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B-regularity of certain domains in $\mathbb{C}^n$

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Abstract. We study the B-regularity of some classes of domains in $\mathbb{C}^n$. The results include a complete characterization of B-regularity in the class of Reinhardt domains, we also give some sufficient conditions for Hartogs domains to be B-regular. The last result yields sufficient conditions for preservation of B-regularity under holomorphic mappings.

I. Introduction. Let Ω be a bounded domain in $\mathbb{R}^n$. We denote by $\mathcal{C}(\partial\Omega)$ and $\mathcal{C}(\overline{\Omega})$ the spaces of real-valued continuous functions on $\partial\Omega$ and $\overline{\Omega}$, respectively. An important problem of (real) potential theory is whether every function $f \in \mathcal{C}(\partial\Omega)$ can be extended to a function $u \in \mathcal{C}(\overline{\Omega})$ which is harmonic on Ω. It is a classical fact that the function u, if it exists, can be computed as follows:

$$u = u_{f,\Omega} := \sup\{v \in \mathrm{SH}(\Omega) : v^* \leq f \text{ on } \partial\Omega\},$$

where $\mathrm{SH}(\Omega)$ denotes the cone of subharmonic functions on Ω and v^* is the upper regularization of v, which is defined on $\overline{\Omega}$. The function $u_{f,\Omega}$ is called the *Perron envelope* of f and it is easy to check that it is always harmonic on Ω even if f is only assumed to be bounded. A beautiful result of potential theory states that the Perron envelope solves the above problem *if and only if* for each boundary point $x \in \partial\Omega$ there exists a *barrier* at x, i.e. $u \in \mathrm{SH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $u(x) = 0$ whereas $u < 0$ elsewhere. In particular, this condition is satisfied if $\partial\Omega$ is smooth.

It is natural to consider a similar problem in the *complex setting*. Namely, if Ω is a bounded domain in $\mathbb{C}^n$, under what conditions can every $f \in \mathcal{C}(\partial\Omega)$ be extended to a maximal plurisubharmonic function u on Ω which is continuous on $\overline{\Omega}$? Here we recall that a plurisubharmonic function u is

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maximal if for every relatively compact subdomain Ω' of Ω and every plurisubharmonic function v on Ω' such that $v^* \le u$ on $\partial\Omega'$ we have $v \le u$ on Ω'. Using an argument analogous to the real case, Bremermann shows in [Br] that such a function u, if it exists, can be written as

$$u = u_{f,\Omega} := \sup\{v \in \mathrm{PSH}(\Omega) : v^* \le f \text{ on } \partial\Omega\},$$

where $\mathrm{PSH}(\Omega)$ is the cone of plurisubharmonic functions on Ω. However, here we encounter a difficulty as the Perron–Bremermann envelope $u_{f,\Omega}$ may not be upper semicontinuous on Ω. To check the continuity of $u_{f,\Omega}$ on $\overline{\Omega}$, according to a result of Walsh in [Wa], it suffices to verify that $\lim_{z\to x} u_{f,\Omega}(z) = f(x)$ for every boundary point $x \in \partial\Omega$.

Building upon the work of Bremermann and Walsh, Sibony in [Si] has given some characterizations of Ω so that the above mentioned problem always has a solution (see also [Bl] and [Wi]). The goal of this note is to apply the results and methods of [Si] to study B-regularity of some concrete classes of domains in $\mathbb{C}^n$. The first result of the paper gives a full description of B-regularity for Reinhardt domains. Next, we give some sufficient conditions for B-regularity of Hartogs domains and the inverse of B-regular domains.

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II. Preliminaries. We start off with a standard notion:

DEFINITION 2.1. A bounded open set Ω in $\mathbb{C}^n$ is said to be *hyperconvex* if there is a negative plurisubharmonic exhaustion function for Ω.

It is a well known result of Kerzman and Rosay (see [KR]) that every bounded pseudoconvex open set with $\mathcal{C}^1$ smooth boundary in $\mathbb{C}^n$ is hyperconvex. Moreover, they also prove that every hyperconvex open set in $\mathbb{C}^n$ admits a negative $\mathcal{C}^\infty$ smooth strictly plurisubharmonic exhaustion function. For a more refined version of the latter result see Theorem 6.2 in [Bl].

The next concepts are crucial to our paper:

DEFINITIONS 2.2.

(a) A compact set K in $\mathbb{C}^n$ is called *B-regular* if every continuous function on K can be approximated uniformly on K by continuous plurisubharmonic functions on neighbourhoods of K. A locally closed set K is said to be *locally B-regular* if for every $a \in K$ there is a ball U centred at a such that $K \cap \overline{U}$ is B-regular,

(b) A bounded open set Ω is called *B-regular* if every (real-valued) continuous function on $\partial\Omega$ can be extended to a function plurisubharmonic on Ω and continuous on $\overline{\Omega}$.

REMARKS 2.3. (i) These definitions are taken from [Si, p. 301] and [Bl, p. 721] respectively. Notice that in [Si], Sibony mainly studies the class of bounded pseudoconvex domains with $\mathcal{C}^1$ smooth and B-regular boundaries. See Theorem 2.4 below for some connections between these domains and B-regular ones.

(ii) It is immediate that every compact subset of a B-regular compact set is also B-regular. Moreover by Proposition 1.4 in [Si], a compact set K in $\mathbb{C}^n$ is B-regular if and only if it is locally B-regular.

(iii) By an *analytic disk* in $\mathbb{C}^n$, we mean the image of a holomorphic mapping from the open unit disk Δ to $\mathbb{C}^n$. Using the maximum principle, we see that if a compact set K is B-regular then K *has no analytic structure*, i.e., contains no non-constant analytic disk. The converse is not true; indeed, by Proposition 1.11 in [Si] we know that every compact set in $\mathbb{C}$ is B-regular if and only if its fine interior is empty.

