$ is a boundedly compact metric space and that $\mu$ satisfies condition* (10). *Then there exists a function $f_0 \in \mathcal{F}$ such that*

$$\varphi(f_0) = \|\mu\|_{\mathcal{F}}. \tag{11}$$

*Moreover, if a function $f_0 \in \mathcal{F}$ satisfies* (11) *then it fulfills at least one of the following two conditions*:

$1^{\circ}$ *There exist $x, y \in X$, $x \neq y$, such that*

$$|f_0(x) - f_0(y)| = \varrho(x, y). \tag{12}$$

$2^{\circ}$ *The function $f_0$ has the following properties*:

$$f_0(x) = 1 \qquad \textit{for } x \in \operatorname{supp} \mu_+, \tag{13}$$

$$f_0(x) = -1 \qquad \textit{for } x \in \operatorname{supp} \mu_-. \tag{14}$$

*Proof.* Condition (10) implies that for every $f \in \mathcal{F}$ the integral (9) exists and

$$|\varphi(f)| < \infty. \tag{15}$$

It follows immediately that there exists a sequence $(f_n)$, $f_n \in \mathcal{F}$, satisfying

$$\lim_{n\to\infty} \varphi(f_n) = \sup_{f\in\mathcal{F}} \varphi(f) < \infty. \tag{16}$$

According to the Ulam theorem (see [3]) we can choose an increasing sequence of compact sets $K_s \subset X$ such that

$$|\mu|(X \setminus K_s) < 1/s \qquad \text{for } s = 1, 2 \ldots. \tag{17}$$

Using the Arzelà–Ascoli theorem and the diagonal Cantor process we find a subsequence $(f_{\alpha_n})$ which converges pointwise on the set

$$\widehat{K} = \bigcup_{s=1}^{\infty} K_s \tag{18}$$

to a function $\widehat{f} : \widehat{K} \to \mathbb{R}$. Evidently $\widehat{f}$ satisfies the Lipschitz condition with constant 1. According to the McShane theorem (see [14]) there exists an extension $f_0$ of $\widehat{f}$ defined on $X$ which satisfies the Lipschitz condition with the same constant. From the construction it follows that $(f_{\alpha_n})$ converges to $f_0$ on $\widehat{K}$ and $|\mu|(X \setminus \widehat{K}) = 0$. By the Lebesgue dominated convergence theorem we have

$$\lim_{n\to\infty} \varphi(f_{\alpha_n}) = \varphi(f_0).$$

This and (16) imply (11). Now we are going to show that every $f_0 \in \mathcal{F}$ satisfying (11) fulfils ($1^{\circ}$) or ($2^{\circ}$). Suppose, on the contrary, that there exists a contractive $f_0 \in \mathcal{F}$ such that

$$\varphi(f_0) = \|\mu\|_{\mathcal{F}} \quad \text{and} \quad f_0(x_0) < 1 \quad \text{for some } x_0 \in \operatorname{supp} \mu_+. \tag{19}$$

Let $X = X_+ \cup X_-$ be the Hahn decomposition for $\mu$. From the continuity of $f_0$ there is a closed ball $B(x_0, r_0)$ such that

$$f_0(x) < 1 \qquad \text{for } x \in B(x_0, r_0).$$

Moreover

$$\mu_+(B(x_0, r_0)) > 0.$$

According to the Ulam theorem there is a compact set $K \subseteq B(x_0, r_0) \cap X_+$ such that

$$\mu_+(K) > 0. \tag{20}$$

Evidently $\mu_-(K) = 0$ and $f_0(x) < 1$ for $x \in K$.

Define

$$K_\delta = \{x \in X : \varrho(x, K) < \delta\}.$$

Using the compactness of $K$ we can find a $\delta > 0$ such that

$$\mu_-(K_\delta \setminus K) \le \mu_+(K)/2 \quad \text{and} \quad \sup_{x \in K_\delta} f_0(x) < 1. \tag{21}$$

