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A counterexample to the regularity of the degenerate complex Monge–Ampère equation

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Abstract. We modify an example due to X.-J. Wang and obtain some counterexamples to the regularity of the degenerate complex Monge–Ampère equation on a ball in $\mathbb{C}^n$ and on the projective space $\mathbb{P}^n$.

1. Introduction. The Monge–Ampère operator of a smooth plurisubharmonic function u is given by

$$(dd^c u)^n = 4^n n! \det(u_{i\bar{j}}) d\mathcal{L}, \quad \text{where} \quad u_{i\bar{j}} = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$$

and $\mathcal{L}$ is the $2n$-dimensional Lebesgue measure. For an arbitrary continuous plurisubharmonic function u one can define $(dd^c u)^n$ as a regular Borel measure. Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$ (throughout the note we always assume $n \geq 2$). Then for any nonnegative f which is continuous in Ω and for φ continuous on $\partial\Omega$ the Dirichlet problem

$$\begin{cases} u \in \mathrm{PSH}(\Omega) \cap \mathcal{C}(\overline{\Omega}), \\ (dd^c u)^n = f d\mathcal{L} \quad \text{in } \Omega, \\ u = \varphi \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

has a unique solution (see [B-T]).

Below we list some regularity results for solutions of (1.1):

(1) $\partial\Omega \in \mathcal{C}^\infty$, $\varphi \in \mathcal{C}^\infty(\partial\Omega)$, $f \in \mathcal{C}^\infty(\overline{\Omega})$, $f > 0 \Rightarrow u \in \mathcal{C}^\infty(\overline{\Omega})$ (Caffarelli, Kohn, Nirenberg and Spruck [C-K-N-S]);
(2) $\partial\Omega \in \mathcal{C}^{3,1}$, $\varphi \in \mathcal{C}^{3,1}(\partial\Omega)$, $f^{1/n} \in \mathcal{C}^{1,1}(\overline{\Omega})$, $f \geq 0 \Rightarrow u \in \mathcal{C}^{1,1}(\overline{\Omega})$ (Krylov [Kr1, Kr2]).

There are analogous regularity theorems for the real Monge–Ampère equations on a strongly convex domain Ω in $\mathbb{R}^n$ (see [C-K-N] and [G-T-W]). In a forthcoming paper about the degenerate Monge–Ampère equation on

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strongly pseudo-convex domains in $\mathbb{C}^n$, the author proves the following complex version of a result from [G-T-W]:

(3) $\partial\Omega \in \mathcal{C}^{3,1}$, $\varphi \in \mathcal{C}^{3,1}(\partial\Omega)$, $f^{1/(n-1)} \in \mathcal{C}^{1,1}(\overline{\Omega})$, $f \geq 0 \Rightarrow u$ is almost $\mathcal{C}^{1,1}$ (i.e. mixed complex derivatives $u_{i\bar{j}}$ are bounded).

Similar results are true for Kähler manifolds. Let M be a compact Kähler manifold of complex dimension n, with the Kähler form ω. We will say that a continuous function ϕ on M is *admissible* if $dd^c\phi + \omega \geq 0$. For any nonnegative f which is continuous on M and satisfies the necessary condition

$$\int_M f\omega^n = \int_M \omega^n \tag{1.2}$$

the Monge–Ampère equation

$$\begin{cases} \phi \text{ is admissible,} \\ (dd^c\phi + \omega)^n = f\omega^n \quad \text{in } M, \end{cases} \tag{1.3}$$

has a unique (up to a constant) continuous solution (see [K1, K2, B3]).

We have the following results about regularity of (1.3):

(4) $f \in \mathcal{C}^\infty(M)$, $f > 0 \Rightarrow \phi \in \mathcal{C}^\infty$ (Yau [Y]);
(5) $f^{1/(n-1)} \in \mathcal{C}^{1,1}(M)$, $f \geq 0 \Rightarrow \phi$ is almost $\mathcal{C}^{1,1}$ (Błocki [B2]).

In [W] Wang proved that the exponent $1/(n-1)$ is optimal for analogous results in the real case. In this paper we show that in (3) and (5) the constant $1/(n-1)$ is also optimal. Our examples are very similar to Example 3 in [W] but in the proof of Lemma 2.1 below, although the idea is also similar to [W], at some point we proceed quite differently than in the real case.