(iv) If Ω is a bounded open set such that there is a sequence of holomorphic mappings $\varphi_j : \Delta \to \mathbb{C}^n$ such that $\varphi_j(\Delta) \subset \Omega$ and φ_j converges locally uniformly on Δ to a non-constant holomorphic mapping $\varphi : \Delta \to \mathbb{C}^n$ satisfying $\varphi(\Delta) \subset \partial\Omega$ then Ω is *not* B-regular. Indeed, since φ is non-constant, we can find $0 < r < 1$ such that $\varphi(0) \neq \varphi(z)$ for all $z \in \partial\Delta_r$, where $\Delta_r = \{z : |z| < r\}$. Choose $u \in \mathcal{C}(\partial\Omega)$ such that $(u \circ \varphi)(0) = 1/2$ and $(u \circ \varphi)(z) = 0$ if $z \in \partial\Delta_r$. Assume that there is some $\widetilde{u} \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $\widetilde{u} = u$ on $\partial\Omega$. Then by applying the maximum principle to the subharmonic functions $\widetilde{u} \circ \varphi_j$ we obtain $(\widetilde{u} \circ \varphi_j)(0) \leq \sup_{\partial\Delta_r} \widetilde{u} \circ \varphi_j$. Letting j tend to ∞ we get a contradiction to the choice of u. In particular, if Ω is B-regular and if $\partial\Omega$ is $\mathcal{C}^1$ smooth then $\partial\Omega$ contains no non-constant analytic disk, for otherwise, by translating the disk along some inward normal vector and applying the above remark, we get a contradiction to the B-regularity of Ω.

The next theorem of Sibony (see Theorem 2.1 in [Si] or Theorem 1.7 in [Bl]) describes the main relationships between the concepts given in Definitions 2.2.

THEOREM 2.4. *Let Ω be a bounded open set in $\mathbb{C}^n$. If Ω is hyperconvex and $\partial\Omega$ is B-regular then Ω is B-regular. Conversely, if Ω is B-regular then Ω is hyperconvex, and if in addition $\partial\Omega$ is of class $\mathcal{C}^1$ then $\partial\Omega$ is B-regular.*

Notice that the first assertion of the theorem was proven in [Si] under the stronger assumption that Ω is a bounded pseudoconvex domain having a smooth B-regular boundary. However, the proof given there works also for hyperconvex domains with B-regular boundary (see Remarks 2.10(ii)). In Lemma 4.2 we will show that if $\partial\Omega$ is $\mathcal{C}^1$ smooth near a point $a \in \partial\Omega$ and if there is a barrier at a (with respect to Ω) then $J_a(\partial\Omega) = \{\delta_a\}$ (for the notation see Definition 2.6). The last assertion of Theorem 2.4 will follow immediately from this result.

The most convenient tool in verifying B-regularity of bounded domains in $\mathbb{C}^n$ is perhaps the following theorem.

THEOREM 2.5. *Let Ω be a bounded open set in $\mathbb{C}^n$. Then the following statements are equivalent.*

(i) *Ω is B-regular.*

(ii) *For each $z_0 \in \partial\Omega$ there exists a barrier at z_0 with respect to Ω, i.e., there is $u \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $u(z_0) < 0$ and $u < 0$ on $\overline{\Omega} \setminus \{z_0\}$.*

(iii) *For each $z_0 \in \partial\Omega$ there exists a local barrier at z_0, i.e., there exist a neighbourhood U of z_0 and $\varphi \in \mathrm{PSH}(\Omega \cap U) \cap \mathcal{C}(\overline{\Omega} \cap \overline{U})$ such that $\varphi(z_0) = 0$ and $\varphi < 0$ on $\overline{\Omega} \cap (\overline{U} \setminus \{z_0\})$.*

Proof. This theorem is implicitly contained in [Si]. For the sake of completeness we sketch a proof. It suffices to show (iii)⇒(i) since the other implications are trivial. After shrinking U we can find $\varepsilon > 0$ so small that $\varphi \le -\varepsilon$ on $\overline{\Omega} \cap \partial U$. Define

$$\widetilde{\varphi} = \begin{cases} \max(\varphi, -\varepsilon) & \text{on } \overline{\Omega} \cap \overline{U}, \\ -\varepsilon & \text{on } \overline{\Omega} \setminus \overline{U}. \end{cases}$$

It is easy to see that $\widetilde{\varphi} \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$, $\widetilde{\varphi}(z_0) = 0$ and $\widetilde{\varphi} < 0$ on $\overline{\Omega} \setminus \{z_0\}$. Applying Theorem 1.7 of [Bl] (see also Theorem 2.1 in [Si]) we conclude that Ω is B-regular.

It follows easily from this theorem that every strictly pseudoconvex domain is B-regular. A useful approach to the existence of a barrier at a given boundary point $z_0 \in \partial\Omega$ of a bounded domain Ω is via the concept of Jensen measures with barycentre at z_0.

DEFINITION 2.6. Let K be a compact set in $\mathbb{C}^n$ and $z_0 \in K$. The set of *Jensen measures with barycentre at* z_0, denoted by $J_{z_0}(K)$, is the collection of all positive, regular Borel measures μ supported in K such that $\mu(K) = 1$ and for every plurisubharmonic function u on a neighbourhood of K we have

$$u(z_0) \le \int_K u \, d\mu. \tag{1}$$

Some remarks seem to be appropriate at this point.

REMARKS 2.7. (i) By a standard regularization, it is no difference to require only that (1) holds for *smooth* plurisubharmonic functions on neighbourhoods of K.

(ii) By Proposition 1.3 in [Si], K is B-regular if and only if $J_z(K) = \{\delta_z\}$ for every $z \in K$, where δ_z is the Dirac mass at z.

(iii) In [Po, p. 416], another definition of $J_{z_0}(K)$ is given where the inequality (1) is required to hold for all plurisubharmonic functions on K. Here, by a plurisubharmonic function on K, Poletsky means, roughly speaking, an upper semicontinuous function u on K satisfying the submean value inequality on the cluster set of any sequence of uniformly bounded analytic disks that "converge" towards K. For the precise definition, we refer to Section 3 of [Po]. In Lemma 4.1 we will show that the two classes of Jensen measures introduced by Sibony and Poletsky are the same. This fact will enable us to use a basic theorem of Poletsky saying that every $\mu \in J_{z_0}(K)$ can be approximated in the weak-$*$ topology by holomorphic measures.