Since $K \subset K_\delta$ and the set $K_\delta$ is open, according to Theorem 1 there exists an $\varepsilon > 0$ such that $\varepsilon < 1 - \sup_{x\in K_\delta} f_0(x)$, and a contractive function $\widetilde{f_0} : X \to \mathbb{R}$ satisfying conditions (6) and (7) with $G = K_\delta$. From (6), (7) and the equality $\mu_-(K) = 0$ it follows that

$$\begin{aligned}\langle \widetilde{f_0}, \mu\rangle - \langle f_0, \mu\rangle = & \int_{X\setminus K_\delta} (\widetilde{f_0}(x) - f_0(x))\,\mu(dx) \\ & + \varepsilon \int_K \mu_+(dx) + \int_{K_\delta\setminus K} (\widetilde{f_0}(x) - f_0(x))\,\mu(dx) \\ & \ge \varepsilon\mu_+(K) - \int_{K_\delta\setminus K} (\widetilde{f_0}(x) - f_0(x))\,\mu_-(dx) \\ & \ge \varepsilon\mu_+(K) - \varepsilon\mu_-(K_\delta \setminus K).\end{aligned}$$

Now using (21) we obtain

$$\langle \widetilde{f_0}, \mu\rangle \ge \langle f_0, \mu\rangle + \varepsilon\mu_+(K)/2.$$

Since $\widetilde{f}$ is a contractive function, this contradicts (19) and finishes the proof in the case when $f_0(x_0) < 1$ for some $x_0 \in \operatorname{supp}\mu_+$. If $f(x_0) > -1$ for some $x_0 \in \operatorname{supp}\mu_-$, the argument is similar. It is based on Remark 2. ∎

Using Theorem 2 it is easy to prove the following

COLOLLARY 1. *Let $\mu_1$ and $\mu_2$ be two distinct distributions. Assume that*

$$\operatorname{dist}(\operatorname{supp}(\mu_1 - \mu_2)_+, \operatorname{supp}(\mu_1 - \mu_2)_-) < 2. \tag{22}$$

*Then every $f_0 \in \mathcal{F}$ satisfying* (11) *fulfills condition* 1°.

*Proof.* Suppose on the contrary that there exists a contractive $f_0 \in \mathcal{F}$ such that

$$\varphi(f_0) = \sup_{g\in\mathcal{F}} \varphi(g). \tag{23}$$

Using (22) we can find points $x_0 \in \operatorname{supp}(\mu_1 - \mu_2)_+$ and $y_0 \in \operatorname{supp}(\mu_1 - \mu_2)_-$ such that $\varrho(x_0, y_0) \le 2$. On the other hand, by condition 2° of the maximum principle we have $f_0(x_0) - f_0(y_0) = 2$, which is impossible. ∎

**4. Applications.** Let $(X, \varrho)$ be a boundedly compact metric space. This assumption will not be repeated in what follows.

An operator $P : \mathcal{M} \to \mathcal{M}$ is called a *Markov operator* if it satisfies the following three conditions:

(i) $P$ is positively linear:

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2 \quad \text{for } \lambda_1, \lambda_2 \ge 0 \text{ and } \mu_1, \mu_2 \in \mathcal{M},$$

(ii) $P$ preserves the measure of the space:

$$P\mu(X) = \mu(X) \quad \text{for } \mu \in \mathcal{M}.$$

(iii) There exists an operator $U : B(X) \to B(X)$ such that

$$\langle Uf, \mu\rangle = \langle f, P\mu\rangle \quad \text{for } f \in B(X),\ \mu \in \mathcal{M}. \tag{24}$$

The operator $U$ is called *dual* to $P$. If in addition $Uf \in C(X)$ for $f \in C(X)$, then the Markov operator $P$ is called *Fellerian*.

Remark 3. Every Markov operator $P$ can be uniquely extended as a linear operator to the space of all signed measures.

Setting $\mu = \delta_x$ in (24) we obtain

$$(Uf)(x) = \langle f, P\delta_x\rangle \quad \text{for } f \in B(X),\ x \in X, \tag{25}$$

where $\delta_x \in \mathcal{M}_1$ is the point (Dirac) measure supported at $x$.

From formula (24) it follows immediately that $U$ is a linear operator satisfying the following conditions:

$$Uf \ge 0 \quad \text{for } f \ge 0,\ f \in B(X), \tag{26}$$

$$U\mathbf{1}_X = \mathbf{1}_X, \tag{27}$$

$$Uf_n \downarrow 0 \quad \text{for } f_n \downarrow 0,\ f_n \in B(X). \tag{28}$$

Remark 4. The dual operator $U$ has the unique extension to the set of all Borel measurable, nonnegative, not necessarily bounded functions on $X$, such that formula (24) holds.

