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On convex and $*$-concave multifunctions

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Abstract. A continuous multifunction $F : [a,b] \to \mathrm{clb}(Y)$ is $*$-concave if and only if the inclusion

$$\frac{1}{t-s}\int_s^t F(x)\,dx \subset \frac{F(s) \overset{*}{+} F(t)}{2}$$

holds for every $s,t \in [a,b]$, $s < t$.

1. It is known that a real convex function f defined on $[a,b]$ satisfies the Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

(cf. [5, pp. 196–197]). The inequality was first shown by Ch. Hermite in Mathesis in 1883. Independently it was proved by J. Hadamard in 1893, so it is usually called the Hermite–Hadamard inequality. This inequality can be used to characterize real convex functions. More exactly, we have

Theorem 0 (cf. e.g. [9, p. 15]). *If $f : [a,b] \to \mathbb{R}$ is continuous, then f is convex if and only if*

$$\frac{1}{t-s}\int_s^t f(x)\,dx \le \frac{f(s)+f(t)}{2} \quad \textit{for all } a \le s < t \le b.$$

It is not clear who presented it first. More information on the subject may be found in the paper of D. S. Mitrinović and I. B. Lacković [7].

Our main goal is to give a similar characterization of $*$-concave and convex multifunctions, continuous with respect to the Hausdorff metric. In that characterization the Riemann integral of multifunctions will be used. A multivalued counterpart of inequality (1) for convex multifunctions and the Aumann integral was studied by E. Sadowska [10, Theorem 1]. In the proof of

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the inclusion

$$\frac{1}{b-a}\int_a^b F(x)\,dx \subset F\left(\frac{a+b}{2}\right)$$

the integral Jensen inequality for convex multifunctions was applied. The last result is due to J. Matkowski and K. Nikodem (see [6, p. 350, Theorem]).

2. Let $(Y, \|\cdot\|)$ be a real Banach space. Denote by $\mathrm{clb}(Y)$ the set of all nonempty convex closed bounded subsets of Y. For given $A, B \in \mathrm{clb}(Y)$, we set $A + B = \{a + b : a \in A,\, b \in B\}$, $\lambda A = \{\lambda a : a \in A\}$ for $\lambda \geq 0$ and $A \stackrel{*}{+} B = \mathrm{cl}(A+B) = \mathrm{cl}(\mathrm{cl}\, A + \mathrm{cl}\, B)$, where $\mathrm{cl}\, A$ means the closure of A in Y. It is easy to see that $(\mathrm{clb}(Y), \stackrel{*}{+}, \cdot)$ has the following properties:

$$\lambda(A \stackrel{*}{+} B) = \lambda A \stackrel{*}{+} \lambda B,\ (\lambda+\mu)A = \lambda A \stackrel{*}{+} \mu A,\ \lambda(\mu A) = (\lambda\mu)A,\ 1\cdot A = A$$

for any $A, B \in \mathrm{clb}(Y)$ and $\lambda, \mu \geq 0$. If $A, B, C \in \mathrm{clb}(Y)$, then the equality $A \stackrel{*}{+} C = B \stackrel{*}{+} C$ implies $A = B$ (see e.g. [2, Theorem II-17, p. 48]). Thus the cancellation law holds in $\mathrm{clb}(Y)$ for the operation $\stackrel{*}{+}$.

The set $\mathrm{clb}(Y)$ is a metric space with the Hausdorff metric h defined by

$$h(A, B) = \inf\{t > 0 : A \subset B + tS,\ B \subset A + tS\},$$

where S denotes the closed unit ball in Y. The metric space $(\mathrm{clb}(Y), h)$ is complete (see e.g. [2, Theorem II-3, p. 40]). Moreover, h is translation invariant since

$$h(A \stackrel{*}{+} C, B \stackrel{*}{+} C) = h(A + C, B + C) = h(A, B),$$

and positively homogeneous, i.e.,

$$h(\lambda A, \lambda B) = \lambda h(A, B)$$

for all $\lambda \geq 0$ and $A, B, C \in \mathrm{clb}(Y)$ (cf. [1, Lemma 2.2]).

Let F be a multifunction defined on an interval $[a, b]$ with values in $\mathrm{clb}(Y)$. It is said to be $*$-*concave* (resp. *convex*) if

$$F(\lambda x + (1-\lambda)y) \subset \lambda F(x) \stackrel{*}{+} (1-\lambda)F(y)$$
$$(\text{resp. } \lambda F(x) + (1-\lambda)F(y) \subset F(\lambda x + (1-\lambda)y))$$

for all $x, y \in [a, b]$ and $\lambda \in (0, 1)$.