(iv) In the case $K = \overline{\Omega}$, where Ω is a bounded domain in $\mathbb{C}^n$, Wikström introduces in [Wi] (see also Definition 2.1 in [CCW]) the class $J^{\mathrm{c}}_{z_0}(K)$ of Jensen measures by requiring (1) to be true for the set of continuous functions on $\overline{\Omega}$ which are plurisubharmonic on Ω. Obviously we have $J^{\mathrm{c}}_{z_0}(K) \subset J_{z_0}(K)$. The inclusion might be strict; for instance, when $\Omega = \Delta \setminus [-1/2, 1/2]$, where Δ is the unit disk in $\mathbb{C}$, it is not hard to check that $J^{\mathrm{c}}_0(K) = \{\delta_0\}$ while $J_0(K)$ contains m, the normalized Lebesgue measure on the unit circle. On the other hand, if $\partial\Omega$ is $\mathcal{C}^1$ smooth, then according to Theorem 1 in [FW], every continuous function on $\overline{\Omega}$ which is plurisubharmonic on Ω can be approximated uniformly on $\overline{\Omega}$ by continuous plurisubharmonic functions on neighbourhoods of $\overline{\Omega}$. Therefore, in this case, $J^{\mathrm{c}}_{z_0}(K) = J_{z_0}(K)$ for every $z_0 \in K$.

We also need the following fact which is probably known:

LEMMA 2.8. *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and $z_0 \in \partial\Omega$. Assume that $J_{z_0}(\partial\Omega) = \{\delta_{z_0}\}$. Then there exists a barrier at z_0, i.e., there is a $u \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ so that $u(z_0) = 0$ while $u < 0$ on $\overline{\Omega} \setminus \{z_0\}$.*

For the proof, we require

LEMMA 2.9. *Let Ω be a bounded hyperconvex open set in $\mathbb{C}^n$ and u be a $\mathcal{C}^2$ smooth plurisubharmonic function on a neighbourhood U of $\partial\Omega$. Then there exists $u' \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $u' = u$ on $\partial\Omega$.*

Proof. Choose a negative $\mathcal{C}^\infty$ smooth strictly plurisubharmonic exhaustion function φ for Ω. Let θ be a $\mathcal{C}^\infty$ smooth function with compact support in U such that $\theta = 1$ on a neighbourhood of $\partial\Omega$. Then we can choose $u' = C\varphi + \theta u$ for some large constant C.

Proof of Lemma 2.8. We set

(2) $\mathcal{A} = \{f \in \mathcal{C}(\partial\Omega) : \text{there is } u \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\overline{\Omega}) \text{ with } u|_{\partial\Omega} \equiv f\}.$

We are going to prove the four assertions below, where (iii) is used to prove (iv).

(i) $\mathcal{A}$ is closed in $\mathcal{C}(\partial\Omega)$.
(ii) $\mathcal{A}$ is a convex cone that contains all constants.
(iii) If $f, g \in \mathcal{A}$ then $\max\{f, g\} \in \mathcal{A}$ and $\chi \circ f \in \mathcal{A}$ for every increasing convex function $\chi : \mathbb{R} \to \mathbb{R}$.
(iv) For every neighbourhood U of z_0 there exists $\varphi \in \mathcal{A}$ such that $\varphi(z_0) = -1$, $\varphi \le -2$ on $(\partial\Omega) \setminus U$ and $\varphi \le 0$ on $\partial\Omega$.

Assuming for the moment that (i), (ii), (iv) are proved, we can apply *the proof* of Lemma 3.2 in [Po] to obtain $\psi \in \mathcal{A}$ such that $\psi(z_0) = 0$ and $\psi < 0$ elsewhere. Extend ψ to a function $u \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$; by the maximum principle we have $u < 0$ on $\overline{\Omega} \setminus \{z_0\}$ whereas $u(z_0) = 0$. The desired conclusion follows.

Now it remains to prove (i)–(iv). For (i) we just apply Corollary 3.9 of [Wi], and (ii), (iii) follow immediately from (2). For (iv), we let $\mathcal{J}$ denote the set of positive, regular Borel measures μ supported on $\partial\Omega$ satisfying

$$u(z_0) \le \int_{\partial\Omega} u \, d\mu \quad \text{for all } u \in \mathcal{A}.$$

By applying the Edwards duality theorem (see [Ed] or Theorem 2.1 in [Wi]) to the data $(\partial\Omega, \mathcal{A}, \mathcal{J})$ we get for every $h \in \mathcal{C}(\partial\Omega)$,

(3) $$\sup\{\varphi(z_0) : \varphi \in \mathcal{A},\ \varphi \le h\} = \inf\Big\{ \int_{\partial\Omega} h \, d\mu : \mu \in \mathcal{J} \Big\}.$$

We claim that $\mathcal{J} \subset J_{z_0}(\partial\Omega)$. Indeed, let $\mu \in \mathcal{J}$ and u be an arbitrary $\mathcal{C}^2$ smooth plurisubharmonic function on a neighbourhood of $\partial\Omega$. By Lemma 2.9 there is $u' \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $u' = u$ on $\partial\Omega$. It follows that

$$u(z_0) = u'(z_0) \le \int_{\partial\Omega} u' \, d\mu = \int_{\partial\Omega} u \, d\mu.$$

This proves the claim. Thus, by our assumption, $\mathcal{J} = \{\delta_{z_0}\}$. Combining this with (3) we get for every $h \in \mathcal{C}(\partial\Omega)$,

$$\sup\{\varphi(z_0) : \varphi \in \mathcal{A},\ \varphi \le h\} = h(z_0).$$

Now let U be an arbitrary neighbourhood of z_0 and $h \in \mathcal{C}(\partial\Omega)$ so that $h \le 2$, $h(z_0) = 1$ and $h \equiv -1$ on $(\partial\Omega) \setminus U$. Using the topological Choquet lemma and (iii), we get an increasing sequence $\{\varphi_j\} \subset \mathcal{A}$ such that $\varphi_j \le h$ on $\partial\Omega$ and $\{\varphi_j(z_0)\}$ converges to 1. It follows that there is j_0 large enough such that $\varphi_{j_0} \le 2$ on $\partial\Omega$, $2/3 < a := \varphi_{j_0}(z_0) < 3/2$ and $\varphi_{j_0} \le -1$ on

$(\partial\Omega) \setminus U$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be the function

$$\chi(x) = \begin{cases} \dfrac{x-a}{a+1}, & x < a, \\ \dfrac{x-2}{2-a} + 1, & x \geq a. \end{cases}$$

Then $\chi(a) = 0$, $\chi(2) = 1$, $\chi(-1) = -1$, and it is easy to see that χ is convex and increasing; thus by (iii), $\chi \circ \varphi_{j_0} \in \mathcal{A}$. Moreover, from the choice of φ_{j_0} we can check that the function $\varphi := \chi \circ \varphi_{j_0} - 1$ satisfies (iv). The proof is complete.