2. Examples. Let

$$f(z) = A\eta\left(\frac{|z_n|}{|z'|^\alpha}\right)|z'|^\beta, \qquad (z_1,\ldots,z_{n-1},z_n) = (z',z_n) = z \in \mathbb{C}^n, \tag{2.1}$$

where $\alpha, \beta > 2$, $A > 0$ and

$$\eta(t) = \begin{cases} e^{-1/(1-t^2)}, & |t| < 1, \\ 0, & |t| \geq 1. \end{cases}$$

Let B be the unit ball in $\mathbb{C}^n$.

We need the following lemma:

LEMMA 2.1. *Let $u \in \mathrm{PSH} \cap \mathcal{C}(B)$ be such that $u|_{\{0\}\times\mathbb{C}\cap B}$ is not constant,*

$$(dd^c u)^n = f\,d\mathcal{L} \quad \text{in } B$$

where f is given by (2.1) *and*

$$u(z',z_n) = u(w',w_n) \quad \text{if } |z'| = |w'| \text{ and } |z_n| = |w_n|. \tag{2.2}$$

Then

$$u(0,\varepsilon_k) - u(0) \ge C\varepsilon_k^{(2\alpha+2(n-1)+\beta)/n\alpha}$$

for some sequence $\varepsilon_k \searrow 0$ *and* $C > 0$ *depending only on* α, β *and* n. *In particular, if* $\beta < 2(n-1)(\alpha-1)$, *then* u *is not* $\mathcal{C}^{1,a}$ *smooth for some* $a < 1$.

Proof. 1. By the maximum principle, if $|z'| \le |w'|$ and $|z_n| \le |w_n|$, then $u(z', z_n) \le u(w', w_n)$.

2. Let $h(x) = u(0, e^x)$ for $x \in [-\infty, 0)$. Then h is convex and reaches its strict minimum at $x = -\infty$. Indeed, suppose that $m = \sup\{x : h(x) = h(-\infty)\} \ne -\infty$. There are $\mu > 0$ such that

$$h(m+2\mu) < u(e^{(m-2\mu)/\alpha}, 0, \ldots, 0, e^{m-2\mu})$$

and an affine function l such that $l(m-\mu) = h(m-\mu)$ and $l(m+\mu) = h(m+\mu)$. So

$$\overline{\{u(z', z_n) < l(\log|z_n|)\}} \subset V = \{\max\{e^{m-2\mu}, |z'|^\alpha\} < |z_n| < e^{m+2\mu}\}.$$

But u is maximal in V, so this is impossible.

3. There is a sequence $\varepsilon_k \searrow 0$ such that $u(z', z_n) = u(0, \varepsilon_k)$ for $|z'|^\alpha < |z_n| = \varepsilon_k$. Indeed, there is a sequence $\varepsilon_k \searrow 0$ such that h is strictly convex in ε_k for $k \in \mathbb{N}$, i.e. there is an affine function l such that $\{l(x) < h(x)\} = [-\infty, 0) \setminus \{\varepsilon_k\}$. From continuity and maximality of u (in $\operatorname{int}\{f = 0\}$) it is clear that for every small $s > 0$ we have $l(\log|z_n|) + s \ge u(z)$ for $z = (z', z_n)$ whenever $|z_n| = \varepsilon_k$ and $|z'|^\alpha < |z_n|$.

4. Let $\varepsilon = \varepsilon_k$ and $\lambda = \varepsilon^{1/\alpha}$. Let T denote the transformation $(w', w_n) = T(z', z_n) = (z'/\lambda, z_n/\varepsilon)$, and let

$$v(w', w_n) = \frac{u(\lambda w', \varepsilon w_n) - u(0,\varepsilon)}{(\varepsilon\lambda^{n-1})^{2/n}}.$$

Then $v < 0$ in B and $(dd^c v)^n = f \circ T^{-1}$.

5. Let $\psi = |z|^2 - 1$. Then $\psi \in \mathrm{PSH}^-(B)$, $\lim_{z\to\partial B} \psi = 0$ and $\psi < -1/2$ on $\frac{1}{2}B$. So [B1, Corollary 2.3] gives us

$$\frac{1}{2^n} \int\limits_{\frac{1}{2}B} (dd^c v)^n \le \int\limits_B |\psi|^n (dd^c v)^n \le \|v\|_B^n \int\limits_B (dd^c \psi)^n = C_1 \|v\|_B^n.$$

6. Let

$$D = \left\{ z : |z_n| < \frac{1}{2}\left(\frac{\lambda}{8}\right)^\alpha, \ \frac{\lambda}{8} < |z'| < \frac{\lambda}{4} \right\}.$$