A set $\Delta = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \cdots < x_n = b$, is said to be a *partition* of $[a, b]$. For given partition Δ we put $\delta(\Delta) := \max\{x_i - x_{i-1} : i \in \{1, \ldots, n\}\}$ and form the approximating sum

$$S(\Delta, \tau) = (x_1 - x_0)F(\tau_1) \stackrel{*}{+} \cdots \stackrel{*}{+} (x_n - x_{n-1})F(\tau_n),$$

where τ is a system $(\tau_1, \ldots, \tau_n)$ of intermediate points ($\tau_i \in [x_{i-1}, x_i]$). If for every sequence (Δ^ν, τ^ν), $\nu \in \mathbb{N}$, where Δ^ν are partitions of $[a, b]$

and τ^ν are systems of intermediate points, such that $\lim_{\nu\to\infty}\delta(\Delta^\nu) = 0$, the sequence $(S(\Delta^\nu,\tau^\nu))$ of approximating sums always tends to the same limit $I \in \mathrm{clb}(Y)$, then F is said to be *Riemann integrable* over $[a,b]$ and $\int_a^b F(x)\,dx := I$.

The Riemann integral for multifunctions with compact convex values was investigated by A. Dinghas [3] and M. Hukuhara [4]. In [8] the above integral was introduced and its properties were studied for $F : [a,b] \to \mathrm{clb}(Y)$. Continuous multifunctions (with respect to the Hausdorff metric) are Riemann integrable.

3. Let Y^* denote the space of all continuous linear functionals on Y. For $A \in \mathrm{clb}(Y)$ we define A^ξ by

$$A^\xi = \sup\{\xi(a) : a \in A\}.$$

Of course the number A^ξ can be defined for any bounded set $A \subset Y$ and it is easily seen that $A^\xi = (\mathrm{cl}\,A)^\xi$.

We note that

$$(\lambda A)^\xi = \lambda A^\xi, \tag{2}$$

$$(A \overset{*}{+} B)^\xi = A^\xi + B^\xi \tag{3}$$

for every $\lambda \geq 0$, $A, B \in \mathrm{clb}(Y)$ and $\xi \in Y^*$. The first equality is clear. To obtain the second one we observe that

$$(A \overset{*}{+} B)^\xi = (A+B)^\xi = \sup_{\substack{a\in A\\ b\in B}} \xi(a+b) = \sup_{a\in A}\xi(a) + \sup_{b\in B}\xi(b) = A^\xi + B^\xi.$$

For given $A, B \in \mathrm{clb}(Y)$ we have

$$A \subset B \quad \text{if and only if} \quad A^\xi \leq B^\xi \text{ for all } \xi \in Y^*. \tag{4}$$

To prove the "if" part assume that $a \in A \setminus B$. Then by the Separation Theorem, there is a functional $\xi \in Y^*$ and a real number c such that

$$B^\xi = \sup_{b\in B}\xi(b) < c < \xi(a) \leq A^\xi.$$

The "only if" part is obvious.

For any multifunction $F : [a,b] \to \mathrm{clb}(Y)$ and $\xi \in Y^*$, a real function $F^\xi : [a,b] \to \mathbb{R}$ is defined by

$$F^\xi(x) = F(x)^\xi, \qquad x \in [a,b].$$

The first lemma below is an immediate consequence of (2)–(4).

LEMMA 1. *A multifunction $F : [a,b] \to \mathrm{clb}(Y)$ is concave (resp. convex) if and only if $F^\xi : [a,b] \to \mathbb{R}$ is convex (resp. concave) for every $\xi \in Y^*$.*

Lemma 2. *If* $F : [a,b] \to \mathrm{clb}(Y)$ *is continuous, then so is* $x \mapsto F^\xi(x)$ *and*

$$\Big(\int_a^b F(x)\,dx\Big)^\xi = \int_a^b F^\xi(x)\,dx \quad \textit{for all } \xi \in Y^*. \tag{5}$$