REMARKS 2.10. (i) In Theorem 2.1 of [Wi] the author works with a slightly different notion of Jensen measures, namely he considers just positive measures (not necessarily Borel and regular). However, since the space $\mathbb{C}^n$ is locally compact, by the Riesz representation theorem, the measure s that appears in the *proof* of Theorem 2.1 in [Wi] can be chosen to be Borel and regular. This justifies the use of Theorem 2.1 of [Wi] in our context. Observe that similar uses of the Edwards theorem have been made in Theorem 2.8 of [CCW] and Corollary 2.2 of [Wi].

(ii) It follows immediately from Lemma 2.8 that every hyperconvex bounded domain with B-regular boundary is B-regular.

III. Results. We first discuss the B-regularity of Reinhardt domains in $\mathbb{C}^n$; a weaker version of the result below is stated in Proposition 2.4 of [Si].

PROPOSITION 3.1. *Let Ω be a bounded Reinhardt domain in $\mathbb{C}^n$. Then Ω is B-regular if and only if Ω is hyperconvex and $\partial\Omega$ has no analytic structure.*

Recall that a domain Ω is called *Reinhardt* if for every $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ and $(z_1, \ldots, z_n) \in \Omega$ we have $(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in \Omega$. We also let

$$V_j = \{(z_1, \ldots, z_n) : z_j = 0\}, \qquad V = \bigcup_{1 \leq j \leq n} V_j,$$

$$\log \Omega_* = \{(\log|z_1|, \ldots, \log|z_n|) : (z_1, \ldots, z_n) \in \Omega \setminus V\}.$$

The following characterizations of pseudoconvexity and hyperconvexity of bounded Reinhardt domains play essential roles in the proof of Proposition 3.1. It should be remarked that while Lemma 3.2 below is well known (see [Zw1], [Zw2]), Lemma 3.3 is a recent result due to Zwonek (see [Zw2]).

LEMMA 3.2. *Let Ω be a bounded Reinhardt domain. Then Ω is pseudoconvex if and only if the set* $\log \Omega_*$ *is convex in* $\mathbb{R}^n$ *and for each if $\Omega \cap V_j \neq \emptyset$ then the condition $(z_1, \ldots, z_j, \ldots, z_n) \in \Omega$ implies $(z_1, \ldots, \lambda z_j, z_{j+1}, \ldots, z_n) \in \Omega$ for all $|\lambda| < 1$.*

LEMMA 3.3. *Let Ω be a bounded pseudoconvex Reinhardt domain. Then Ω is hyperconvex if and only if $\overline{\Omega} \cap V_j \neq \emptyset$ implies $\Omega \cap V_j \neq \emptyset$.*

Proof of Proposition 3.1. It is enough to prove the implication "$\Leftarrow$". Let $a = (a_1, \dots, a_n)$ be an arbitrary point in $\partial\Omega$. According to Theorem 2.5 we need to show that there exists a local barrier at a. After a linear change of coordinates, we may assume that $a = (1, \dots, 1, 0, \dots, 0)$, where there are k 1's and $0 \le k \le n$. Since Ω is hyperconvex, by Lemma 3.3 we have $1 \le k \le n$. There are two cases to be considered.

CASE 1: $k = n$. Since Ω is Reinhardt and pseudoconvex, the domain $\log \Omega_*$ is convex. Obviously the origin is in $\partial(\log \Omega_*)$. So we can find a hyperplane passing through the origin and disjoint from $\log \Omega_*$. Hence there is $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and a small neighbourhood U of a such that the function

$$\varphi(z) = \sum_{j=1}^{n} \alpha_j \log |z_j|$$

is plurisubharmonic on $\Omega \cap U$, continuous on $\overline{\Omega} \cap \overline{U}$ and satisfies $\varphi < 0$ on $\Omega \cap U$ whereas $\varphi(a) = 0$.

Now we claim that $\varphi < 0$ on $(\partial\Omega) \cap (U \setminus \{a\})$. Otherwise, there is some $b = (b_1, \dots, b_n) \in (\partial\Omega) \setminus V$ such that $b \neq a$ and $\varphi(b) = 0$. Define the holomorphic map

$$\psi : \mathbb{C} \to \mathbb{C}^n, \qquad z \mapsto (e^{\log |b_1| z}, \dots, e^{\log |b_n| z}).$$

Since $\log \Omega_*$ is convex, the segment $\{(t \log |b_1|, \dots, t \log |b_n|) : 0 \le t \le 1\}$ is contained in $\partial(\log \Omega_*)$. This implies that $\psi(H) \subset \partial\Omega$, where H is the strip $\{z : 0 < \Re z < 1\}$. Notice that $\partial\Omega$ has no analytic structure, so ψ is constant, which is clearly absurd. The claim follows. It implies that φ is a local barrier at a.

CASE 2: $1 \le k < n$. By Lemma 3.3 we have $\Omega \cap V_j \neq \emptyset$ for every $k+1 \le j \le n$. Let π denote the projection

$$\pi : \mathbb{C}^n \to \mathbb{C}^k, \qquad (z_1, \dots, z_n) \mapsto (z_1, \dots, z_k).$$

By Lemmas 3.2 and 3.3, $\pi(\Omega)$ is a bounded hyperconvex Reinhardt domain in $\mathbb{C}^k$.