Conditions (26)–(28) allow us to reverse the roles of $P$ and $U$. Namely if an operator $U$ satisfying (26)–(28) is given we may define a Markov operator $P : \mathcal{M} \to \mathcal{M}$ by setting

$$P\mu(A) = \langle U\mathbf{1}_A, \mu\rangle \quad \text{for } \mu \in \mathcal{M},\ A \in \mathcal{B}. \tag{29}$$

A mapping $\pi : X \times \mathcal{B} \to [0, 1]$ is called a *transition function* if $\pi(x, \cdot)$ is a probability measure for every $x \in X$ and $\pi(\cdot, A)$ is a measurable function for every $A \in \mathcal{B}$.

Having a transition function $\pi$ we may define the corresponding Markov operator $P : \mathcal{M}_{\mathrm{sig}} \to \mathcal{M}_{\mathrm{sig}}$ by the formula

$$P\mu(A) = \int_X \pi(x, A)\,\mu(dx) \quad \text{for } \mu \in \mathcal{M}_{\mathrm{sig}},\ A \in \mathcal{B}, \tag{30}$$

and its dual operator $U : B(x) \to B(X)$ by

$$Uf(x) = \int_X f(u)\,\pi(x,du). \tag{31}$$

Conversely, having a Markov operator $P$ we may define a function $\pi : X \times \mathcal{B} \to [0,1]$ by setting

$$\pi(x,A) = P\delta_x(A). \tag{32}$$

Clearly, $\pi$ is a transition function such that (30) is satisfied.

Thus, conditions (30), (32) show a one-to-one correspondence between the Markov operators and the transition functions.

Note that a Markov operator $P$ is Fellerian if and only if its transition function has the following property:

$$x_n \to x \quad \text{implies} \quad \pi(x_n,\cdot) \to \pi(x,\cdot) \text{ weakly.}$$

If this condition is satisfied the transition function $\pi$ is also called Fellerian.

A Markov operator $P$ is called *Lipschitzian* with a constant $k > 0$ if

$$\|P\mu_1 - P\mu_2\|_{\mathcal{F}} \le k\|\mu_1 - \mu_2\|_{\mathcal{F}} \quad \text{for } \mu_1,\mu_2 \in \mathcal{M}_1. \tag{33}$$

If $k \le 1$ then $P$ is a nonexpansive operator.

A family $(P^t)_{t\ge 0}$ of Markov operators is called a *semigroup* if

$$P^{t+s} = P^t P^s \quad \text{for } t,s \in \mathbb{R}^+$$

and $P^0 = I$ is the identity operator on $\mathcal{M}_{\text{sig}}$.

If the Markov operators $P^t$ for $t \in \mathbb{R}^+$ are Fellerian, we say that $(P^t)_{t\ge 0}$ is a *Markov–Feller semigroup*. We denote by $(U^t)_{t\ge 0}$ the semigroup of the dual operators to $(P^t)_{t\ge 0}$.

A Markov semigroup $(P^t)_{t\ge 0}$ is called *locally Lipschitzian* if there exists a locally bounded function $k : \mathbb{R}^+ \to \mathbb{R}^+$ such that for every $t \in \mathbb{R}^+$ the Markov operator $P^t$ is Lipschitzian with constant $k(t)$. If $k(t) \le 1$ for $t \in \mathbb{R}^+$, then $(P^t)_{t\ge 0}$ is a *nonexpansive semigroup*.

A nonexpansive semigroup $(P^t)_{t\ge 0}$ is called *strongly contracting* on $\mathcal{M}_1$ if for every $\mu_1,\mu_2 \in \mathcal{M}_1$, $\mu_1 \ne \mu_2$, there is a $t_0 \in \mathbb{R}^+$ such that

$$\|P^{t_0}\mu_1 - P^{t_0}\mu_2\|_{\mathcal{F}} < \|\mu_1 - \mu_2\|_{\mathcal{F}}.$$

A measure $\mu_* \in \mathcal{M}$ is called *stationary* (or *invariant*) for a Markov semigroup $(P^t)_{t\ge 0}$ if

$$P^t\mu_* = \mu_* \quad \text{for } t \in \mathbb{R}^+.$$

A Markov semigroup $(P^t)_{t\ge 0}$ is called *asymptotically stable* if there is a stationary distribution $\mu_*$ such that

$$\lim_{t\to\infty} \|P^t\mu - \mu_*\|_{\mathcal{F}} = 0 \quad \text{for } \mu \in \mathcal{M}_1. \tag{34}$$

The distribution $\mu_*$ satisfying (34) is unique.

A continuous $V : X \to [0, \infty)$ is called a *Lyapunov function* if

$$\lim_{\varrho(x,x_0)\to\infty} V(x) = \infty \tag{35}$$

for some $x_0 \in X$. Of course this definition is meaningful only in the case when $X$ is unbounded. It is evident that the validity on (35) does not depend on the choice of $x_0$.

