A counterexample to the regularity of the degenerate complex Monge–Ampère equation

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Abstract. We modify an example due to X.-J. Wang and obtain some counterexamples to the regularity of the degenerate complex Monge–Ampère equation on a ball in \( \mathbb{C}^n \) and on the projective space \( \mathbb{P}^n \).

1. Introduction. The Monge–Ampère operator of a smooth plurisubharmonic function \( u \) is given by

\[
(dd^c u)^n = 4^n n! \det(u_{ij}) d\mathcal{L}, \quad \text{where} \quad u_{ij} = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}
\]

and \( \mathcal{L} \) is the 2n-dimensional Lebesgue measure. For an arbitrary continuous plurisubharmonic function \( u \) one can define \((dd^c u)^n\) as a regular Borel measure. Let \( \Omega \) be a strictly pseudoconvex domain in \( \mathbb{C}^n \) (throughout the note we always assume \( n \geq 2 \)). Then for any nonnegative \( f \) which is continuous in \( \Omega \) and for \( \varphi \) continuous on \( \partial \Omega \) the Dirichlet problem

\[
\begin{align*}
\begin{cases}
  u \in \text{PSH}(\Omega) \cap \mathcal{C}(\overline{\Omega}), \\
  (dd^c u)^n = f d\mathcal{L} \quad \text{in} \ \Omega, \\
  u = \varphi \quad \text{on} \ \partial \Omega,
\end{cases}
\end{align*}
\]

(1.1)

has a unique solution (see [B-T]).

Below we list some regularity results for solutions of (1.1):

(1) \( \partial \Omega \in \mathcal{C}^\infty, \varphi \in \mathcal{C}^\infty(\partial \Omega), f \in \mathcal{C}^\infty(\overline{\Omega}), f > 0 \Rightarrow u \in \mathcal{C}^\infty(\overline{\Omega}) \) (Caffarelli, Kohn, Nirenberg and Spruck [C-K-N-S]);

(2) \( \partial \Omega \in \mathcal{C}^{3,1}, \varphi \in \mathcal{C}^{3,1}(\partial \Omega), f^{1/n} \in \mathcal{C}^{1,1}(\overline{\Omega}), f \geq 0 \Rightarrow u \in \mathcal{C}^{1,1}(\overline{\Omega}) \) (Krylov [Kr1, Kr2]).

There are analogous regularity theorems for the real Monge–Ampère equations on a strongly convex domain \( \Omega \) in \( \mathbb{R}^n \) (see [C-K-N] and [G-T-W]). In a forthcoming paper about the degenerate Monge–Ampère equation on

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strongly pseudo-convex domains in \( \mathbb{C}^n \), the author proves the following complex version of a result from [G-T-W]:

\(3\) \( \partial \Omega \subset C^{3,1}, \varphi \in C^{3,1}(\partial \Omega), f^{1/(n-1)} \in C^{1,1}(\overline{\Omega}), f \geq 0 \Rightarrow u \) is almost \( C^{1,1} \) (i.e. mixed complex derivatives \( u_{ij} \) are bounded).

Similar results are true for Kähler manifolds. Let \( M \) be a compact Kähler manifold of complex dimension \( n \), with the Kähler form \( \omega \). We will say that a continuous function \( \phi \) on \( M \) is admissible if \( dd^c \phi + \omega \geq 0 \). For any nonnegative \( f \) which is continuous on \( M \) and satisfies the necessary condition

\[(1.2) \quad \int_M f \omega^n = \int_M \omega^n,\]

the Monge–Ampère equation

\[(1.3) \quad \begin{cases} \phi \text{ is admissible}, \\ (dd^c \phi + \omega)^n = f \omega^n \quad \text{in } M, \end{cases}\]

has a unique (up to a constant) continuous solution (see [K1, K2, B3]).

We have the following results about regularity of (1.3):

\[(4) \quad f \in C^\infty(M), f > 0 \Rightarrow \phi \in C^\infty \quad \text{(Yau [Y])};\]
\[(5) \quad f^{1/(n-1)} \in C^{1,1}(M), f \geq 0 \Rightarrow \phi \text{ is almost } C^{1,1} \quad \text{(Błocki [B2])}.\]

In [W] Wang proved that the exponent \( 1/(n-1) \) is optimal for analogous results in the real case. In this paper we show that in (3) and (5) the constant \( 1/(n-1) \) is also optimal. Our examples are very similar to Example 3 in [W] but in the proof of Lemma 2.1 below, although the idea is also similar to [W], at some point we proceed quite differently than in the real case.