Now we claim that $\partial(\pi(\Omega))$ has no analytic structure. Otherwise we can find a non-constant holomorphic mapping $F : \Delta \to \mathbb{C}^k$ such that $F(\Delta) \subset \partial(\pi(\Omega))$. By Lemma 3.2 we infer that the non-constant holomorphic mapping $\widetilde{F} = (F, 0, \dots, 0)$ satisfies $\widetilde{F}(\Delta) \subset \partial\Omega$. This contradicts the assumption on $\partial\Omega$.

Since Ω is hyperconvex Reinhardt, using Lemma 3.3 we deduce that $\pi(a) \notin \pi(\Omega)$. It follows that $\pi(a) \in \partial(\pi(\Omega))$. From the result proven in Case 1 we get a neighbourhood U (in $\mathbb{C}^k$) of $\pi(a)$ and a barrier u at $\pi(a)$

with respect to $U \cap \pi(\Omega)$. It is clear that $u \circ \pi$ is a barrier at a with respect to $\pi^{-1}(U) \cap \Omega$.

COROLLARY 3.4. *Let $a_1, \ldots, a_n$ be positive numbers. Then the domain*

$$\Omega = \{(z_1, \ldots, z_n) : |z_1|^{a_1} + \cdots + |z_n|^{a_n} < 1\}$$

is B-regular.

Proof. Since $a_i > 0$, from Lemmas 3.2 and 3.3 we infer that Ω is a bounded hyperconvex Reinhardt domain in $\mathbb{C}^n$.

We claim that $\partial\Omega$ has no analytic structure. Indeed, suppose $\Phi = (\varphi_1, \ldots, \varphi_n)$ is a non-constant holomorphic mapping from Δ to $\mathbb{C}^n$ such that $\Phi(\Delta) \subset \partial\Omega$. We may assume that the first k components of Φ are non-vanishing on some disk $\Delta' \subset \Delta$ for some $1 \le k \le n$. For each $1 \le j \le k$, we write $\varphi_j = e^{\psi_j}$ where ψ_j is holomorphic on Δ'. Then

$$\sum_{j=1}^{k} e^{a_j \Re \psi_j} = 1, \qquad \forall z \in \Delta'.$$

Applying the operator $\partial^2/\partial z \partial \overline{z}$ to both sides we get

$$\sum_{j=1}^{k} a_j^2 \left| \frac{\partial \psi_j}{\partial z} \right|^2 e^{a_j \Re \psi_j} = 0, \qquad \forall z \in \Delta'.$$

This is clearly absurd. Thus $\partial\Omega$ has no analytic structure. This implies that Ω is B-regular, in view of Proposition 3.1.

Before formulating the next result we recall that a plurisubharmonic function u is called *strictly plurisubharmonic* at a point a if there is some neighbourhood U of z and $\lambda > 0$ so that $\varphi(z) - \lambda|z|^2$ is plurisubharmonic on U. Observe that the set of points where u is strictly plurisubharmonic is open.

PROPOSITION 3.5. *Let Ω be a bounded domain in $\mathbb{C}^n$ and φ be an upper semicontinuous function on Ω which is bounded from below. Let* $\Omega_\varphi = \{(z, w) : z \in \Omega, \log|w| + \varphi(z) < 0\}$.

(a) *If Ω_φ is B-regular then*

(i) *Ω is B-regular.*

(ii) $\varphi \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\Omega)$ *and* $\lim_{z \to \xi} \varphi(z) = \infty$ *for all* $\xi \in \partial\Omega$.

(iii) *For every non-constant analytic disk S contained in Ω, the restriction of φ to S is not harmonic.*

(b) *If Ω and φ satisfy conditions* (i), (ii) *and if the set*

$$X = \{z \in \Omega : \varphi \text{ is not strictly plurisubharmonic at } z\}$$

is locally connected and locally B-regular then Ω_φ is B-regular.

We need some lemmas; the first two of them are rather elementary.

LEMMA 3.6. *Let θ, $\{\theta_j\}_{j\geq 1}$ be continuous mappings from an open set U in $\mathbb{C}^n$ to $\mathbb{C}^p$. Assume that $\{\theta_j\}$ converges to θ uniformly on U. Then for every open set $V \subset \theta(U)$ and $a \in \theta^{-1}(V)$, there exist an open subset U' of $\theta^{-1}(V)$ and j_0 so large that $a \in U' \subset \theta_j^{-1}(V)$ for all $j \geq j_0$.*

Proof. Assume that the conclusion of the lemma is false; then we can find a sequence $\{z_j\} \subset \theta^{-1}(V)$ with $z_j \to a$ such that $\theta_j(z_j) \notin V$. It follows that there exist $\varepsilon > 0$ and $j_0 \geq 1$ such that $|\theta_j(z_j) - \theta(a)| > \varepsilon$ for all $j \geq j_0$. This is absurd since θ_j converges uniformly to θ on U.

LEMMA 3.7. *Let X be a connected compact set in $\mathbb{C}^n$ and $\varphi \in \mathcal{C}(X)$. Then the compact set*

$$X_\varphi = \{(z,w) : \log|w| + \varphi(z) = 0,\ z \in X\}$$

is connected.

Proof. Assume that X_φ is not connected. Then we can find non-empty disjoint open subsets U_1, U_2 of $\mathbb{C}^n$ such that $X_\varphi = V_1 \cup V_2$, where $\emptyset \neq V_i := X_\varphi \cap U_i, i = 1, 2$. Let π be the projection $(z,w) \mapsto z$. Then $X \subset \pi(V_1) \cup \pi(V_2)$. Since X is connected, there exists $z_0 \in \pi(V_1) \cap \pi(V_2)$. Observing that $\pi^{-1}(z_0) \cap X_\varphi = \{z_0\} \times \{w : |w| = e^{-\varphi(z_0)}\}$ is a connected compact set, we get a contradiction.

The next lemma is of independent interest.