A family $\varPi$ of probability measures on $X$ is said to be *tight* if for every positive $\varepsilon$ there exists a compact set $K$ such that

$$\mu(K) \geq 1 - \varepsilon \quad \text{for all } \mu \in \varPi. \tag{36}$$

Using the Lyapunov function, it is easy to give a sufficient condition for the tightness of trajectories of a Markov semigroup.

LEMMA 1. *Let $(P^t)_{t\geq 0}$ be a Markov–Feller semigroup and $(U^t)_{t\geq 0}$ its dual semigroup. Assume that there exists a Lyapunov function $V$ such that for $t \geq 0$,*

$$U^t V(x) \leq AV(x) + B \quad \text{for } x \in X, \tag{37}$$

*where $A, B$ are nonnegative constants. Then for every $\mu \in \mathcal{M}_1$ the family $(P^t\mu)_{t\geq 0}$ of distributions is tight.*

*Proof.* Fix an $\varepsilon > 0$ and a $\mu \in \mathcal{M}_1$. By the Ulam theorem we may choose a compact set $K \subset X$ such that

$$\mu(K) \geq 1 - \varepsilon/2.$$

Set $V_K = \sup_{x\in K} V(x)$. We define a new measure $\overline{\mu}$ by $\overline{\mu}(E) = \mu(E \cap K)$, where $E \in \mathcal{B}$. Let $Y = V^{-1}([0, q])$, where $q$ is a positive number satisfying

$$q \geq \frac{2}{\varepsilon}(AV_K + B). \tag{38}$$

Using the Chebyshev inequality and the definition of $\overline{\mu}$ we have

$$\begin{aligned} P^t\mu(Y) \geq P^t\overline{\mu}(Y) &\geq 1 - \frac{\varepsilon}{2} - \frac{1}{q}\int_X V(x)\, P^t\overline{\mu}(dx) \\ &= 1 - \frac{\varepsilon}{2} - \frac{1}{q}\int_X U^t V(x)\, \overline{\mu}(dx). \end{aligned}$$

Now using inequality (37) we obtain

$$P^t\mu(Y) \geq 1 - \frac{\varepsilon}{2} - \frac{1}{q}\Big[A\int_X V(x)\,\overline{\mu}(dx) + B\overline{\mu}(K)\Big].$$

From this and (38) it follows that

$$P^t\mu(Y) \geq 1 - \frac{\varepsilon}{2} - \frac{1}{q}[AV_K + B] \geq 1 - \varepsilon \quad \text{for } t \geq 0.$$

Since the set $Y$ is bounded and closed, it is compact. This completes the proof. ■

THEOREM 3. *Let $(P^t)_{t\geq 0}$ be a Markov–Feller semigroup and $(U^t)_{t\geq 0}$ its dual semigroup. Assume that*:

(i) *There is $t_0 \in \mathbb{R}^+$ such that for every $f \in \mathcal{F}$*,

$$|U^{t_0} f(x) - U^{t_0} f(y)| < \varrho(x,y) \quad \text{for } x,y \in X,\ x \neq y, \tag{39}$$

$$|U^t f(x) - U^t f(y)| \leq \varrho(x,y) \quad \text{for } x,y \in X \text{ and } t \in \mathbb{R}^+. \tag{40}$$

(ii) *For every $\mu_1,\mu_2 \in \mathcal{M}_1$, $\mu_1 \neq \mu_2$, there exists $t_1 \in \mathbb{R}^+$ such that*

$$\operatorname{dist}(\operatorname{supp}(P^{t_1}(\mu_1-\mu_2))_+, \operatorname{supp}(P^{t_1}(\mu_1-\mu_2))_-) < 2. \tag{41}$$

(iii) *There is a Lyapunov function $V$ such that*

$$U^t V(x) \leq AV(x) + B \quad \text{for } x \in X \text{ and } t \geq 0, \tag{42}$$

*where $A, B$ are nonnegative constants.*

*Then the semigroup $(P^t)_{t\geq 0}$ is asymptotically stable.*

*Proof.* From (40), it follows immediately that $U^t(\mathcal{F}) \subset \mathcal{F}$ for $t \in \mathbb{R}^+$ and that the Markov–Feller semigroup $(P^t)_{t\geq 0}$ is nonexpansive. Indeed, for $\mu_1,\mu_2 \in \mathcal{M}_1$ and $t \in \mathbb{R}^+$ we have

$$\|P^t\mu_1 - P^t\mu_2\|_{\mathcal{F}} = \sup\{|\langle U^t f, \mu_1-\mu_2\rangle| : f \in \mathcal{F}\} \leq \|\mu_1-\mu_2\|_{\mathcal{F}}. \tag{43}$$