2. Examples. Let

\[(2.1) \quad f(z) = A \eta \left( \frac{|z_n|}{|z'|^\alpha} \right) |z'|^\beta, \quad (z_1, \ldots, z_{n-1}, z_n) = (z', z_n) = z \in \mathbb{C}^n,\]

where \( \alpha, \beta > 2, A > 0 \) and

\[\eta(t) = \begin{cases} e^{-1/(1-t^2)}, & |t| < 1, \\ 0, & |t| \geq 1. \end{cases}\]

Let \( B \) be the unit ball in \( \mathbb{C}^n \).

We need the following lemma:

**Lemma 2.1.** Let \( u \in \text{PSH} \cap C(B) \) be such that \( u|_{\{0\} \times \mathbb{C} \cap B} \) is not constant,

\[(dd^c u)^n = f d\mathcal{L} \quad \text{in } B\]

where \( f \) is given by (2.1) and

\[(2.2) \quad u(z', z_n) = u(w', w_n) \quad \text{if } |z'| = |w'| \text{ and } |z_n| = |w_n|,\]
Then

$$u(0, \varepsilon_k) - u(0) \geq C\varepsilon_k^{(2\alpha+2(n-1)+\beta)/n\alpha}$$

for some sequence $\varepsilon_k \searrow 0$ and $C > 0$ depending only on $\alpha$, $\beta$ and $n$. In particular, if $\beta < 2(n-1)(\alpha - 1)$, then $u$ is not $C^{1,\alpha}$ smooth for some $\alpha < 1$.

**Proof.** 1. By the maximum principle, if $|z'| \leq |w'|$ and $|z_n| \leq |w_n|$, then $u(z', z_n) \leq u(w', w_n)$.

2. Let $h(x) = u(0, e^x)$ for $x \in [-\infty, 0)$. Then $h$ is convex and reaches its strict minimum at $x = -\infty$. Indeed, suppose that $m = \sup\{x : h(x) = h(-\infty)\} \neq -\infty$. There are $\mu > 0$ such that

$$h(m + 2\mu) < u(e^{(m-2\mu)/\alpha}, 0, \ldots, 0, e^{m-2\mu})$$

and an affine function $l$ such that $l(m - \mu) = h(m - \mu)$ and $l(m + \mu) = h(m + \mu)$. So

$$\{u(z', z_n) < l(\log |z_n|)\} \subset V = \{\max\{e^{m-2\mu}, |z'|^\alpha\} < |z_n| < e^{m+2\mu}\}.$$  

But $u$ is maximal in $V$, so this is impossible.

3. There is a sequence $\varepsilon_k \searrow 0$ such that $u(z', z_n) = u(0, \varepsilon_k)$ for $|z'|^\alpha < |z_n| = \varepsilon_k$. Indeed, there is a sequence $\varepsilon_k \searrow 0$ such that $h$ is strictly convex in $\varepsilon_k$ for $k \in \mathbb{N}$, i.e. there is an affine function $l$ such that $\{l(x) < h(x)\} = [-\infty, 0) \setminus \{\varepsilon_k\}$. From continuity and maximality of $u$ (in $\{f = 0\}$) it is clear that for every small $s > 0$ we have $l(\log |z_n|) + s \geq u(z)$ for $z = (z', z_n)$ whenever $|z_n| = \varepsilon_k$ and $|z'|^\alpha < |z_n|$.

4. Let $\varepsilon = \varepsilon_k$ and $\lambda = \varepsilon^{1/\alpha}$. Let $T$ denote the transformation $(w', w_n) = T(z', z_n) = (z'/\lambda, z_n/\varepsilon)$, and let

$$v(w', w_n) = \frac{u(\lambda w', \varepsilon w_n) - u(0, \varepsilon)}{(\varepsilon \lambda^{n-1})^{2/n}}.$$  

Then $v < 0$ in $B$ and $(dd^c v)^n = f \circ T^{-1}$.

5. Let $\psi = |z|^2 - 1$. Then $\psi \in \text{PSH}^-(B)$, $\lim_{z \to \partial B} \psi = 0$ and $\psi < -1/2$ on $1/2 B$. So [B1, Corollary 2.3] gives us

$$\frac{1}{2^n} \int_{1/2 B} (dd^c v)^n \leq \int_B |\psi^n (dd^c v)^n \leq \|v\|^n_B \int_B (dd^c \psi)^n \leq C_1 \|v\|^n_B.$$  

6. Let

$$D = \left\{ z : |z_n| < \frac{1}{2} \left( \frac{\lambda}{8} \right)^\alpha, \frac{\lambda}{8} < |z'| < \frac{\lambda}{4} \right\}.$$  