LEMMA 3.8. *Let X, Y be compact sets in $\mathbb{C}^n$ and $\mathbb{C}^p$ respectively. Assume that θ is a holomorphic mapping from a neighbourhood U of X to $\mathbb{C}^p$ such that*

(i) *$Y = \theta(X)$ is B-regular.*
(ii) *$\theta^{-1}(z) \cap X$ is B-regular for every $z \in Y$.*

Then X is B-regular if one of the following conditions holds:

(iii) *X is connected and θ can be approximated uniformly on U by holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$.*
(iii′) *X is locally connected.*

REMARKS 3.9. (i) A stronger version of the above result, where X is neither supposed to be connected nor locally connected and $\theta \in \mathcal{C}(X)$ is merely assumed to be approximated uniformly on X by holomorphic functions on neighbourhoods of X, was claimed in Proposition 1.10 of [Si]. The proof of that result, in our opinion, contains a gap. Namely, we do not understand why the B-regularity of Y implies the density of the set of continuous plurisubharmonic function on neighbourhoods of X in the set of continuous plurisubharmonic functions on neighbourhoods of $\theta^{-1}(\theta(x))$ where x is some point in X. By employing some deep results of Poletsky in [Po] we are able to give a proof of Lemma 3.8. It remains, however, an open problem whether the above mentioned claim of Sibony is correct.

(ii) The image of a B-regular compact set under a holomorphic mapping need not be B-regular. Indeed, let X be the compact set $\{(z, \overline{z}) : |z| \le 1\}$ and π be the projection $(z, w) \mapsto z$. By the Stone–Weierstrass theorem, every continuous function on X can be approximated uniformly by polynomials in $\mathbb{C}^2$. This implies that X is B-regular. On the other hand, $\pi(X) = \{z : |z| \le 1\}$ is obviously not B-regular.

Proof of Lemma 3.8. We first show that X is B-regular under assumption (iii). Let $\{\theta_j\}$ be a sequence of holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$ such that $\{\theta_j\}$ converges uniformly to θ on U. Let $\xi_0 \in X$ and $\mu \in J_{\xi_0}(X)$. According to Theorem 3.2 in [Po], we can approximate μ by holomorphic measures in the following sense: There exists a sequence $L = \{f_j\}$ of uniformly bounded holomorphic mappings from Δ to $\mathbb{C}^n$ such that

(4) $(f_j^*)_* m \to \mu$ in the weak-$*$ topology, where f_j^* is the radial limit values of f_j, and $(f_j^*)_* m$ is the direct image of m under f_j^*, i.e., for every Borel set $E \subset X$, $((f_j^*)_* m)(E) = m((f_j^*)^{-1}(E) \cap \partial\Delta)$, where m is the normalized Lebesgue measure on $\partial\Delta$.

(5) $\lim_{j\to\infty} f_j(0) = \xi_0$.

(6) $\operatorname{cl} L \subset X$, where $\operatorname{cl} L$ is the set of points z in $\mathbb{C}^n$ such that every neighbourhood of z intersects $f_j(\Delta)$ for infinitely many j.

(7) For every $z \in \operatorname{cl} L$ and every neighbourhood V of z we have

$$\limsup_{j\to\infty} \omega(0, f_j^{-1}(V), \Delta) > 0,$$

where $\omega(\cdot, E, \Delta)$ denotes the harmonic measure of the set E with respect to Δ.

Next we set $L' = \{\theta_j \circ f_j\}$. Since θ_j is a holomorphic map from $\mathbb{C}^n$ to $\mathbb{C}^p$, we infer that L' is also a sequence of uniformly bounded holomorphic mappings from Δ to $\mathbb{C}^p$. Further, from (4)–(7) we get

(8) $((\theta_j \circ f_j)^*)_* m \to \theta_* \mu$ in the weak-$*$ topology.

(9) $\lim_{j\to\infty} (\theta_j \circ f_j)(0) = \theta(\xi_0)$.

(10) $\operatorname{cl} L' = \pi(\operatorname{cl} L) \subset Y$.

(11) For every $z' \in \operatorname{cl} L'$ and every neighbourhood V' of z' we have

$$\limsup_{j\to\infty} \omega(0, (\theta_j \circ f_j)^{-1}(V'), \Delta) > 0.$$

Notice that (11) follows from Lemma 3.6 and (7).

Now we claim that $\operatorname{cl} L \subset \theta^{-1}(\theta(\xi_0)) \cap X$. Otherwise, there is some point $\xi_0' \in \operatorname{cl} L \setminus \theta^{-1}(\theta(\xi_0))$. It follows from (10) that $\theta(\xi_0') \in \operatorname{cl} L'$ and $\theta(\xi_0') \ne \theta(\xi_0)$. Since the sequence L' satisfies (8)–(11), we may invoke Lemma 4.1 of [Po] to deduce that $\theta(\xi_0)$ is not a plurisubharmonic peak point with respect to Y in the sense of Poletsky (see [Po, p. 416]). This contradicts the B-regularity of Y.

Since the claim is valid, we have $\mu \in J_{\xi_0}(\theta^{-1}(\theta(\xi_0)) \cap X)$. From (ii) we infer that $\mu = \delta_{\xi_0}$. Thus X is B-regular by Remarks 2.7(ii).

Now if X satisfies (iii′), then we let z_0 be an arbitrary point in X, and choose a small neighbourhood U of z_0 such that $X \cap \overline{U}$ is connected and that θ can be approximated uniformly on U by holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$. It follows that $X \cap \overline{U}$ satisfies (iii), so $X \cap \overline{U}$ is B-regular. This implies that X is locally B-regular, thus X is in fact B-regular.

Proof of Proposition 3.5. (a) If Ω_φ is B-regular then in particular it is hyperconvex. It is well known that this is the case if and only if $\varphi \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\Omega)$. Now assume that there is some point $\xi \in \partial\Omega$ and a sequence $\{z_j\} \subset \Omega$ such that $z_j \to \xi$ and $\lim_{j\to\infty} \varphi(z_j) = \alpha < \infty$. Set $S = \{w : |w| < e^{-\alpha-1}\}$. It is easy to check that $\{z_j\} \times S \subset \Omega_\varphi$ for j large enough and $\{\xi\} \times \{w : |w| < e^{-\alpha-1}\} \subset \partial\Omega_\varphi$. This contradicts the B-regularity of Ω_φ, by Remarks 2.3(iii). Thus (ii) follows.