We claim that $(P^t)_{t\geq 0}$ is also strongly contracting. For the proof fix $\mu_1,\mu_2 \in \mathcal{M}_1$, $\mu_1 \neq \mu_2$. According to the maximum principle for the Fortet–Mourier norm there exists a function $f_0 \in \mathcal{F}$ such that

$$\langle f_0, P^{t_0+t_1}\mu_1 - P^{t_0+t_1}\mu_2\rangle = \|P^{t_0+t_1}\mu_1 - P^{t_0+t_1}\mu_2\|_{\mathcal{F}}. \tag{44}$$

This equality may be rewritten in the form

$$\langle U^{t_0} f_0, P^{t_1}\mu_1 - P^{t_1}\mu_2\rangle = \|P^{t_0+t_1}\mu_1 - P^{t_0+t_1}\mu_2\|_{\mathcal{F}}. \tag{45}$$

The function $U^{t_0} f$ satisfies (39), so according to Corollary 1 and part 2° of the maximum principle applied to the measures $P^{t_1}\mu_1 - P^{t_1}\mu_2$ we obtain

$$\|P^{t_0+t_1}\mu_1 - P^{t_0+t_1}\mu_2\|_{\mathcal{F}} < \|P^{t_1}\mu_1 - P^{t_1}\mu_2\|. \tag{46}$$

From this and inequality (43) it follows that the Markov–Feller semigroup $(P^t)_{t\geq 0}$ is strongly contracting.

To complete the proof it is sufficient to verify that for every $\mu \in \mathcal{M}_1$ the trajectory $\{P^t\mu\}_{t\geq 0}$ is compact with respect to the Fortet–Mourier norm. Let $(t_n)$ be a sequence of integers such that $t_n \to \infty$ and $t_n \in \mathbb{R}^+$ for $n \in \mathbb{N}$. From Lemma 1 and condition (42) it follows that the family $(P^{t_n}\mu)_{n\in\mathbb{N}}$ of distributions is tight. Further, from the Prokhorov theorem it follows immediately that there exists a subsequence $(P^{t_{k_n}}\mu)_{n\in\mathbb{N}}$ which converges weakly to a measure $\mu_0 \in \mathcal{M}_1$.

We have verified that the semigroup $(P^t)_{t\geq 0}$ is strongly contracting and that the orbits are compact. According to the variational principle (see [9]) the semigroup $(P^t)_{t\geq 0}$ is asymptotically stable. ■

For locally Lipschitzian Markov semigroups the following version of Theorem 3 can be proved similarly:

THEOREM 4. *Let $(P^t)_{t\geq 0}$ be a locally Lipschitzian Markov semigroup on $\mathcal{M}_{\mathrm{sig}}$ and $(U^t)_{t\geq 0}$ its dual semigroup. Assume that*:

(i) *There is $t_0 \in \mathbb{R}^+$ such that for every $f \in \mathcal{F}$,*

$$|U^{t_0}f(x) - U^{t_0}f(y)| < \varrho(x,y) \quad \text{for } x,y \in X,\ x \neq y. \tag{47}$$

(ii) *For every $\mu_1,\mu_2 \in \mathcal{M}_1$, $\mu_1 \neq \mu_2$, there exists $n_0 \in \mathbb{N}$ such that*

$$\operatorname{dist}(\operatorname{supp}(P^{n_0t_0}(\mu_1-\mu_2))_+, \operatorname{supp}(P^{n_0t_0}(\mu_1-\mu_2))_-) < 2. \tag{48}$$

(iii) *There is a Lyapunov function $V$ such that for $n \geq 0$,*

$$U^{nt_0}V(x) \leq AV(x) + B \quad \text{for } x \in X, \tag{49}$$

*where $A, B$ are nonnegative constants.*

*Then $(P^t)_{t\geq 0}$ is asymptotically stable.*

We complete this series of sufficient conditions for the asymptotic stability of Markov semigroups by the following

THEOREM 5. *Let $(P^t)_{t\geq 0}$ be a Markov–Feller semigroup and $(U^t)_{t\geq 0}$ its dual semigroup. Assume that*:

(i) *There is $t_0 \in \mathbb{R}^+$ such that for every $f \in \mathcal{F}$,*

$$|U^{t_0}f(x) - U^{t_0}f(y)| < \varrho(x,y) \quad \text{for } x,y \in X,\ x \neq y, \tag{50}$$

$$|U^{t}f(x) - U^{t}f(y)| \leq \varrho(x,y) \quad \text{for } x,y \in X \text{ and } t \in \mathbb{R}^+. \tag{51}$$

(ii) *There exist $t_0, t_1, t_2 \in \mathbb{R}^+$ such that for every $f \in \mathcal{F}$, either*

$$U^{t_0+t_1}f(x) \in (-1,1] \quad \text{for } x \in X,$$

*or*

$$U^{t_0+t_2}f(x) \in [-1,1) \quad \text{for } x \in X.$$

(iii) *There is a Lyapunov function $V$ such that for $t \geq 0$,*

$$U^{t}V(x) \leq AV(x) + B \quad \text{for } x \in X, \tag{52}$$

*where $A, B$ are nonnegative constants.*

*Then $(P^t)_{t\geq 0}$ is asymptotically stable.*

*Proof.* We repeat the argument used in the proof of Theorem 3. However, in this case for $\mu_1,\mu_2 \in \mathcal{M}_1$, $\mu_1 \neq \mu_2$, equality (44) should be replaced by

$$\langle f_0, P^{t_0+\widetilde{t}}\mu_1 - P^{t_0+\widetilde{t}}\mu_2\rangle = \|P^{t_0+\widetilde{t}}\mu_1 - P^{t_0+\widetilde{t}}\mu_2\|_{\mathcal{F}}, \tag{53}$$

where $\widetilde{t} = \min(t_1, t_2)$ and $f_0 \in \mathcal{F}$. Again this may be rewritten in the form

$$\langle U^{t_0+\widetilde{t}} f_0, \mu_1 - \mu_2 \rangle = \|P^{t_0+\widetilde{t}}\mu_1 - P^{t_0+\widetilde{t}}\mu_2\|_{\mathcal{F}}. \tag{54}$$

The function $U^{t_0} f$ satisfies (50), so according to (ii) and part 2° of the Fortet–Mourier maximum principle for $\mu_1 - \mu_2$ we obtain

$$\|P^{t_0+\widetilde{t}}\mu_1 - P^{t_0+\widetilde{t}}\mu_2\| < \|\mu_1 - \mu_2\|.$$

This shows that $(P^t)_{t\geq 0}$ is strongly contracting with respect to the Fortet–Mourier norm. The remaining part of the proof is the same as for Theorem 3. ∎

We may simplify the verification of condition (ii). Namely, we have

PROPOSITION 1. *Let $\pi : X \times \mathcal{B} \to [0,1]$ be a Fellerian transition function. Assume that*

$$\operatorname{supp} \pi(x, \cdot) = X \quad \textit{for } x \in X. \tag{55}$$

*Then for every $f \in \mathcal{F}$, either*

$$Uf(x) \in (-1, 1] \quad \textit{for } x \in X,$$

*or*

$$Uf(x) \in [-1, 1) \quad \textit{for } x \in X,$$

*where $U$ is the corresponding dual operator* (31).

*Proof.* Fix $f \in \mathcal{F}$ and suppose that there exists an $x_1 \in X$ such that $Uf(x_1) = 1$. By the properties of the dual operator we have

$$U1_X(x_1) - Uf(x_1) = \int_X [1_X(y) - f(y)]\, \pi(x_1, dy) = 0.$$

From this and (55) it follows that

$$f(x) = 1 \quad \pi(x_1, \cdot)\text{-almost everywhere.} \tag{56}$$

Because $f$ is continuous, this is equivalent to

$$f(x) = 1 \quad \text{for } x \in X.$$

Since $U$ is the dual operator we conclude that

$$Uf(x) = 1 \quad \text{for } x \in X.$$

If there exists an $x_2 \in X$ such that $Uf(x_2) = -1$ the argument is similar. ∎

In order to illustrate the utility of Theorem 5 we show a sufficient condition for the asymptotic stability of a special Markov operator. It is related to mathematical models of the cell cycle [10].

EXAMPLE 1. We will study the asymptotic stability of Markov operators describing the evolution of measures due to the action of randomly chosen transformations. In the classical case the family of transformations is finite. Typical results of this kind may be found in [1, 2, 12, 4]. The case of an infinite family was recently discussed by P. Janoska in [7] and [8]. Our result is based on a quite different technique related to the maximum principle for the Fortet–Mourier norm. The main difference between our approach and the results of P. Janoska is that we do not assume any kind of compactness on the set of indices.