Then $D \subset T^{-1}(1/2 B)$ and $\mathcal{L}(D) \geq C_2 \varepsilon^2 \lambda^{2(n-1)}$ and

$$\min_D f = f \left( \frac{\lambda}{8}, 0, \ldots, 0, \frac{1}{2} \left( \frac{\lambda}{8} \right)^\alpha \right) \geq C_3 \lambda^\beta.$$
7. Let $\widetilde{B} = \frac{1}{2}T^{-1}(B)$. We thus obtain

$$-v(0) = \|v\|_B \geq \left( C_4 \int_{\frac{1}{2}B} (dd^c v)^n \right)^{1/n} = \left( \frac{C_4}{\varepsilon^2 \lambda^{2(n-1)}} \int_{\widetilde{B}} f d\mathcal{L} \right)^{1/n} \geq \left( \frac{C_4}{\varepsilon^2 \lambda^{2(n-1)}} \mathcal{L}(D) \min_D f \right)^{1/n} \geq C_5 \lambda^{\beta/n} = C_5 \varepsilon^{\beta/\alpha n}.$$

8. We therefore conclude

$$u(0, \varepsilon) - u(0) = (\varepsilon \lambda^{n-1})^{2/n} v(0) \geq C_5 \varepsilon^{\frac{2}{n}} \frac{\lambda^{2(n-1)} + \beta}{\alpha n} = C_5 \varepsilon^{\frac{2\alpha + 2(n-1) + \beta}{n\alpha}}. \quad \blacksquare$$

Now we can give our example for the unit ball in $\mathbb{C}^n$.

**Example 2.2.** Let $f$ be given by (2.1) where $A = 1$, $\beta = 2(n-1)(\alpha-1)-1$ and $a > 1/(n-1)$. Choose $\alpha$ such that $a \geq 2\alpha/\beta$. Then $f^a$ is $C^{1,1}$ but the solution $u$ of (1.1) with $\Omega = B$ and $\varphi \equiv 0$ is not $C^{1,1}$.

**Proof.** Since $f$ and $\varphi$ satisfy condition (2.2), by the uniqueness of solution, $u$ also satisfies (2.2). From Lemma 2.1 we conclude that $u$ is not $C^{1,1}$. $\blacksquare$

For $\mathbb{P}^n$ we have the following example:

**Example 2.3.** Let $\varrho : [0, \infty) \to [0, 1]$ be a function of class $C^\infty$ such that $\varrho|_{[0, 1]} \equiv 1$ and $\varrho|_{[2, \infty]} \equiv 0$. View $\mathbb{P}^n$ as a Kähler manifold carrying the Fubini–Study metric $\omega = \sum g_{ij} dz_i \wedge d\bar{z}_j$. Write $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$ where $\mathbb{P}^{n-1}$ is the hyperplane at infinity. Let $\tilde{f} : \mathbb{P}^n \to \mathbb{R}_+$ be a continuous function given in local coordinates in $\mathbb{C}^n$ by

$$\tilde{f} = \frac{\varrho(|z|) f}{4^n n! \det(g_{ij})},$$

where $f$ is given by (2.1), $A$ is such that $\tilde{f}$ satisfies the necessary condition (1.2), $\beta = 2(n-1)(\alpha-1)-1$ and $a > 1/(n-1)$. Choose $\alpha$ such that $a \geq 2\alpha/\beta$. Then $\tilde{f}^a$ is $C^{1,1}$ but the solution $\phi$ of (1.3) with $M = \mathbb{P}^n$ and with $\tilde{f}$ in place of $f$ is not $C^{1,1}$.

**Proof.** In local coordinates in $\mathbb{C}^n$, $g_{ij}$ are given by $g_{ij} = (\frac{1}{2} \log(1 + |z|^2))_{ij}$. Let $u = \phi + \frac{1}{2} \log(1 + |z|^2)$. Then $u$ is a continuous solution of the Monge–Ampère equation

$$(dd^c u)^n = f \varrho \quad \text{in } \mathbb{C}^n.$$ 

Since $\phi$ is bounded, we have $\lim_{|z| \to +\infty} u = +\infty$. Then from the same argument as in the proof of Lemma 2.1 we see that the function $z_n \mapsto v(0, z_n)$ reaches its strict minimum at $z_n = 0$. Since $\omega$ and $\tilde{f}$ satisfy condition (2.2), by the uniqueness of solution $\phi$ satisfies (2.2) and so does $u$. From Lemma 2.1, $u$ is not $C^{1,1}$, so neither is $\phi$. $\blacksquare$
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References


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