Next we let f be an arbitrary real-valued continuous function on $\partial\Omega_\varphi$. Set $\widetilde{f}(z,0) = f(z)$ for $z \in \partial\Omega$. Then $\widetilde{f}$ is continuous on a closed subset of $\partial\Omega_\varphi$. Extend it to a continuous function, still denoted by $\widetilde{f}$, on $\partial\Omega_\varphi$. Since Ω_φ is B-regular there exists $\widetilde{u} \in \mathrm{PSH}(\Omega_\varphi) \cap \mathcal{C}(\overline{\Omega}_\varphi)$ such that $\widetilde{u} = \widetilde{f}$ on $\partial\Omega_\varphi$. Thus $u(z) := \widetilde{u}(z,0)$ belongs to $\mathrm{PSH}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ and satisfies $u \equiv f$ on $\partial\Omega$. This proves (i).

It remains to prove (iii). Assume towards a contradiction that the restriction of φ to S is harmonic for some non-constant analytic disk S in Ω. Then we can find a non-constant holomorphic mapping h from Δ to Ω such that $\varphi \circ h$ is harmonic on Δ. Choose a holomorphic function $\widetilde{\varphi}$ on Δ such that $\Re\widetilde{\varphi} = -\varphi \circ h$. Consider the sequence $\{\varphi_j\}$ of holomorphic mappings from Δ to $\mathbb{C}^{n+1}$ defined by $\varphi_j(\xi) = (h(\xi), (1-1/j)e^{\widetilde{\varphi}(\xi)})$. It is easy to check that $\varphi_j(\Delta) \subset \Omega_\varphi$ and φ_j converges uniformly on Δ to a non-constant holomorphic mapping with image in $\partial\Omega_\varphi$. Using again Remarks 2.3(iv) we get a contradiction to the B-regularity of Ω_φ.

(b) If Ω is B-regular and $\varphi \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\Omega)$ then it is easy to check that Ω_φ is hyperconvex. Let $p = (z_0, w_0)$ be an arbitrary point in $\partial\Omega_\varphi$. According to Theorem 2.5 it suffices to show that there exists a local barrier at p. There are two cases to be considered.

Case 1: $z_0 \in \partial\Omega$. Since $\lim_{\xi\to z_0} \varphi(\xi) = \infty$ we have $w_0 = 0$. Since Ω is B-regular, we can find a barrier u at z_0 in Ω. It follows that $v(z,w) = u(z)$ is a barrier at p.

Case 2: $z_0 \in \Omega$. We claim that there exists a ball W centred at p so that $\overline{W} \cap \partial\Omega_\varphi$ is B-regular. Indeed, choose a relatively compact neighbourhood

U of z_0 in Ω such that $X \cap \overline{U}$ is connected and B-regular. Let $U' = U \times \mathbb{C}$. Since $\varphi \in \mathcal{C}(\Omega)$ we have

$$\widetilde{U} := \overline{U'} \cap \partial\Omega_\varphi = \{(z,w) : z \in \overline{U},\ \log|w| + \varphi(z) = 0\}.$$

Let A be an arbitrary compact subset of $\widetilde{U}$ with $\pi(A) \cap X = \emptyset$, where π is the projection $(z,w) \mapsto z$. Let (z', w') be any point in A. Then we can find a small neighbourhood V of z' and $\psi \in \mathrm{PSH}(V)$, $\lambda > 0$ so that

$$\varphi(z) = \lambda|z|^2 + \psi(z), \qquad \forall z \in V.$$

It follows that

$$0 = \varphi(z) + \log|w| = \psi(z) + \lambda|z|^2 + \log|w|, \qquad \forall (z,w) \in A \cap (\overline{V} \times \mathbb{C}).$$

So $-\lambda|z|^2 = \psi(z) + \log|w|$ for every $(z,w) \in A \cap (\overline{V} \times \mathbb{C})$. This implies that the function $-|z|^2$ is the restriction to the compact set $A \cap (\overline{V} \times \mathbb{C})$ of some plurisubharmonic function on a neighbourhood of it. So A is locally B-regular, and by Remarks 2.3(ii) we conclude that the compact set A is B-regular. Next we let

$$\widetilde{A} = \widetilde{U} \cap \pi^{-1}(X) = \{(z,w) : z \in X \cap \overline{U},\ \log|w| + \varphi(z) = 0\}.$$

Since $X \cap \overline{U}$ is connected, by Lemma 3.7 the set $\widetilde{A}$ is connected. Notice that $\pi(\widetilde{A})$ is B-regular. Lemma 3.8 shows that $\widetilde{A}$ is B-regular. It follows that $\widetilde{U}$ can be written as a countable union of compact sets of type A and $\widetilde{A}$, so using Proposition 1.9 in [Si] we deduce that $\widetilde{U}$ is B-regular. Choose a ball W centred at p such that $W \subset U'$. Then $\overline{W} \cap \partial\Omega_\varphi$, being contained in $\widetilde{U}$, is B-regular. This proves the claim.

Since ∂W is B-regular, so is $\partial(W \cap \Omega_\varphi)$. Finally, noticing that $W \cap \Omega_\varphi$ is hyperconvex, by Theorem 2.4 we infer that $W \cap \Omega_\varphi$ is B-regular. In particular, there exists a local barrier at p with respect to $W \cap \Omega_\varphi$. The theorem is proven.

Proposition 3.9. *Let Ω be a domain in $\mathbb{C}^n$ and suppose that $f : \Omega \to \mathbb{C}^n$ is a non-degenerate holomorphic mapping. Let Ω', Ω'' be bounded B-regular subdomains of $\Omega, f(\Omega)$ respectively. Set $\Omega''' = f^{-1}(\Omega'') \cap \Omega'$ and*

$$S(f) = \{a \in \Omega : a \text{ is not an isolated point of } f^{-1}(f(a))\}.$$

Assume that there exist an open neighbourhood U of $S(f)$ and a B-regular compact set K of $U \cap \partial\Omega'''$ such that

(i) *$S(f) \cap \overline{U} \cap \partial\Omega'''$ is B-regular.*
(ii) *$\partial\Omega'''$ is $\mathcal{C}^1$ smooth near every point of $(U \cap \partial\Omega''') \setminus (K \cup \partial\Omega')$.*

Then Ω''' is B-regular.