Again, let $(X,\varrho)$ be a boundedly compact metric space. Further, let $(I,\kappa)$ be a metric space of indices. We consider a continuous transformation $S : X\times I\to X$ and a function $F : X\times\mathcal{B}_I\to[0,1]$, where $\mathcal{B}_I$ denotes the $\sigma$-algebra of Borel subsets of $I$. We assume that $F$ satisfies the following conditions:

(1) For every $x\in X$ the mapping $F(x,\cdot):\mathcal{B}_I\to[0,1]$ is a probability measure.
(2) For every $A\in\mathcal{B}_I$ the function $F(\cdot,A):X\to X$ is measurable.

Now we present an imprecise description of the process considered in this example.

Choose an arbitrary point $x_0\in X$ and randomly select a point $i_0\in I$ according to the distribution $F(x_0,\cdot)$. When the point $i_0$ is drawn we define $x_1=S(x_0,i_0)$. Having $x_1$ we select $i_1\in I$ according to the distribution $F(x_1,\cdot)$ and we define $x_2=S(x_1,i_1)$ and so on. Denoting by $\mu_n, n=0,1,\ldots,$ the distribution of $x_n$, i.e. $\mu_n(A)=\mathrm{prob}(x_n\in A)$, we define $P$ as the transition operator such that $\mu_{n+1}=P\mu_n$.

The above procedure can be easily formalized. To do this fix $x\in X$ and set $\mu_0=\delta_x$. According to the description of our process and from the definition of the dual operator $U$ we have

$$Uf(x)=\langle Uf,\delta_x\rangle=\langle f,P\delta_x\rangle=\langle f,\mu_1\rangle\quad\text{ for } f\in B(X).$$

This means that $Uf(x)$ is the mathematical expectation of $f(x_1)$ if $x_0=x$ is fixed. On the other hand, according to our description, the expectation of $f(x_1)$ is equal to

$$\int_I f(S(x,i))\,F(x,di).$$

Since $x$ was arbitrary this implies

$$Uf(x)=\int_I f(S(x,i))\,F(x,di)\quad\text{ for } x\in X. \tag{57}$$

We take (57) as the precise formal definition of the operator $U$ and we define $P$ as the Markov operator corresponding to $U$. Thus $P$ is the unique operator satisfying

$$\langle f,P\mu\rangle=\langle Uf,\mu\rangle. \tag{58}$$

To formulate sufficient conditions of the asymptotic stability of $P$ we introduce the following notations.

Consider the class $\Phi$ of functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following three conditions:

(a) $\varphi$ is continuous and $\varphi(0) = 0$;
(b) $\varphi$ is nondecreasing and concave;
(c) $\varphi(x) > 0$ for $x > 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$.

We denote by $\Phi_0$ the family of functions satisfying (a), (b).

The inequality

$$\omega(t) + \widetilde{\varphi}(r(t)) \le \widetilde{\varphi}(t) \quad \text{for } t \ge 0 \tag{59}$$

plays an important role in the study of the asymptotic behaviour of the Markov operator $P$. In [12] Lasota and Yorke discussed the cases when the functional inequality (59) has a solution belonging to $\Phi$.

If $r(t) < t$ and $\widetilde{\varphi} \in \Phi$ is a solution of (59) then the function $\mathbb{R}^+ \ni t \mapsto \widetilde{\varphi}(t) + t \in \mathbb{R}^+$ satisfies the strict functional inequality

$$\omega(t) + \varphi(r(t)) < \varphi(t) \quad \text{for } t \ge 0. \tag{60}$$

The transition function $\pi : X \times \mathcal{B} \to [0,1]$ corresponding to $P$ is defined by the formula

$$\pi(x, A) = P\delta_x(A) = \int_I 1_A(S(x,i))\, F(x, di) \quad \text{for } (x,A) \in X \times \mathcal{B}. \tag{61}$$

Finally, denote by $\|\cdot\|_T$ the total variation norm in the space $\mathcal{M}_{\text{sig}}(I)$.