Proof. First we note that the function

$$\mathrm{clb}(Y) \ni A \mapsto A^\xi \in \mathbb{R} \tag{6}$$

is a Lipschitzian functional. Indeed, fix $A, B \in \mathrm{clb}(Y)$, $t > h(A,B)$ and $\xi \in Y^*$. Then

$$A \subset B + tS \quad \text{and} \quad B \subset A + tS.$$

According to (2)–(4) we get

$$A^\xi \le B^\xi + t\|\xi\| \quad \text{and} \quad B^\xi \le A^\xi + t\|\xi\|,$$

where $\|\xi\|$ is the norm of the functional ξ. Hence

$$|A^\xi - B^\xi| \le t\|\xi\|.$$

Letting $t \to h(A,B)$ we obtain

$$|A^\xi - B^\xi| \le \|\xi\| h(A,B).$$

Consequently, (6) is continuous and the function $x \mapsto F^\xi(x)$ is also continuous, being the composition of F and (6).

To show (5) we fix $n \in \mathbb{N}$ and take the partition $\Delta = \{x_0, \ldots, x_n\}$ of $[a,b]$ with $x_i = a + (i/n)(b-a)$, $i \in \{0, 1, \ldots, n\}$. Let $\tau = (x_1, \ldots, x_n)$. The continuity of (6), F, F^ξ and (2)–(4) yield

$$\begin{aligned}\Big(\int_a^b F(x)\,dx\Big)^\xi &= (\lim_{n\to\infty} (x_1 - x_0)F(x_1) \stackrel{*}{+} \cdots \stackrel{*}{+} (x_n - x_{n-1})F(x_n))^\xi \\ &= \lim_{n\to\infty} [(x_1 - x_0)F(x_1) \stackrel{*}{+} \cdots \stackrel{*}{+} (x_n - x_{n-1})F(x_n)]^\xi \\ &= \lim_{n\to\infty} [(x_1 - x_0)F^\xi(x_1) + \cdots + (x_n - x_{n-1})F^\xi(x_n)] = \int_a^b F^\xi(x)\,dx.\end{aligned}$$

This completes the proof.

Theorem 0, Lemmas 1, 2 and relations (2)–(4) allow us to formulate two theorems.

Theorem 1. *Let* $F : [a,b] \to \mathrm{clb}(Y)$ *be a continuous multifunction. Then* F *is* $*$*-concave if and only if for any* $s, t \in [a,b]$ *with* $s < t$ *we have the inclusion*

$$\frac{1}{t-s}\int_s^t F(x)\,dx \subset \frac{1}{2}\,[F(s) \stackrel{*}{+} F(t)]. \tag{7}$$

THEOREM 2. *Let $F : [a,b] \to \mathrm{clb}(Y)$ be a continuous multifunction. Then F is convex if and only if for any $s,t \in [a,b]$ with $s < t$ we have the inclusion*

$$\frac{1}{2}\,[F(s) \overset{*}{+} F(t)] \subset \frac{1}{t-s}\int_s^t F(x)\,dx. \tag{8}$$

The proofs of both go in the same way. We prove the second one.

Proof of Theorem 2. By Lemma 1, F is convex if and only if F^ξ is concave for all $\xi \in Y^*$. Next

$$F^\xi \text{ is concave } \Leftrightarrow\ -F^\xi \text{ is convex } \Leftrightarrow$$

$$\frac{1}{t-s}\int_s^t F^\xi(x)\,dx \geq \frac{F^\xi(s)+F^\xi(t)}{2} \quad \text{for all } a \leq s < t \leq b. \tag{9}$$

The last equivalence follows from Theorem 0. By (2)–(5), the validity of (9) for every $\xi \in Y^*$ is equivalent to (8). The proof of Theorem 2 is complete.

REMARK. It may be proved that ∗-concave (resp. convex) multifunctions $F : [a,b] \to \mathrm{clb}(Y)$ are continuous in the open interval (a,b). But the continuity assumption is essential in the proof of ∗-concavity.

EXAMPLE. Let F be defined as follows:

$$F(x) := \begin{cases} \{0\}, & x \in [0,1/2) \cup (1/2,1], \\ [0,1], & x = 1/2. \end{cases}$$

Clearly $\int_s^t F(x)\,dx = \{0\}$ for any $0 \leq s < t \leq 1$ and inclusion (7) holds true. Nevertheless F is not ∗-concave since

$$[0,1] = F\left(\frac{1}{2}\right) \not\subset \frac{1}{2}[F(0) \overset{*}{+} F(1)] = \{0\}.$$

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