Remarks 3.10. (i) Conditions (i) and (ii) of Proposition 3.9 are obviously satisfied when f is *proper*, since in this case $S(f) = \emptyset$.

(ii) Condition (i) of Proposition 3.9 cannot be removed. Indeed, consider $\Omega = \mathbb{C}^2$, $\Omega' = \{(z,w) : |z|^2 + |w|^2 < 10\}$, $f(z,w) = (z, zw)$ and $\Omega'' = \{(z,w) : |z|^2 + |w-1|^2 < 1\}$. Notice that

$$\Omega''' = \{(z,w) : |z|^2 + |w|^2 < 10,\ |z|^2 + |zw-1|^2 < 1\}.$$

It is easy to check that $\partial\Omega''' \setminus \partial\Omega'$ is $\mathcal{C}^1$ smooth everywhere except at the origin, so (ii) is satisfied. On the other hand, $\{(z,w) : z = 1/n,\ w \in S, n \geq 10\} \subset \Omega'''$ where $S = \{w : \Re w > 1/2,\ |w| < 3\}$. It follows that $S(f) \cap \partial\Omega'''$ contains the analytic disk $0 \times S$. Applying Remarks 2.3(iv) we conclude that Ω''' is not B-regular. On the other hand, we do not know if condition (ii) is really needed.

Proof of Proposition 3.9. Notice that Ω''' is hyperconvex. Let ξ_0 be an arbitrary point of $\partial\Omega'''$. According to Theorem 2.5 it suffices to check that there is a local barrier at ξ_0. There are some cases to be considered:

Case 1: $\xi_0 \in \partial\Omega'$. Then since Ω' is B-regular we can even find a barrier at ξ_0 with respect to Ω'.

Case 2: $\xi_0 \in (\partial\Omega''') \setminus (S(f) \cup \partial\Omega')$. Then $\xi_0 \in \Omega$ and we can find a neighbourhood V of ξ_0 such that $f^{-1}(f(\xi_0)) \cap V = \{\xi_0\}$ and $f(V)$ is a neighbourhood of $f(\xi_0) \in \partial\Omega'$. Since Ω' is B-regular, there exists a barrier u at $f(\xi_0)$. This implies that $u \circ f$ is a barrier at ξ_0 with respect to $V \cap \Omega'''$.

Case 3: $\xi_0 \in S(f) \setminus \partial\Omega'$. Choose a small ball U' centred at ξ_0 such that $U' \cap \partial\Omega' = \emptyset$ and $U' \subset U$. Set $W = U' \cap \Omega'''$. Let $a \in L := \partial W \setminus (\partial U' \cup K \cup S(f))$. Then by the result proven in Case 2, we can find a barrier at a with respect to Ω'''. Observe that $\partial\Omega'''$ is $\mathcal{C}^1$ smooth near a, so by Lemma 4.2 (see the appendix) we have $J_a(\partial\Omega''') = \{\delta_a\}$ for every $a \in L$. It follows that L is locally B-regular. Therefore L can be written as a countable union of B-regular compact sets. Since K, $S(f) \cap \overline{U'} \cap \partial\Omega'''$ and $\partial U'$ are B-regular compact sets, we deduce that ∂W is a countable union of B-regular compact sets. Using Proposition 1.9 of [Si] we conclude that ∂W is B-regular. Notice that W is hyperconvex, so by Theorem 2.4 it is B-regular.

IV. Appendix

Lemma 4.1. *Let K be a compact set in $\mathbb{C}^n$ and u be a plurisubharmonic function (in the sense of Poletsky) on K. Then for every $\mu \in J_{z_0}(K)$ we have $u(z_0) \leq \int_K u\, d\mu$.*

Proof. Let $\{\varphi_j\}$ be a sequence in $\mathcal{C}(K)$ decreasing to u on K. Denote by $\mathrm{PSH}(K)$ the cone of plurisubharmonic functions on K. Set

$$E\varphi_j = \sup\{v \in \mathrm{PSH}(K) : v \leq \varphi_j\}.$$

Clearly $u \leq E\varphi_j \leq \varphi_j$ on K. This implies that $E\varphi_j \downarrow u$ on K. According to Lemma 3.1 in [Po], $E\varphi_j$ is the limit of an increasing sequence of continuous plurisubharmonic functions defined on neighbourhoods of K. Since $\mu \in J_{z_0}(K)$, by the monotone convergence theorem we deduce that $E\varphi_j(z_0) \le \int_K E\varphi_j\, d\mu$. Applying again the monotone convergence theorem we get the desired inequality.

LEMMA 4.2. *Let Ω be a bounded open set in $\mathbb{C}^n$ and $a \in \partial\Omega$. Assume that there is a barrier u at a with respect to Ω, and Ω is $\mathcal{C}^1$ smooth near a. Then $J_a(\partial\Omega) = \{\delta_a\}$.*

Proof. Let U be a small ball around a such that $U \cap \partial\Omega$ is $\mathcal{C}^1$ smooth. Let n be the inward normal vector at a. Then we can find a small ball V around a and $\varepsilon_0 > 0$ such that $V \cap \partial\Omega \subset U$ and $(\overline{V} \cap \partial\Omega) + \varepsilon n \subset \Omega$ for $\varepsilon \in (0, \varepsilon_0)$. It follows that for $\varepsilon \in (0, \varepsilon_0)$, the function $u_\varepsilon(z) = u(z + \varepsilon n)$ is plurisubharmonic on a neighbourhood of $\overline{V} \cap \partial\Omega$. Fix $\mu \in J_a(\partial\Omega \cap \overline{V})$. Then

$$u_\varepsilon(a) \le \int\limits_{\overline{V}\cap\partial\Omega} u_\varepsilon\, d\mu, \qquad \forall \varepsilon \in (0, \varepsilon_0).$$

Letting ε tend to 0 we obtain $u(a) \le \int_{\overline{V}\cap\partial\Omega} u\, d\mu$. Since u is a barrier at a, we must have $\mu = \delta_a$. By Proposition 1.4 in [Si] we have $J_a(\partial\Omega) = \{\delta_a\}$.

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