PROPOSITION 2. *Let $\omega, r \in \Phi_0$ be such that $0 \le r(x) < x$ and that the functional inequality* (60) *has a solution in the class $\Phi$. Moreover, assume that the following conditions are satisfied*:

(i) *we have*

$$\int_I \varrho(S(x,i), S(y,i))\, F(x, di) \le r(\varrho(x,y)) \quad \textit{for } x, y \in X, \tag{62}$$

$$\|F(x,\cdot) - F(y,\cdot)\|_T \le \omega(\varrho(x,y)) \quad \textit{for } x, y \in X. \tag{63}$$

(ii) *There is a point $x_0 \in X$ such that*

$$\sup_{x\in X} \int_I \varrho(x_0, S(x_0,i))\, F(x, di) < \infty. \tag{64}$$

(iii) *The transition function $\pi$ given by* (61) *satisfies*

$$\operatorname{supp} \pi(x,\cdot) = X \quad \textit{for } x \in X. \tag{65}$$

*Then the operator $P$ given by* (57) *and* (58) *is asymptotically stable.*

*Proof.* Consider a solution $\varphi \in \Phi$ of (60) corresponding to the pair $(\omega, r)$. The function

$$\varrho_\varphi(x,y) = \varphi(\varrho(x,y)) \quad \text{for } x, y \in X \tag{66}$$

is again a metric on $X$. Denote by $\|\cdot\|_\varphi$ the Fortet–Mourier norm generated by $\varrho_\varphi$, i.e.

$$\|\mu\|_{\mathcal{F}_\varphi} = \sup\{|\langle f,\mu\rangle| : f \in \mathcal{F}_\varphi\} \quad \text{for } \mu \in \mathcal{M}_{\text{sig}},$$

where $\mathcal{F}_\varphi \subset C(X)$ is the set of all $f$ such that $|f| \le 1$ and

$$|f(x) - f(y)| \le \varrho_\varphi(x,y) \quad \text{for } x,y \in X.$$

Now fix $f \in \mathcal{F}_\varphi$. We are going to show that $Uf$ is a contractive function with respect to the metric $\varrho_\varphi$. Using (57), (63) and the continuity of $S$ it is easy to verify that $Uf \in C(X)$ and $|Uf| \le 1$. Moreover for $x,y \in X$, $x \ne y$, we have

$$\begin{aligned}|Uf(x)-Uf(y)| &= \Big|\int_I f(S(x,i))\,F(x,di) - \int_I f(S(y,i))\,F(y,di)\Big| \\ &\le \|F(x,\cdot) - F(y,\cdot)\|_T + \int_I |f(S(x,i)) - f(S(y,i))|\,F(x,di).\end{aligned}$$

From this and (i) it follows that

$$\begin{aligned}|Uf(x) - Uf(y)| &\le \omega(\varrho(x,y)) + \int_I \varphi(\varrho(S(x,i),S(y,i)))\,F(x,di) \\ &\le \omega(\varrho(x,y)) + \varphi\Big(\int_I \varrho(S(x,i),S(y,i))\,F(x,di)\Big) \\ &\le \omega(\varrho(x,y)) + \varphi(r(\varrho(x,y)))\end{aligned}$$

According to (60), the last inequality implies

$$|Uf(x) - Uf(y)| < \varrho_\varphi(x,y). \tag{67}$$

Now, we will verify that

$$U^n V(x) \le r(1)V(x) + B \quad \text{for } x \in X \text{ and } n \in \mathbb{N}, \tag{68}$$

where $V(x) = \varrho(x,x_0)$ and

$$B = (1-r(1))^{-1}\Big(r(1) + \sup_{x\in X}\int_I \varrho(x_0,S(x_0,i))\,F(x,di)\Big).$$

In fact from (62) it follows that

$$\int_I \varrho(S(x,i),x_0)F(x,di) \le r(\varrho(x,x_0)) + \int_I \varrho(x_0,S(x_0,i))F(x,di). \tag{69}$$

Moreover, since $r$ is nondecreasing, concave and $r(0)=0$, we have

$$r(x) \le r(1)x + r(1).$$

The last inequality, (57) and (69) imply (68).

By Proposition 1 and Theorem 5 the operator $P$ is asymptotically stable with respect to the Fortet–Mourier norm $\|\cdot\|_{\mathcal{F}_\varphi}$ generated by $\varrho_\varphi$.

Finally, because the classes of convergent sequences in both spaces $(\mathcal{M}_{\mathrm{sig}}, \|\cdot\|_{\mathcal{F}_\varphi})$ and $(\mathcal{M}_{\mathrm{sig}}, \|\cdot\|_{\mathcal{F}})$ are the same, the operator is asymptotically stable with respect to the Fortet–Mourier norm $\|\cdot\|_{\mathcal{F}}$. ■

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Institute of Mathematics
Silesian University
Bankowa 14
40-007 Katowice, Poland
E-mail: gacki@ux2.math.us.edu.pl


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