Then $D \subset T^{-1}\left(\frac{1}{2}B\right)$ and $\mathcal{L}(D) \ge C_2 \varepsilon^2 \lambda^{2(n-1)}$ and

$$\min_D f = f\left(\frac{\lambda}{8}, 0, \ldots, 0, \frac{1}{2}\left(\frac{\lambda}{8}\right)^\alpha\right) \ge C_3 \lambda^\beta.$$

7. Let $\widetilde{B} = \frac{1}{2}T^{-1}(B)$. We thus obtain

$$-v(0) = \|v\|_B \geq \Big(C_4 \int\limits_{\frac{1}{2}B} (dd^c v)^n\Big)^{1/n} = \Big(\frac{C_4}{\varepsilon^2\lambda^{2(n-1)}} \int\limits_{\widetilde{B}} f\, d\mathcal{L}\Big)^{1/n}$$
$$\geq \Big(\frac{C_4}{\varepsilon^2\lambda^{2(n-1)}}\, \mathcal{L}(D) \min_D f\Big)^{1/n} \geq C_5\lambda^{\beta/n} = C_5\varepsilon^{\beta/\alpha n}.$$

8. We therefore conclude

$$u(0,\varepsilon) - u(0) = (\varepsilon\lambda^{n-1})^{2/n} v(0) \geq C_5\varepsilon^{\frac{2}{n}+\frac{2(n-1)}{n\alpha}+\frac{\beta}{\alpha n}} = C_5\varepsilon^{\frac{2\alpha+2(n-1)+\beta}{n\alpha}}. \ \blacksquare$$

Now we can give our example for the unit ball in $\mathbb{C}^n$.

Example 2.2. *Let f be given by* (2.1) *where $A = 1$, $\beta = 2(n-1)(\alpha-1)-1$ and $a > 1/(n-1)$. Choose α such that $a \geq 2\alpha/\beta$. Then f^a is $\mathcal{C}^{1,1}$ but the solution u of* (1.1) *with $\Omega = B$ and $\varphi \equiv 0$ is not $\mathcal{C}^{1,1}$.*

Proof. Since f and φ satisfy condition (2.2), by the uniqueness of solution, u also satisfies (2.2). From Lemma 2.1 we conclude that u is not $\mathcal{C}^{1,1}$. ∎

For $\mathbb{P}^n$ we have the following example:

Example 2.3. *Let $\varrho : [0,\infty) \to [0,1]$ be a function of class $\mathcal{C}^\infty$ such that $\varrho|_{[0,1]} \equiv 1$ and $\varrho|_{[2,\infty)} \equiv 0$. View $\mathbb{P}^n$ as a Kähler manifold carrying the Fubini–Study metric $\omega = \sum g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. Write $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$ where $\mathbb{P}^{n-1}$ is the hyperplane at infinity. Let $\widetilde{f} : \mathbb{P}^n \to \mathbb{R}_+$ be a continuous function given in local coordinates in $\mathbb{C}^n$ by*

$$\widetilde{f} = \frac{\varrho(|z|)f}{4^n n! \det(g_{i\bar{j}})},$$

where f is given by (2.1), *A is such that $\widetilde{f}$ satisfies the necessary condition* (1.2), *$\beta = 2(n-1)(\alpha-1)-1$ and $a > 1/(n-1)$. Choose α such that $a \geq 2\alpha/\beta$. Then $\widetilde{f}^a$ is $\mathcal{C}^{1,1}$ but the solution ϕ of* (1.3) *with $M = \mathbb{P}^n$ and with $\widetilde{f}$ in place of f is not $\mathcal{C}^{1,1}$.*

Proof. In local coordinates in $\mathbb{C}^n$, $g_{i\bar{j}}$ are given by $g_{i\bar{j}} = (\frac{1}{2}\log(1+|z|^2))_{i\bar{j}}$. Let $u = \phi + \frac{1}{2}\log(1+|z|^2)$. Then u is a continuous solution of the Monge–Ampère equation

$$(dd^c u)^n = f\varrho \quad \text{in } \mathbb{C}^n.$$

Since ϕ is bounded, we have $\lim_{|z_n|\to+\infty} u = +\infty$. Then from the same argument as in the proof of Lemma 2.1 we see that the function $z_n \mapsto v(0, z_n)$ reaches its strict minimum at $z_n = 0$. Since ω and $\widetilde{f}$ satisfy condition (2.2), by the uniqueness of solution ϕ satisfies (2.2) and so does u. From Lemma 2.1, u is not $\mathcal{C}^{1,1}$, so neither is ϕ. ∎

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